Random matrix theory of dissipative quantum chaos

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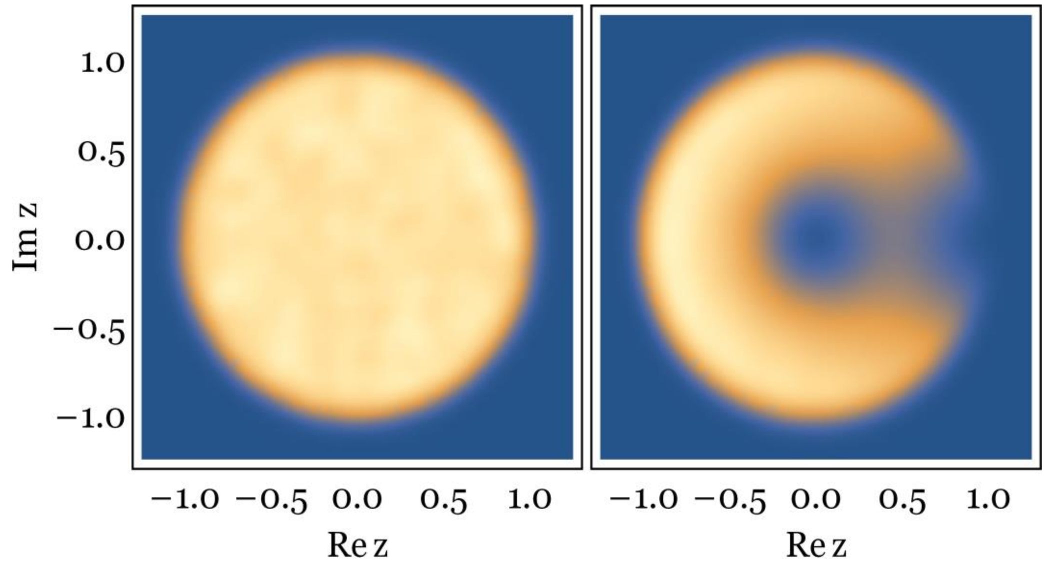
arXiv:1905.02155 [J. Phys. A (2020)]
arXiv:1911.02136
arXiv:2006.xxxxx



TÉCNICO LISBOA

CeFEMA
Center of Physics and Engineering of Advanced Materials

FCT Fundação para a Ciência e a Tecnologia

with P. Ribeiro, T. Prosen, and T. Can

Outline

- Motivation and preliminaries
  - Random matrix theory
  - Quantum chaos
  - Non-Hermitian quantum physics

- Statistical mechanics of complex non-Hermitian systems
  - Spectral properties
  - Steady-state properties

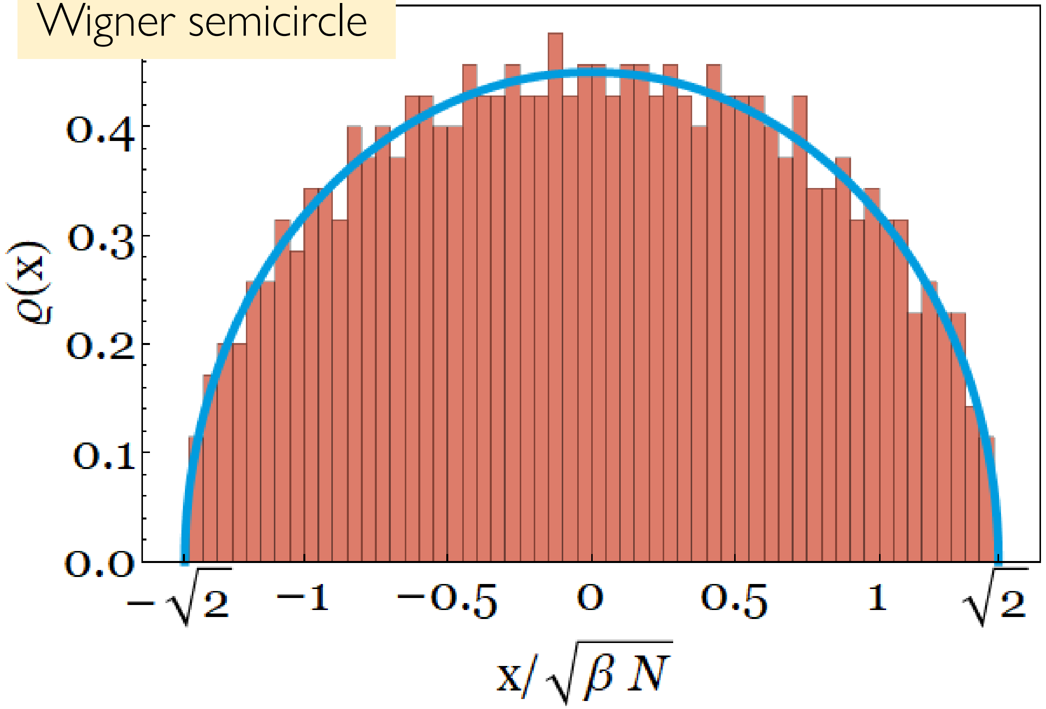
- Signatures of dissipative quantum chaos
  - Complex spacing ratios
  - Toric unitary ensemble

General setup: complex systems & RMT

- How to describe a complicated quantum system?
  - We do not know how to solve most Hamiltonians H
  - Sometimes we cannot even write down H
  - e.g. heavy nuclei (Wigner '55)
- Principle of maximal entropy (ignorance) $\Longrightarrow$ model H as large (hermitian) random matrix

$$P(H)dH = e^{-\mathrm{Tr}(V(H))}|\Delta(\Lambda)|^{\beta} d\Lambda\, dU_{\mathrm{Haar}} \qquad H = U\Lambda U^{-1}$$

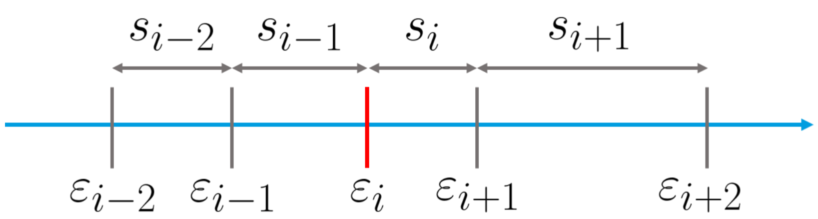
Wigner semicircle

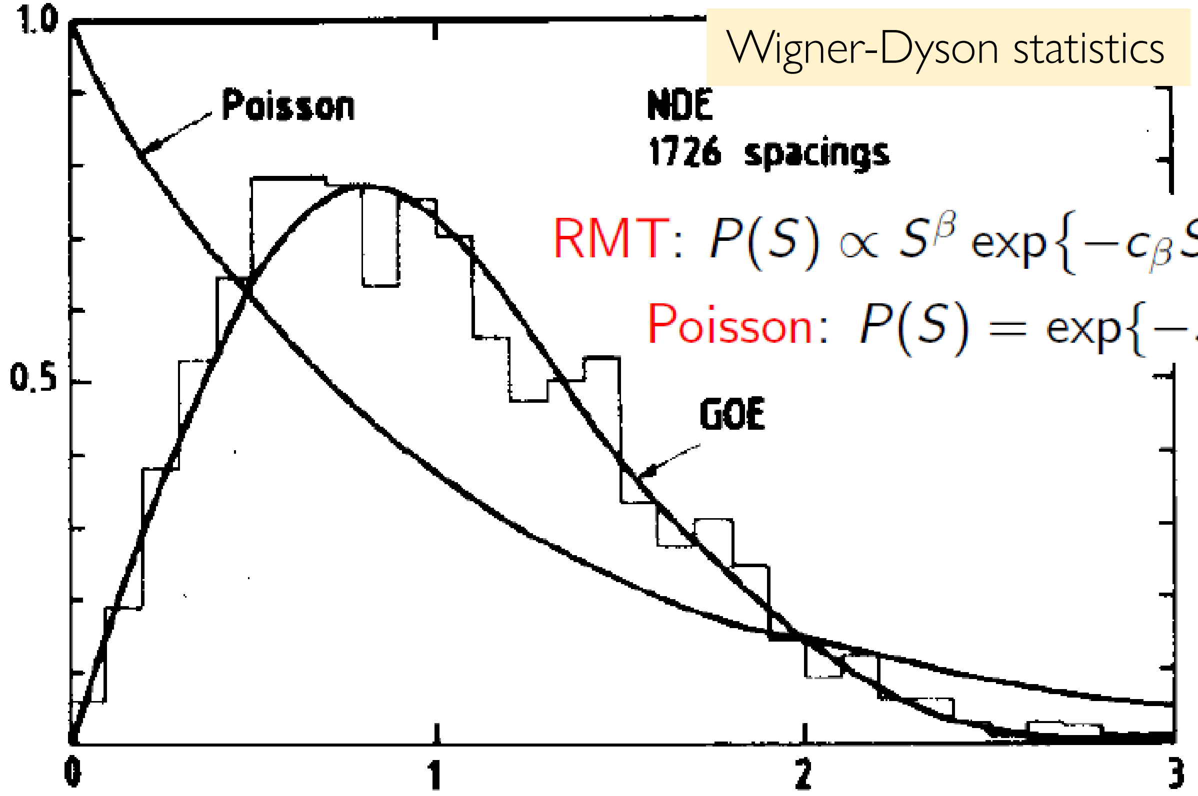


- β – Dyson index
- Statistical properties fully determined by behaviour under time-reversal symmetry T
  - $T = 0 \Longrightarrow \beta = 2 \Longrightarrow$ unitary symmetry (GUE)
  - $T^2 = +1 \Longrightarrow \beta = 1 \Longrightarrow$ orthogonal symmetry (GOE)
  - $T^2 = -1 \Longrightarrow \beta = 4 \Longrightarrow$ symplectic symmetry (GSE)

Level spacing statistics: RMT vs Poisson

Nuclear energy level correlations:





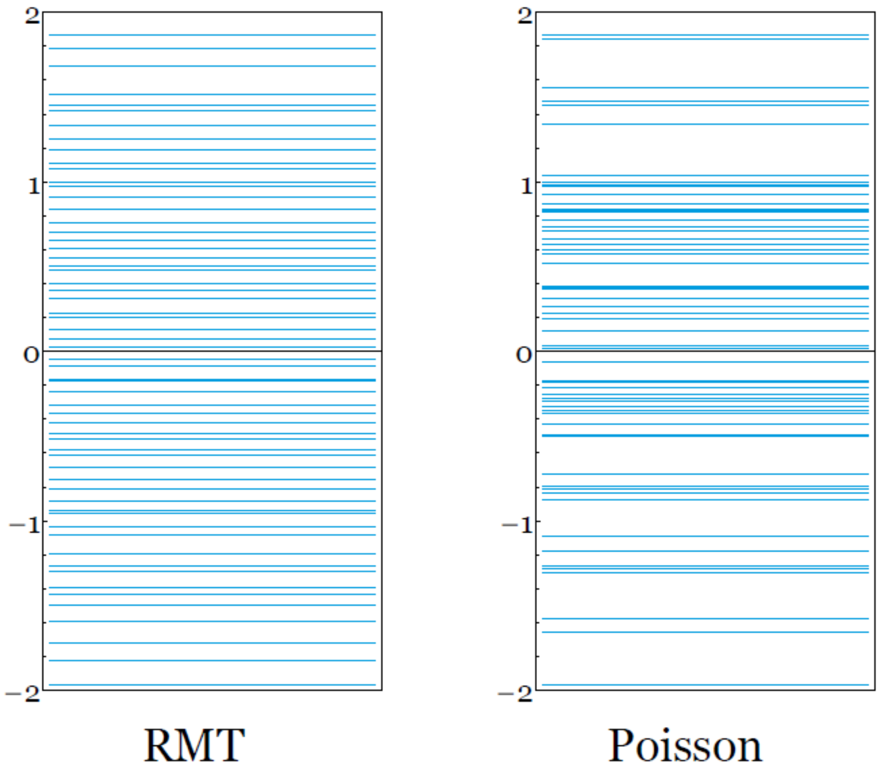
RMT: $P(S) \propto S^{\beta} \exp\{-c_{\beta} S^2\}$

Poisson: $P(S) = \exp\{-S\}$

O. Bohigas, *et al.* in *Nuclear Data for Science and Technology*, K. H Böchhoff (Reidel, 1983), p. 809.

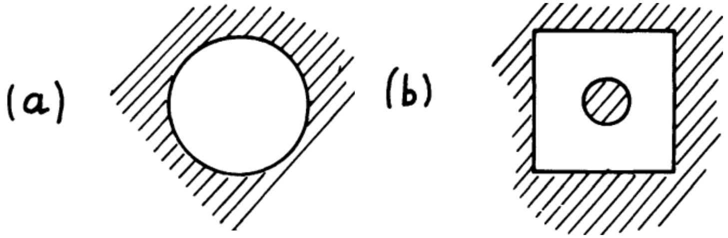
Attractiveness of RMT: **Universality**

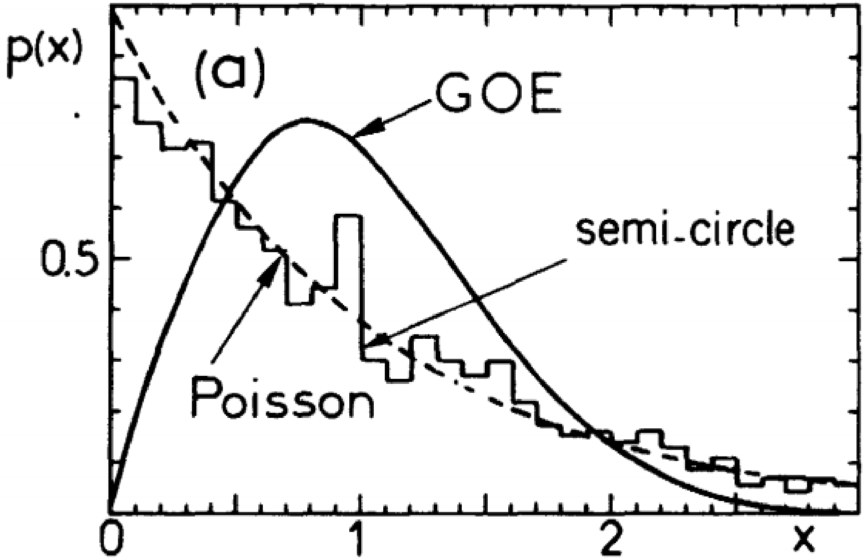
- Statistical properties largely independent of RMT ensemble (only symmetry)
- Statistical properties of complex systems independent of microscopic details

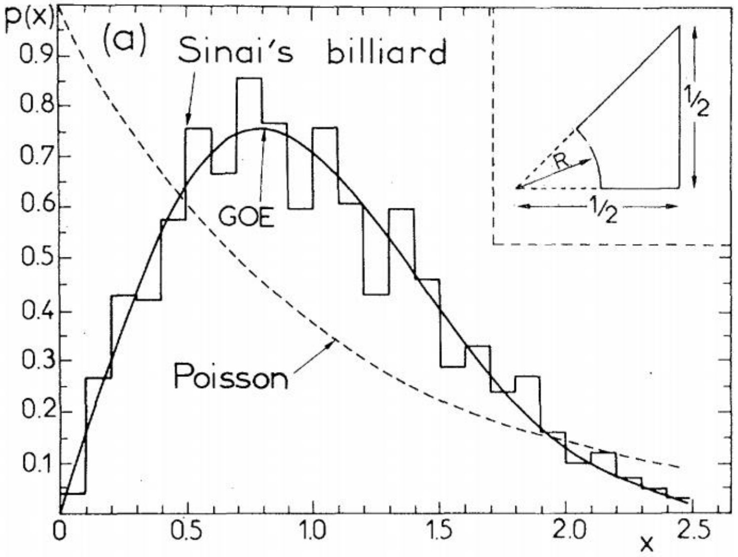


Quantum chaos conjecture

Quantum chaos in semiclassical billiards







[Bohigas, Giannoni, Schmit, PRL **52** 1 (1984)], [Bohigas, Giannoni, Schmit, J. Physique Lett. **45** L-1015 (1984)]

Quantum chaos conjecture

Berry & Tabor '77: classical integrable systems ⟺ Poisson quantum level statistics

Bohigas, Giannoni, & Schmit '84: chaotic semiclassical limit ⟺ RMT quantum level statistics

RMT statistics has become the working definition of quantum non-ergodic behaviour

- Single-particle semiclassical systems (classical chaotic limit)
- Quantum many-body systems (non-integrable)

[Kos, Ljubotina, Prosen, PRX 8 021062 (2018)]
[Chan, De Luca, Chalker, PRX 8 041019 (2018)]

Non-Hermitian (quantum) physics

- Lindbladian Dynamics
  - Continuous-in-time, Markovian, CPTP
  - Quantum optics, monitored systems

$$\partial_t \rho = \mathcal{L}(\rho)$$

$$\mathcal{L}(\rho) = -i\,[H,\rho] + \sum_\mu \left(L_\mu \rho L_\mu^\dagger - \frac{1}{2} L_\mu^\dagger L_\mu \rho - \frac{1}{2} \rho L_\mu^\dagger L_\mu \right)$$

- Kraus Maps
  - Discrete-in-time, most general maps
  - Floquet Physics

$$\rho_t = \Phi^t(\rho_0)$$

$$\Phi(\rho) = \sum_\mu K_\mu \rho K_\mu^\dagger, \quad \sum_\mu K_\mu^\dagger K_\mu = \mathbb{I}$$

- Effective non-Hermitian Hamiltonians
  - Dissipative (many-body) localization
  - Scattering

- Classical Asymmetric Markov Processes

$$\partial_t \mathbf{P}(t) = M\mathbf{P}(t)$$

- Nonhermitian topological phases/symmetry classes

[Gong et al. PRX 8 (2018)]
[Kawabata et al. PRX 9 041015 (2019)]

Random matrix theory of dissipative quantum chaos

Two-fold program:

- Statistical mechanics of complex non-Hermitian systems
  - RMT model of non-Hermitian generators ($\mathcal{L}$, Φ, H_{eff}, M)
  - Spectral properties
  - Spectral gap (relaxation time)
  - Steady-state properties

[LS, Ribeiro, Prosen, JPhysA '20], [LS, Ribeiro, Can, Prosen '20], [Bruzda et al, PhysLettA '09], [Timm PRE '09], [Denisov et al, PRL '19], [Can, JPhysA '19], [Can et al, PRL '19], [Wang, Piazza, Luitz PRL'20]

- Signatures of dissipative quantum chaos
  - Classification of different phases/symmetries
  - Spectral correlations in non-Hermitian RMT (angular correlations)
  - Toolbox of signatures of dissipative quantum chaos (new non-Hermitian RMT ensembles)

[LS, Ribeiro, Prosen, PRX '20], [Grobe et al, PRL '88], [Akemann et al, PRL '19], [Hamazaki, PRR '20]

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Construction of a random Liouvillian

[LS, Ribeiro, Prosen, JPhysA '20]

$$\partial_t \rho = \mathcal{L}(\rho)$$

$$\mathcal{L}(\rho) = -i[H,\rho] + g^2 \sum_{j,k=1}^{N^2-1} d_{jk} \left\{ G_j \rho G_k^\dagger - \frac{1}{2}\left[\rho G_k^\dagger G_j + G_k^\dagger G_j \rho\right] \right\}$$

Gaussian Unitary Ensemble

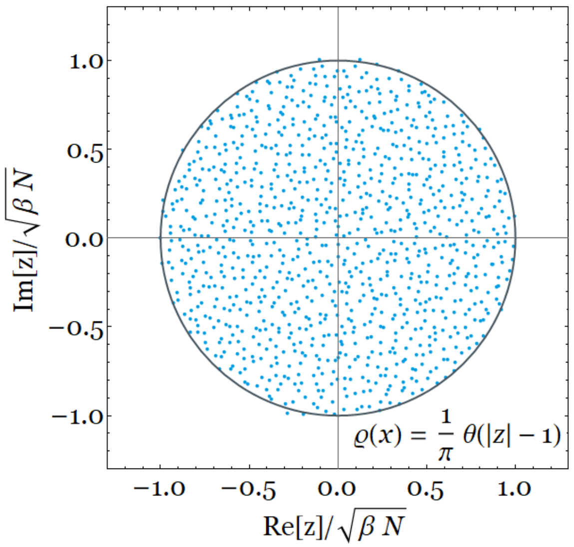
$$P_N(H) \propto \exp\left\{-\frac{1}{2}\mathrm{Tr}\left(H^2\right)\right\}$$

$$d_{jk} = \sum_{\ell=1}^{r} w_{j\ell} w_{k\ell}^* = (ww^\dagger)_{jk}$$

Ginibre Unitary Ensemble [Ginibre JMP '65]

$$P_{(N^2-1,r)}(w) \propto \exp\left\{-\frac{1}{2}\mathrm{Tr}\left(w^\dagger w\right)\right\}$$

- Spectral properties largely independent of particular sampling scheme of dissipator
- Hints towards universality [Denisov et al, PRL '19]



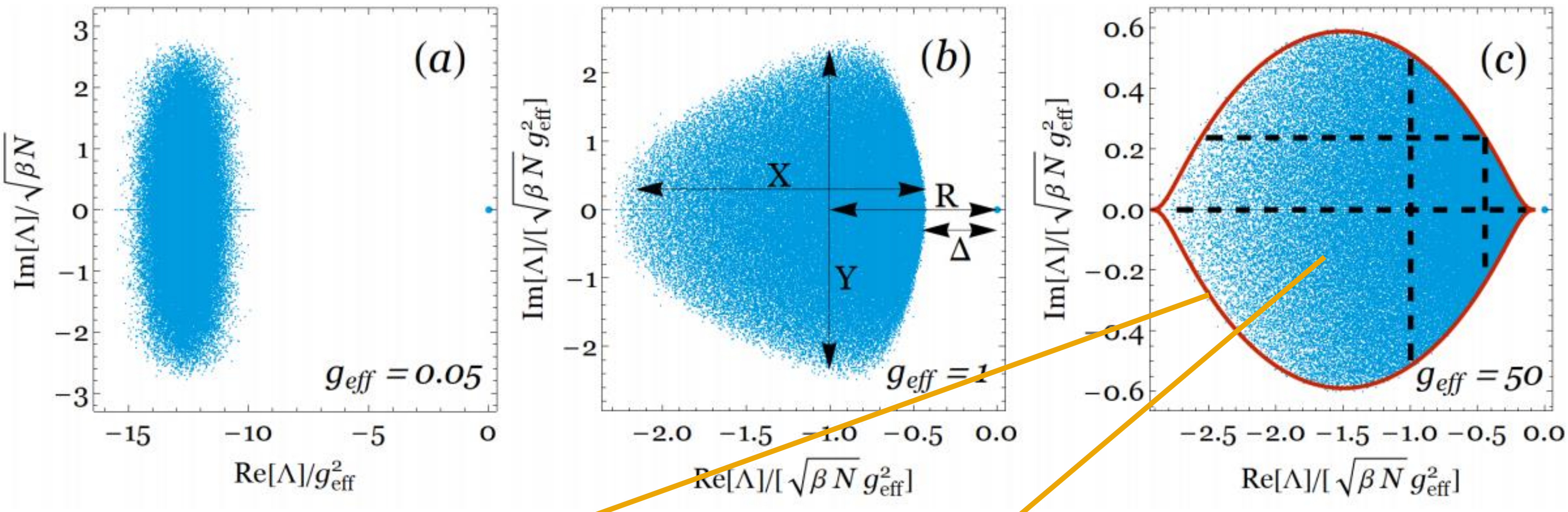
Parameters of the model:

N – system size
g – dissipation strength
r – rank of dissipator (# jump operators)

The spectrum of a random Liouvillian

$$\rho(t) = \sum_\alpha c_\alpha e^{\Lambda_\alpha t} \rho_\alpha$$

- Numerical diagonalization



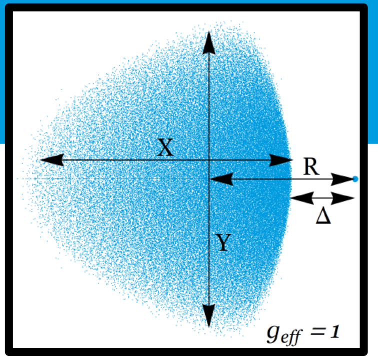
$$g_{\mathrm{eff}} = (2r\beta N)^{1/4}\, g$$

Boundary analytically computed using quaternionic free probability
[Denisov et al, PRL **123**, 140403 (2019)]

Bulk spectral density?
Open question

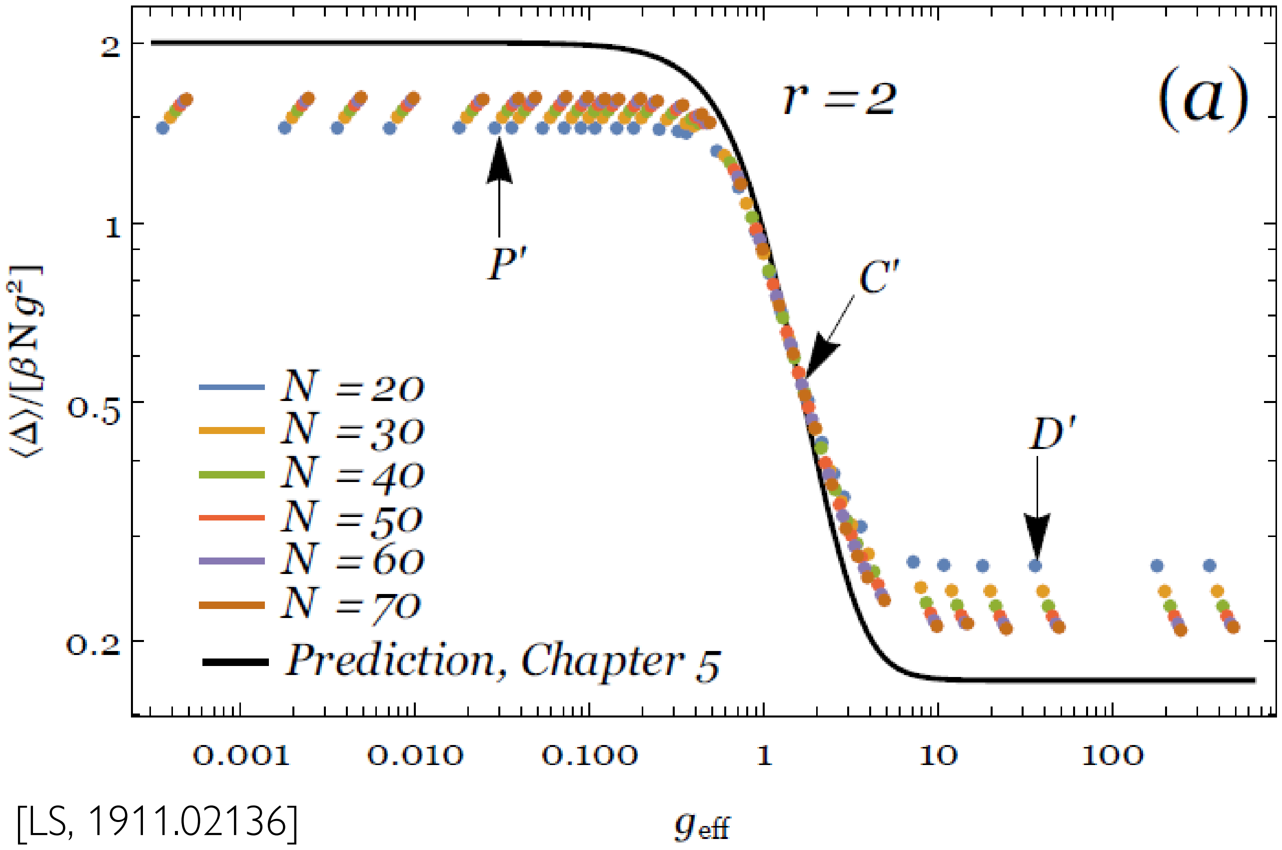
$$\varrho(\Lambda) = \left\langle \sum_\alpha \delta(\Lambda - \Lambda_\alpha) \right\rangle$$

The spectral gap of the random Liouvillian



$\Delta = \min_{\alpha>0} \mathrm{Re}\,(-\Lambda_\alpha)$

- The spectral gap, Δ, is a particularly important spectral feature, since it determines the long-time relaxation asymptotics.
- Although there are three regimes, there is a single scaling function, which can be computed exactly for $N \to \infty$, using holomorphic Green's function methods.



$$G(z) = \frac{1}{N}\mathrm{Tr}\left\langle \frac{1}{z-\mathcal{L}} \right\rangle = \frac{1}{N}\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \mathrm{Tr}\left\langle \mathcal{L}^n \right\rangle$$

- Diagrammatic expansion
- Resummation of non-crossing planar diagrams
- Classical convolution of two non-Hermitian Hamiltonian spectra

$\Delta = -2\sqrt{\beta N}\tilde{y}$, where $\tilde{y}$ is the smallest real solution of

$$\frac{4r}{\tilde{g}\tilde{y}} = \left(\frac{r}{\tilde{y}} - \frac{\tilde{g}}{1+\tilde{g}\tilde{y}} + \frac{1}{\tilde{g}}\right)^2$$

with $\tilde{g} = -\sqrt{\beta N} g^2/2$.

[LS, 1911.02136]

Construction of a random Kraus map

[LS, Ribeiro, Can, Prosen '20]

$$\rho_t = \Phi^t(\rho_0) \qquad \Phi(\rho) = \sum_\mu K_\mu \rho K_\mu^\dagger, \quad \sum_\mu K_\mu^\dagger K_\mu = \mathbb{I}$$

$$\Phi = (1-p)\, U \otimes U^* + p \sum_{j=1}^{d} M_j \otimes M_j^*$$

U: Circular Unitary Ensemble (Haar-random unitary)

M_j: Truncations of Haar unitaries

$$V = \begin{pmatrix} M_1 & V_{12} & \cdots & V_{d1} \\ M_2 & V_{22} & \cdots & V_{d2} \\ \vdots & \vdots & \ddots & \vdots \\ M_d & V_{d2} & \cdots & V_{dd} \end{pmatrix}$$

V: $Nd \times Nd$; M_j: $N \times N$

$$\sum_{j=1}^{d} M_j^\dagger M_j = \mathbb{1}$$

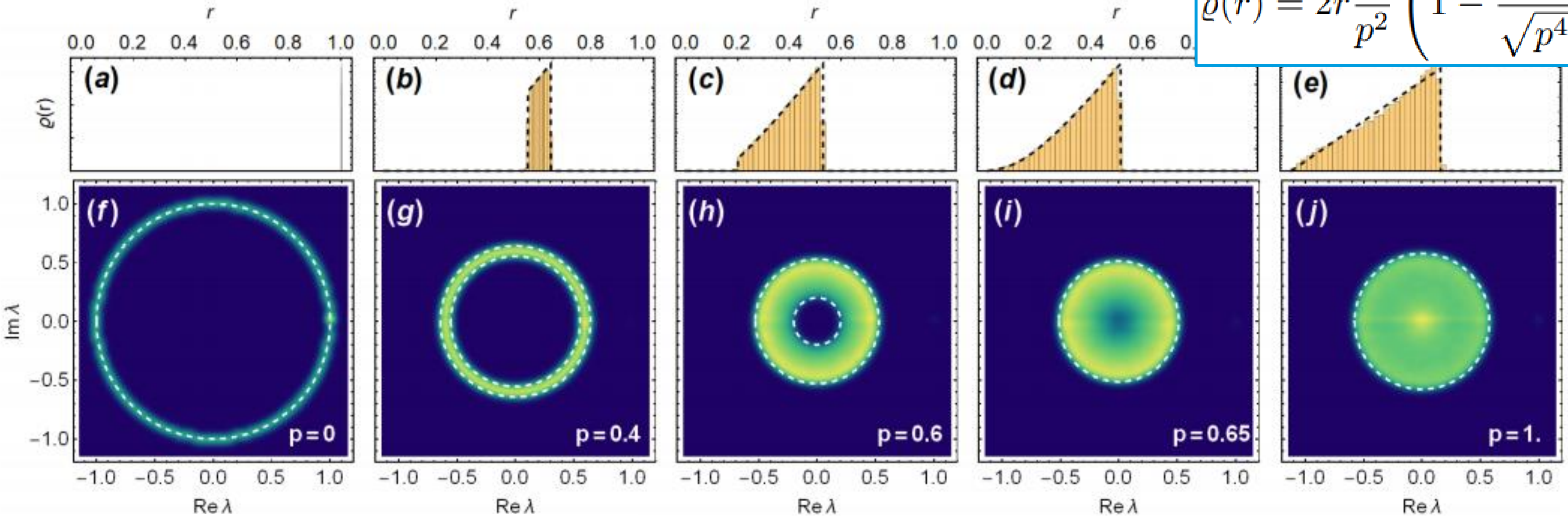
- This is a 0d random Floquet map.

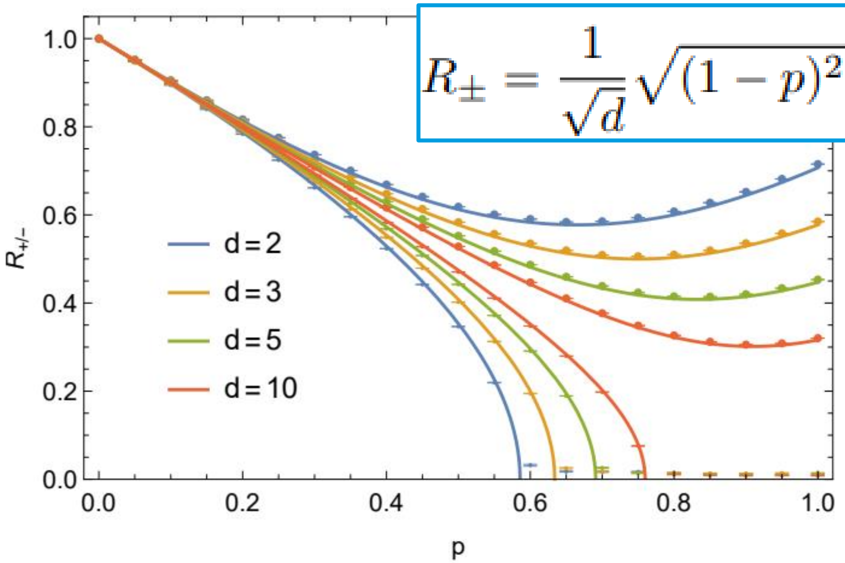
$$\rho_t = \Phi\left(\rho_{t-1}\right) = \Phi^t\left(\rho_0\right)$$

- But qualitatively similar results for a 1d circuit.
- Universality of statistical properties of random maps.

Spectral transition in Kraus maps

$$\varrho(r) = 2r\frac{d}{p^2}\left(1 - \frac{(1-p)^2 d}{\sqrt{p^4 + 4(1-p)^2 d^2 r^2}}\right)$$





$$R_{\pm} = \frac{1}{\sqrt{d}}\sqrt{(1-p)^2 d \pm p^2}$$

Effective RMT model

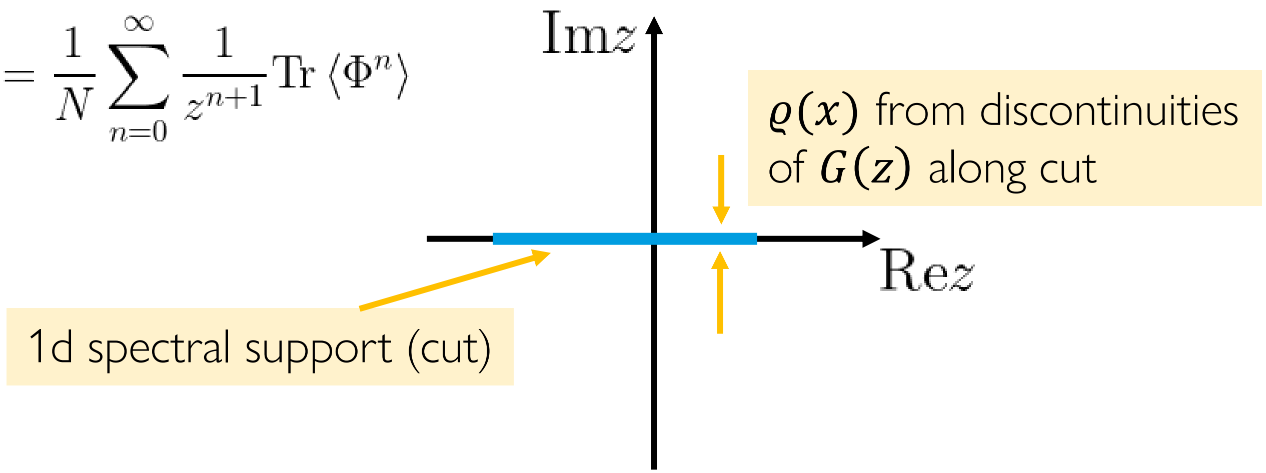
$$\tilde{\Phi} = (1-p)\,\mathbb{U} + \frac{p}{\sqrt{d}}\mathbb{G}$$

Haar-random unitary

Ginibre Unitary Ensemble

RMT & Quaternionic Free Probability

- Holomorphic Green's function $G(z) = \frac{1}{N}\mathrm{Tr}\left\langle \frac{1}{z-\Phi} \right\rangle = \frac{1}{N}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\mathrm{Tr}\left\langle \Phi^n \right\rangle$



- When the random matrix is not normal ($[\Phi, \Phi^\dagger] \neq 0$), spectral support is 2d region
- Analytically continue off complex plane, into quaternions
- Quaternionic Green's function

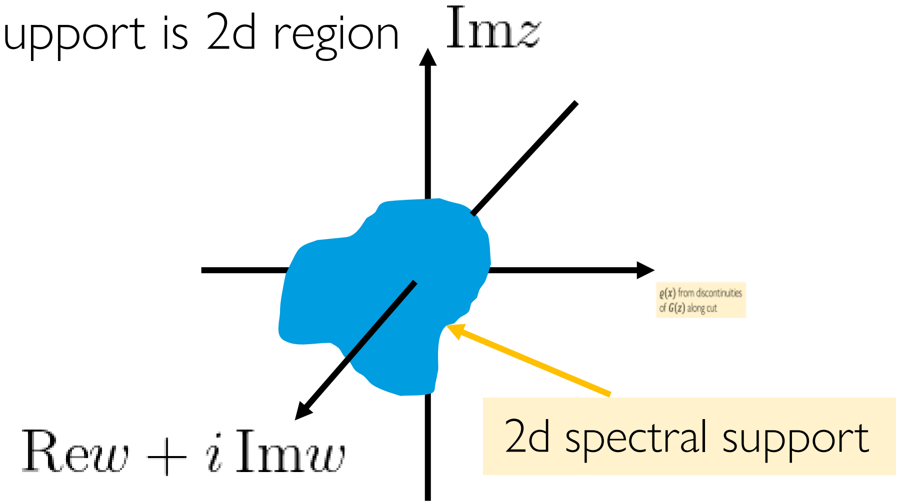
$$\mathcal{G}(Q) = \left\langle \frac{1}{N}\mathrm{bTr}(Q-\mathcal{H})^{-1} \right\rangle$$

$$\varrho(z,\bar{z}) = \frac{1}{\pi}\partial_{\bar{z}}\mathcal{G}_{11}(z,\bar{z})$$

$$Q = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$$

$$\mathcal{G} = \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} \\ -\bar{\mathcal{G}}_{12} & \bar{\mathcal{G}}_{11} \end{pmatrix}$$

$$\mathcal{H} = \begin{pmatrix} \Phi & 0 \\ 0 & \Phi^\dagger \end{pmatrix}$$



Universal Steady-States

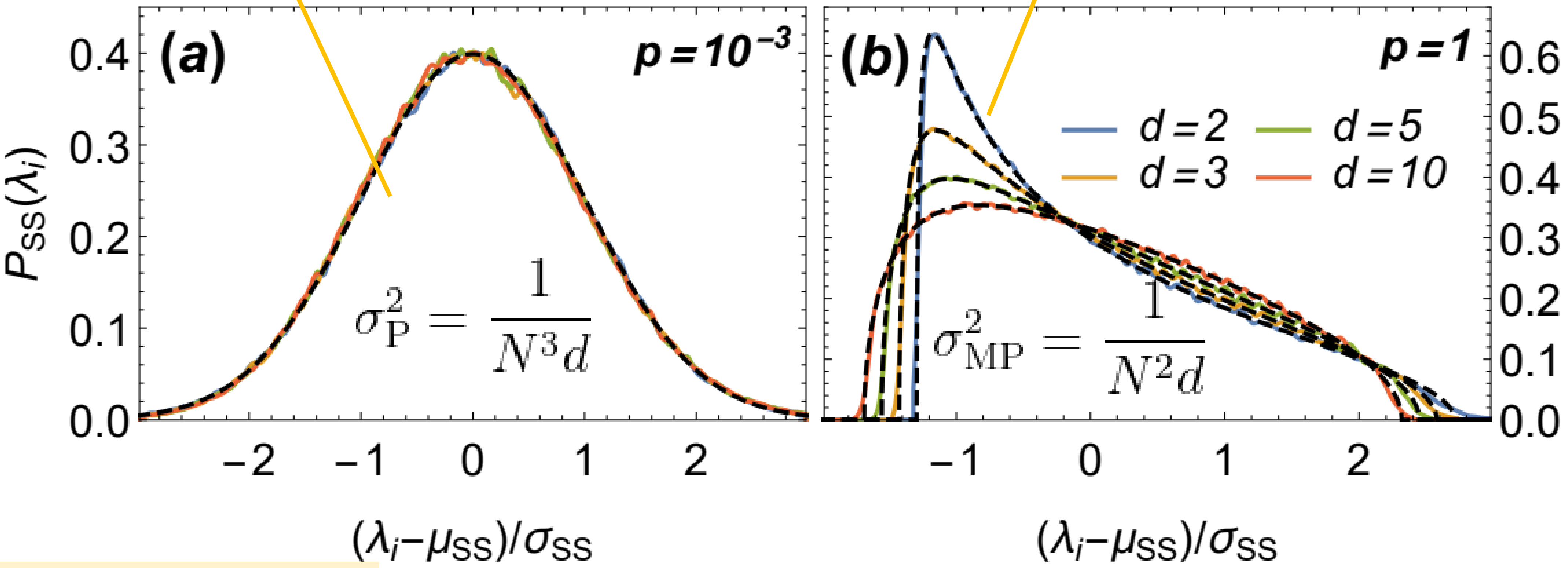
$$\lim_{t\to\infty} \rho(t) = \rho_{\rm SS}$$

Exactly the same steady-state properties for the Liouvillian and Kraus steady-states.

- Gaussian distribution
- Perturbative treatment $\Longrightarrow \rho_{\alpha\alpha} = \sum_{\gamma=1}^{N} T_{\alpha\gamma}\rho_{\gamma\gamma}$ (classical probability eq. in degenerate subspace)

- Marchenko-Pastur distribution (Wishart matrices) $W = XX^\dagger$, X rectangular Ginibre
- Entanglement spectrum of bipartite systems

[Życzkowski, Sommers, JPhysA 34, 7111 (2001)]



$\Longrightarrow$ different purity scalings

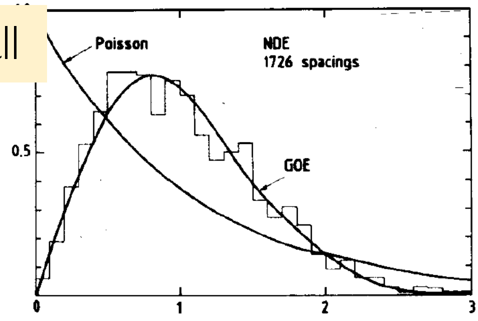
Perturbative crossover in the SS

$$\lim_{t\to\infty} \rho(t) = \rho_{\rm SS}$$

Effective Hamiltonian $\mathcal{H}_{\rm SS} = -\log \rho_{\rm SS}$

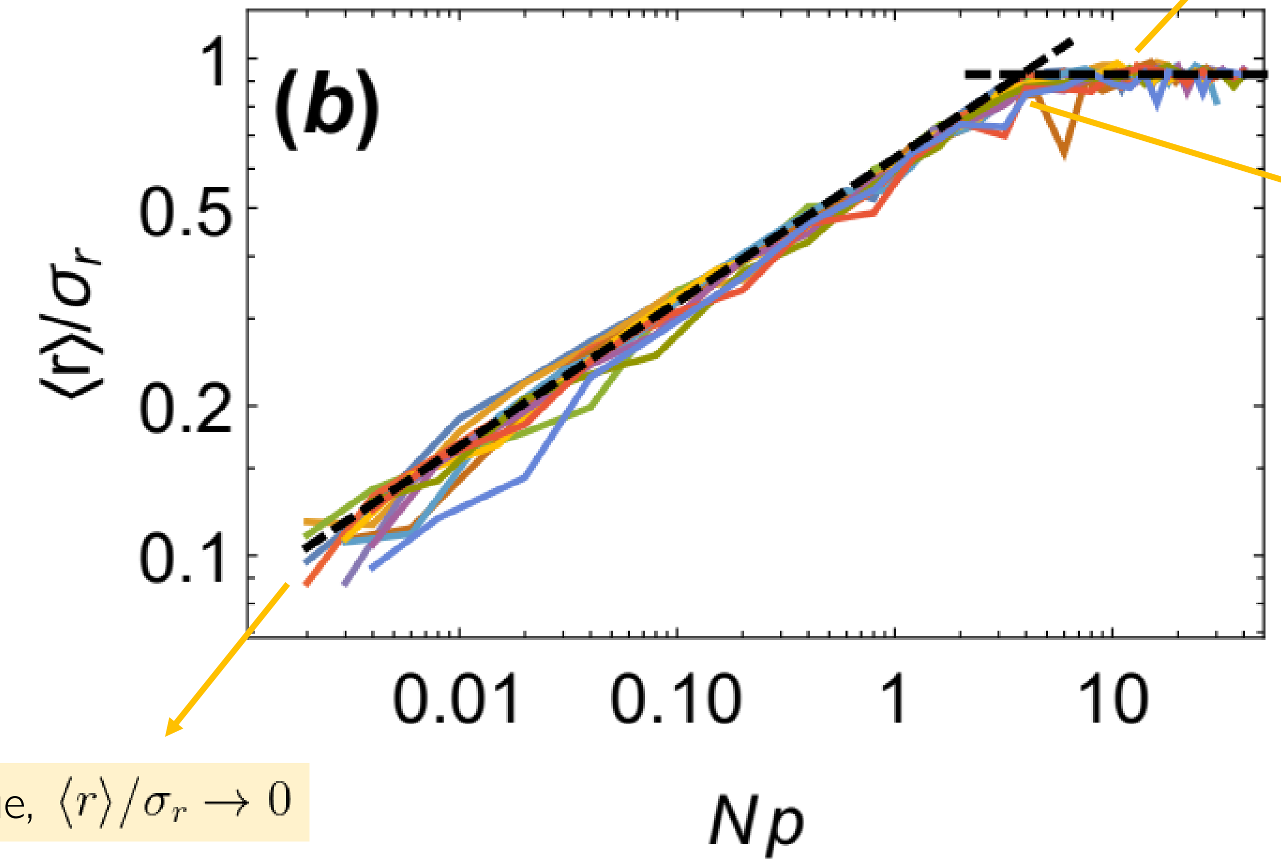
SS correlation statistics $r_i = \dfrac{\varepsilon_{i+1} - \varepsilon_i}{\varepsilon_i - \varepsilon_{i-1}}$

Recall



O. Bohigas, *et al.* in *Nuclear Data for Science and Technology*, K. H. Böchhoff (Reidel, 1983), p. 809.

RMT value, $\langle r \rangle / \sigma_r \approx 0.928$



$p \gtrsim 2\pi/N = \delta^{-1}$

Poisson value, $\langle r \rangle / \sigma_r \to 0$

- No signature of spectral transition in the steady state.
- But, perturbative integrable → ergodic crossover.

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(Hermitian) spacing distributions revisited: three remarks

1. Wigner surmises

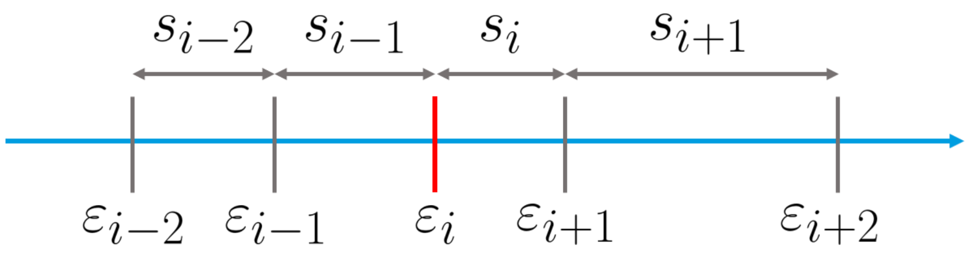
Small-*N* distributions describe large-*N* universal behaviour

2. Unfolding

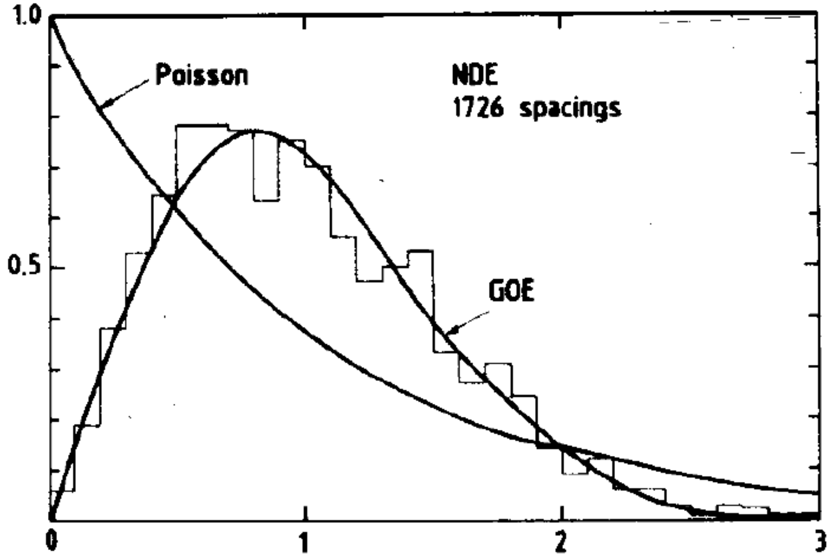
- Removes dependence from spectral density (very non-universal).
- Fix $\langle s \rangle = 1 \Leftrightarrow$ rescale levels using average density $\bar{\varrho}$.
- Difficult procedure if $\bar{\varrho}$ is not known. Can be numerically unreliable.

3. Consecutive spacing ratios

[Oganesyan, Huse, PRB **75**, 155111(2007)]; [Atas, Bogomolny, Giraud, Roux, PRL **110**, 084101(2013)]



$$r_i = \frac{s_{i+1}}{s_i}$$



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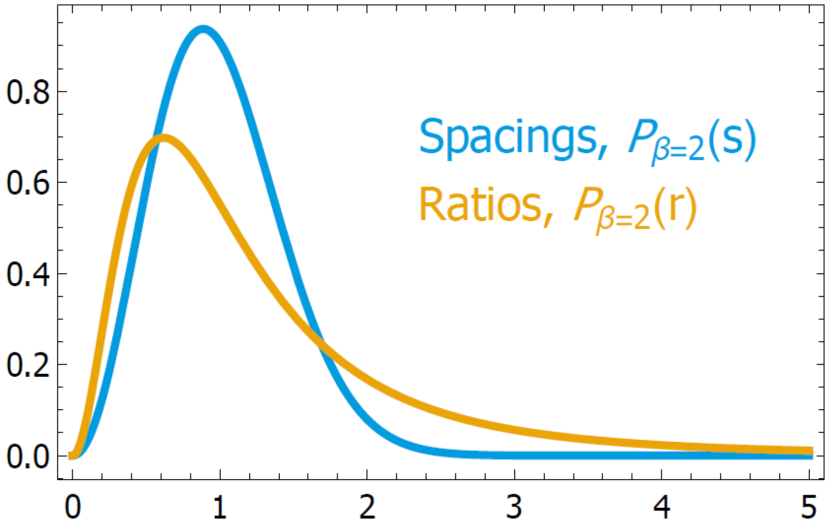
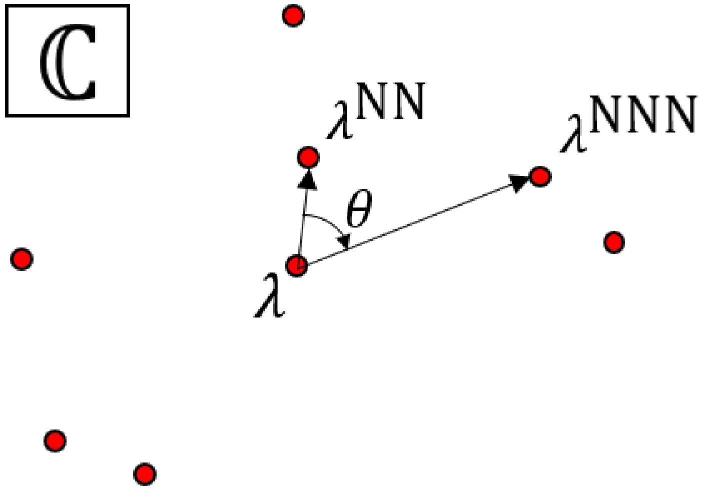


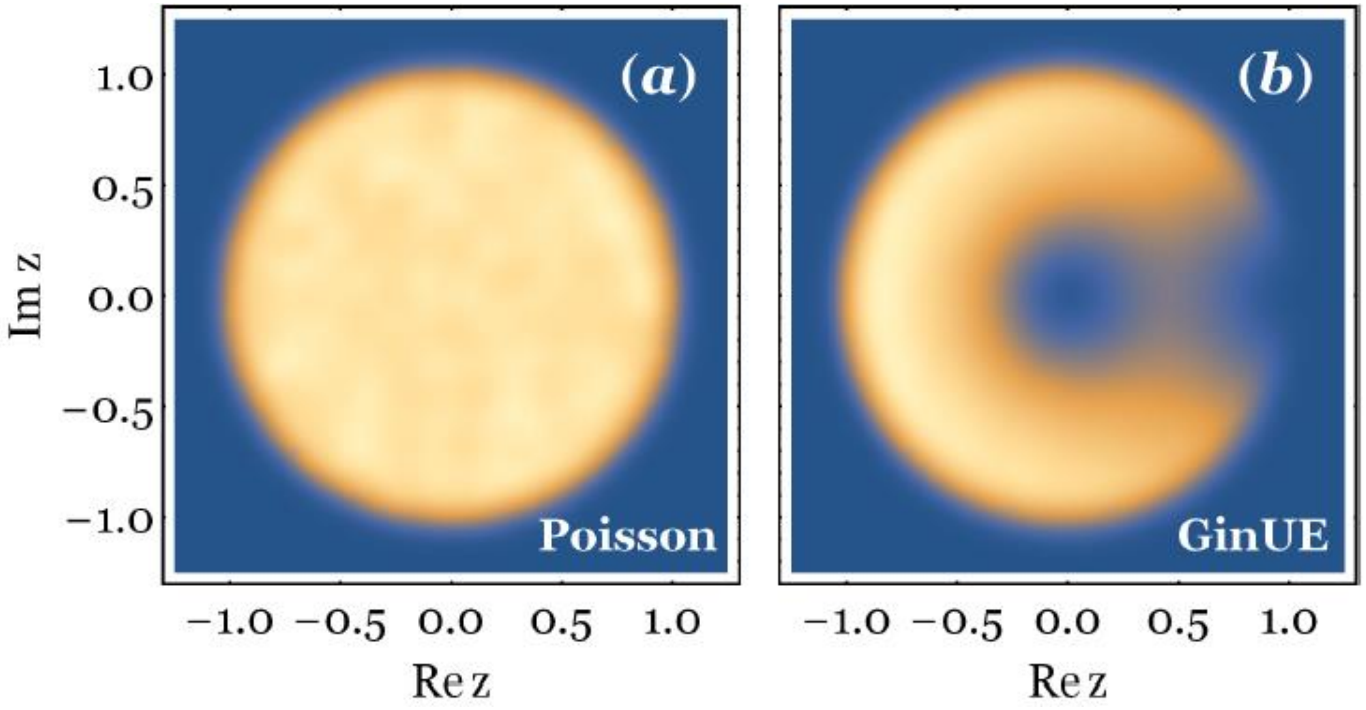
Complex spacing ratios: RMT vs Poisson

Definition



$$z = \frac{\lambda^{\mathrm{NN}} - \lambda}{\lambda^{\mathrm{NNN}} - \lambda} \equiv r e^{i\theta}$$

$$\varrho^{(N)}(z) = ?$$



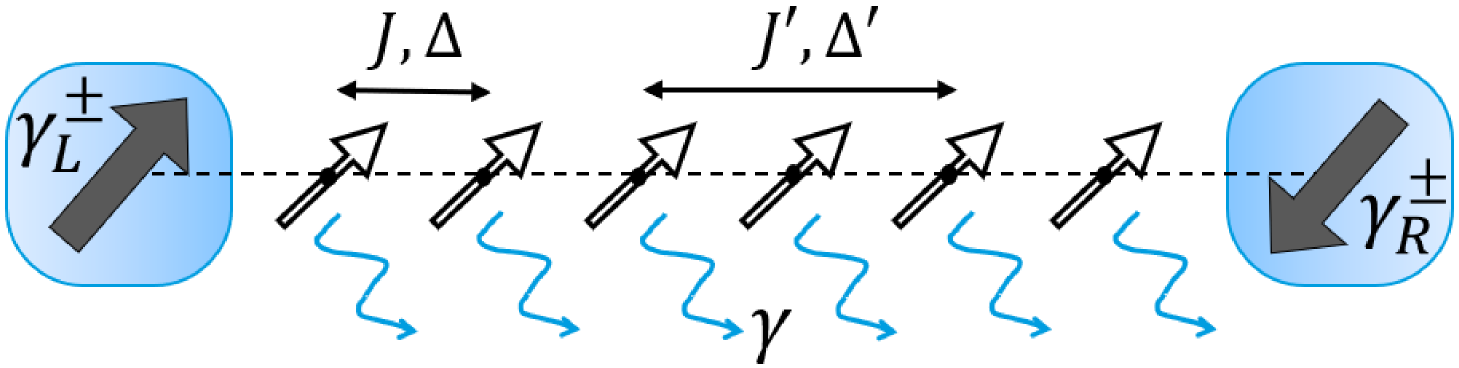
	Poisson	Ginibre
$-\langle \cos\theta \rangle$	0	0.24051(61)
$\langle r \rangle$	2/3	0.73810(18)

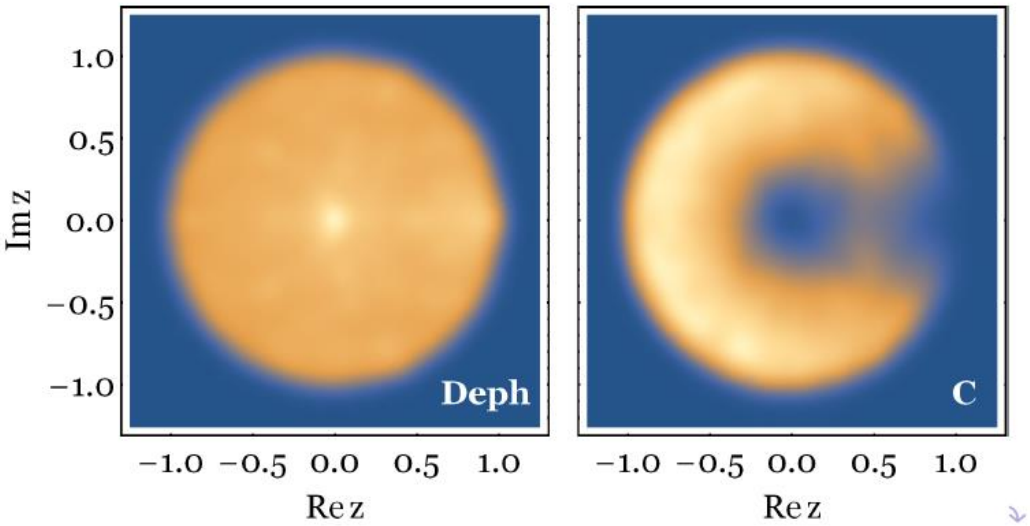
Example I. Liouvillian for boundary-driven spin chain

- Liouvillian time evolution
  - Heisenberg XXZ Hamiltonian (nearest- and second nearest neighbours)
  - Dephasing of all (bulk) spins
  - Spin polarization (driving) at the boundary spins

$$\partial_t \rho = \mathcal{L}(\rho)$$

$$\mathcal{L}(\rho) = -i\,[H,\rho] + \sum_\mu \left(L_\mu \rho L_\mu^\dagger - \frac{1}{2} L_\mu^\dagger L_\mu \rho - \frac{1}{2} \rho L_\mu^\dagger L_\mu \right)$$





- (Deph) – boundary driven XX chain with bulk dephasing (⇔ integrable) [Medvedyeva, Essler, Prosen, PRL **117**, 137202 (2016)]
- (C) – XXZ chain with nearest and next-to-nearest neighbor interactions (arbitrary parameters ⇒ chaotic)

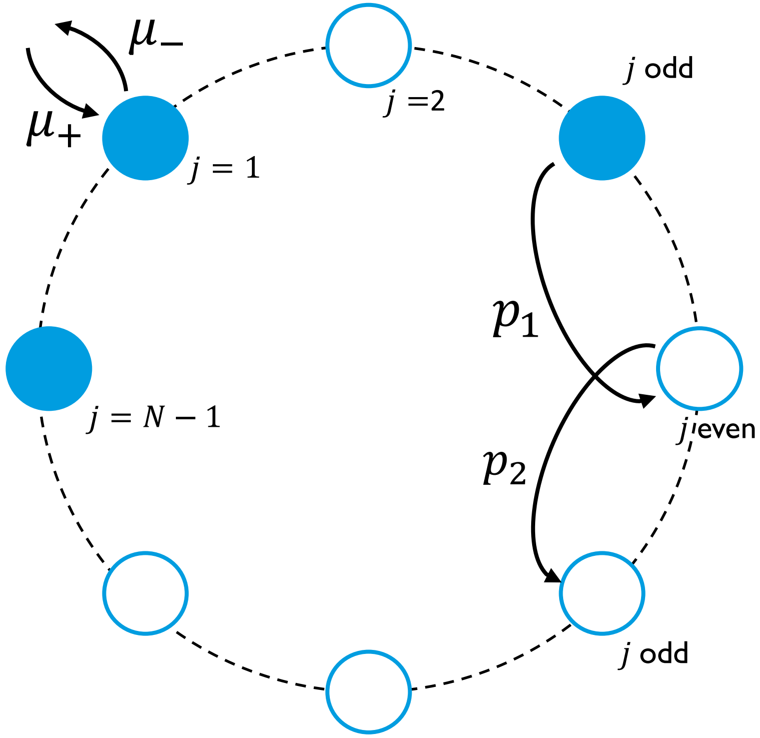
Example II. Markov matrix for exclusion process

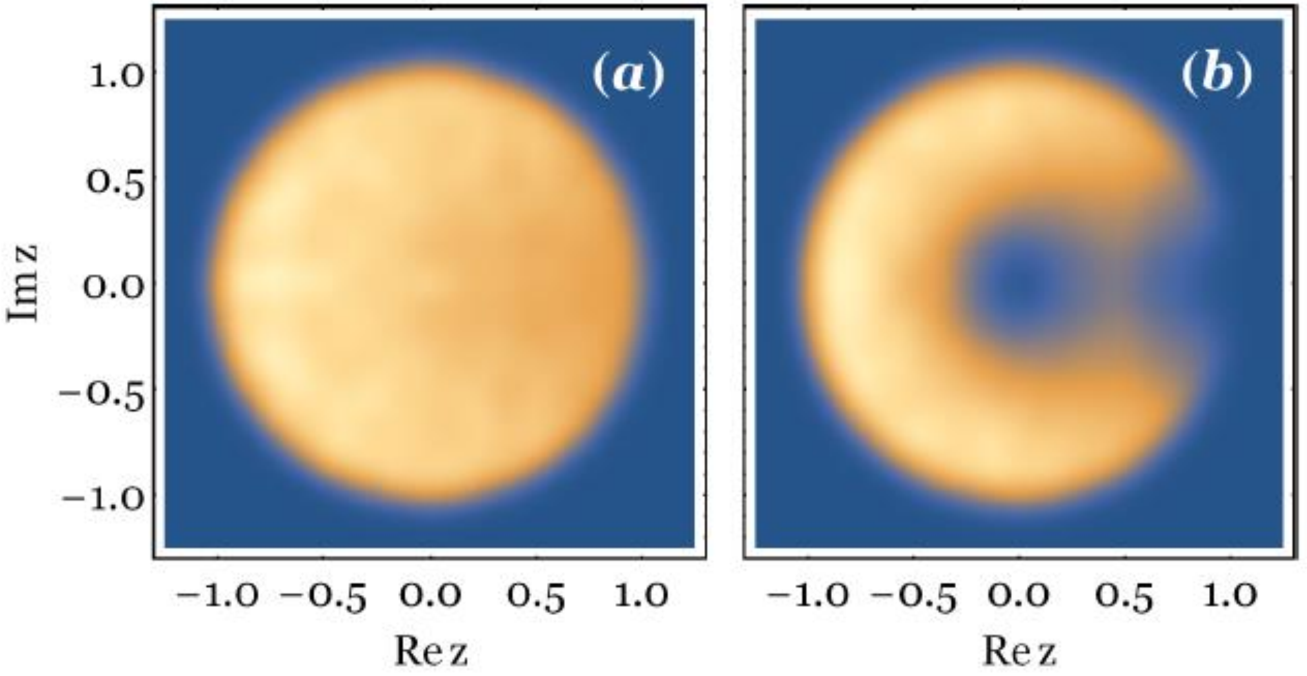
- Classical stochastic process
  - Simple exclusion
  - Particle injection (at $j = 1$)
  - Possibly staggered hopping ($p_1 \neq p_2$)

$$\partial_t \boldsymbol{P}(t) = M\boldsymbol{P}(t)$$

Markov (stochastic) matrix
Specifies transition rates

$$M_{jk} = A_{jk} - \delta_{jk} \sum_m A_{mk}$$





- (a) non-staggered hopping – integrable TASEP
- (b) staggered hopping expected to break integrability

Analytical ratio distribution from Wigner-like surmise

- **Poisson:** Assume isotropy. Easy to show distribution is flat. [LS, Ribeiro, Prosen, PRX **10**, 021019 (2020), App. A]
- RMT (unitary ensembles):

From the definition of spacing ratio and the (known) distribution of eigenvalues: $z = \frac{\lambda^{\mathrm{NN}} - \lambda}{\lambda^{\mathrm{NNN}} - \lambda}$

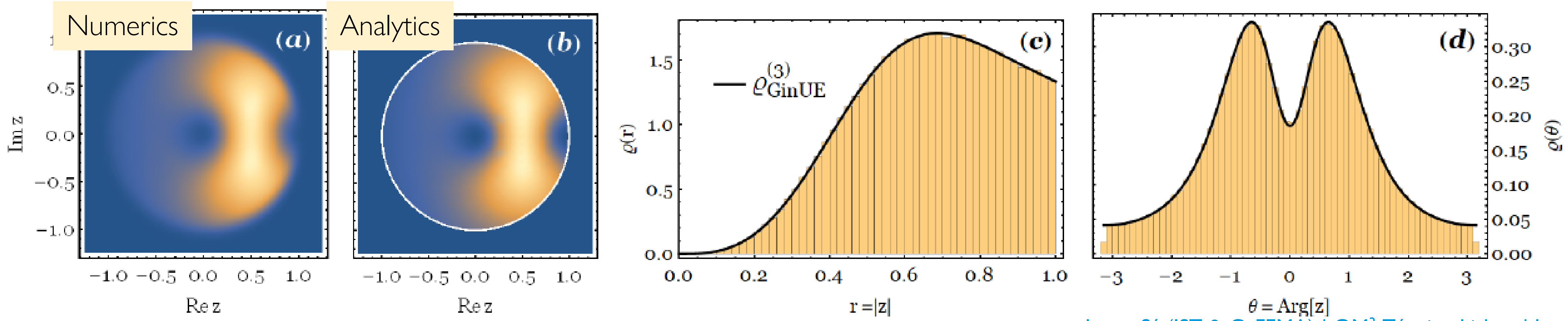
$$\varrho^{(N)}(z) = \int d\lambda_1 \cdots d\lambda_N P^{(N)}(\lambda_1, \ldots \lambda_N)\, \delta^{(2)}\left(z - \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}\right) \Theta\left(|\lambda_3 - \lambda_1|^2 - |\lambda_2 - \lambda_1|^2\right) \prod_{j=4}^{N} \Theta\left(|\lambda_j - \lambda_1|^2 - |\lambda_3 - \lambda_1|^2\right)$$

Eigenvalue distribution (invariant under permutation of λ)

Enforce definition of z

Enforce that λ_2 and λ_3 are λ^{NN} and λ^{NNN}

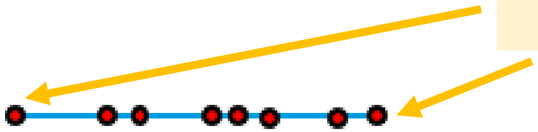
- Ginibre ensemble, $N = 3$:



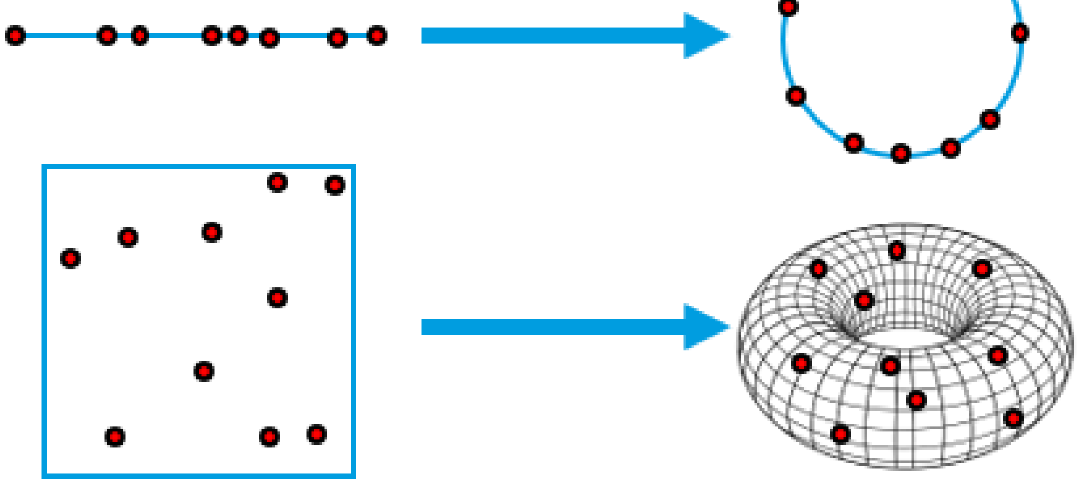
Boundary effects and the Toric Unitary Ensemble

- **Wigner surmise:** Small-N distributions describe large-N universal behaviour.
- But $\varrho^{(3)} \not\approx \varrho^{(N\to\infty)} \implies$ **No GinUE surmise!**
- Why? **Boundary effects:** boundaries favor small angles, but bulk suppresses them.
- Solution: "Periodic boundary conditions" $\implies$ **Toric Ensembles**

Boundary levels have both NN and NNN to the same side



Dyson's circular ensembles



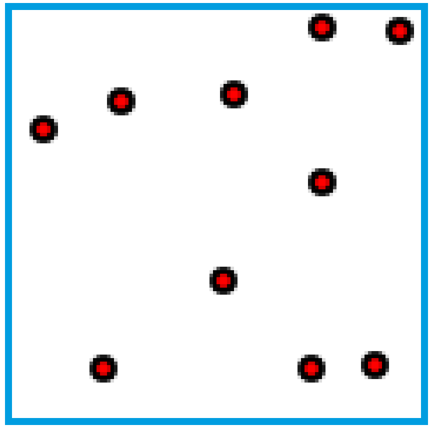
Our toric ensembles

Toric Unitary Ensemble

- Clifford torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \in \mathbb{R}^4$

$$P = \frac{1}{\sqrt{2}}(\cos\vartheta, \sin\vartheta, \cos\varphi, \sin\varphi) \in \mathbb{T}^2$$

$$\varphi \in (-\pi, \pi]$$
$$\vartheta \in (-\pi, \pi]$$



- Toric ensemble distribution $(H = U\Lambda U^{-1})$:

$$P(H)dH = e^{-\mathrm{Tr}(V(H))}|\Delta(\Lambda)|^\beta d\Lambda\, dU_{\mathrm{Haar}}$$

Can be set to 1 (compact support)

Vandermonde interaction $\Delta(\Lambda) = \prod_{j<k}(\lambda_j - \lambda_k)$ (Jacobian)

Haar measure on unitary group. Integrate over $\Longrightarrow$ normalization constant

$$P^{(N)}_{\mathrm{TUE}}(\vartheta_1, \ldots, \vartheta_N; \varphi_1, \ldots, \varphi_N) \propto \prod_{j<k}\left[2 - \cos(\vartheta_j - \vartheta_k) - \cos(\varphi_j - \varphi_k)\right]$$

Open question:

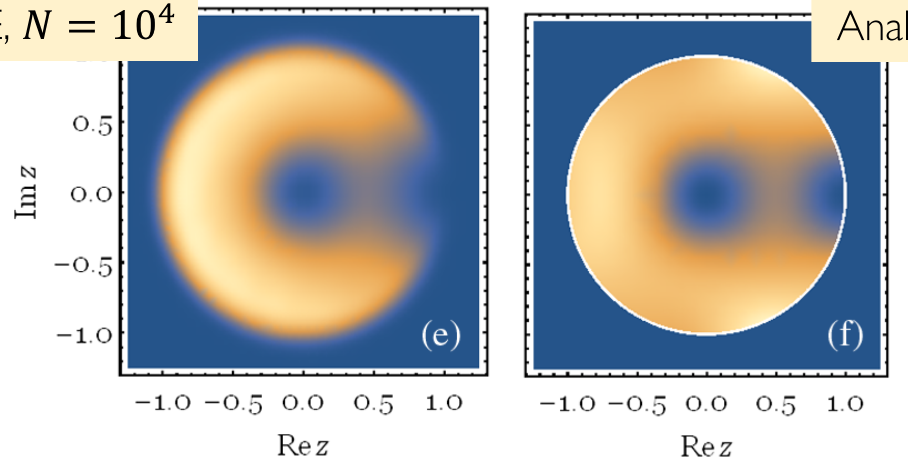
- Matrix model?

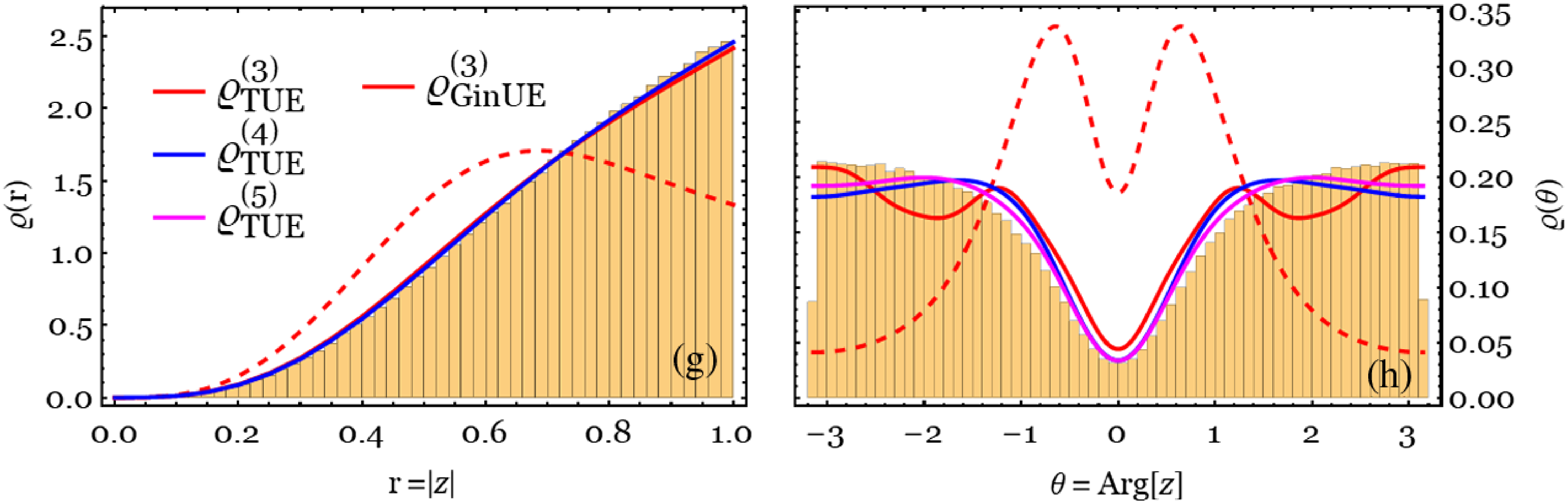
Toric Unitary Ensemble

Wigner-like surmise calculation now works

Numerics GinUE, $N = 10^4$

Analytics TUE, $N = 3$





Random matrix theory of dissipative quantum chaos

Two-fold program:

- Statistical mechanics of complex non-Hermitian systems
  - RMT model of non-Hermitian generators ($\mathcal{L}$, Φ, H_{eff}, M)
  - Spectral properties (refined models for 1-point function?)
  - Spectral gap (relaxation time)
  - Steady-state properties
  - Many-body systems?

How far along are we?

- Signatures of dissipative quantum chaos
  - Classification of different phases/symmetries
  - Spectral correlations in non-Hermitian RMT (angular correlations)
  - Toolbox of signatures of dissipative quantum chaos (new non-Hermitian RMT ensembles)

(i) How well do completely random models of open quantum systems describe real physical systems?
(ii) How universal are the results found for these particular models?

Conclusions & outlook

Main results: random models

- Random non-hermitian generators have (quite) universal spectra (lemon-like, annulus/disk)
- Gap of the Liouvillian exactly computable
- Spectral transition in Kraus maps
- Calculations need quaternionic free probability
- Universal steady-states, with analytic understanding
- Perturbative crossover in steady-state statistics

Main results: dissipative chaos

- Angular correlations give signature of dissipative quantum chaos
- Poisson: flat ratio distribution
- RMT: suppression of small angles + cubic repulsion
- Useful tool for classidying dissipative phases of matter / symmetry classes
- Wigner-like surmise possible with Toric Ensemble

Open questions & outlook

- Spectral density of the random Liouvillian
- Many-body systems
- Are all regimes trully Markovian?
- New universality classes
- RMT-Poisson crossover in complex spacing ratios
- Correlation functions in toric ensembles
- Physical interpretation of the toric ensembles
- Further signatures of dissipative quantum chaos

Thank you.