Representation theory of quivers and its combinatorics

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Setup

- $\mathbf{k}$ is an algebraically closed field
- Q Dynkin quiver
- $\mathbf{k}Q$ path algebra
- $m \in \mathbb{Z} \setminus \{0, 1\}$

$\text{mod}(\mathbf{k}Q)$	fin. gen. $\mathbf{k}Q$-module category
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$D^b(\mathbf{k}Q)$	derived module category
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$B_m(\mathbf{k}Q)$	orbit category of the derived category

- $B_m(\mathbf{k}Q)$ is a $(1-m)$-Calabi-Yau triangulated category.

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Calabi-Yau property

Definition

Let T be a **k**-linear triangulated category with shift functor Σ.
T is *m-Calabi-Yau* (m-CY) if there is a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathsf{T}}(s,t) \simeq D\,\mathrm{Hom}_{\mathsf{T}}(t,\Sigma^m s),$$

for every $s, t \in \mathsf{T}$.

Notion important in:

- Theoretical physics,
- Algebraic and sympletic geometry,
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Classical tilting theory: Used to compare module categories of different algebras

Cluster-tilting theory:

- generalises tilting theory
- takes place in the m-cluster category $\mathrm{B}_m(\mathbf{k}Q)$, for $m \leqslant -1$
- theory extended to positive CY triangulated categories

Aim: To understand the structure of negative CY triangulated categories.

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Representation of quivers

- $Q: \ 1 \longleftarrow 2 \longleftarrow 3$

Definition

A *representation of Q* consists of:

- a **k**-vector space at each vertex of Q, and
- a linear map for each arrow of Q.

Example

$$M = \ \mathbf{k}^2 \xleftarrow{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}} \mathbf{k}^2 \xleftarrow{\begin{bmatrix}1\\ 0\end{bmatrix}} k$$

$\mathbf{\underline{dim}}\, M = (2, 2, 1)$

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A *morphism of representations* is a commutative diagram of linear maps at each vertex:

$$\begin{array}{ccc} V_i & \xrightarrow{\varphi_a} & V_j \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} \\ W_i & \xrightarrow[\gamma_a]{} & W_j \end{array}$$

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- rep(Q): category of finite-dimensional representations of quivers,
- mod$(\mathbf{k}Q)$: category of finitely generated modules over $\mathbf{k}Q$.

Theorem

Given a finite connected quiver, we have rep$(Q) \simeq$ mod$(\mathbf{k}Q)$.

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The *Auslander–Reiten quiver* of an algebra A has as vertices the indecomposable A-modules and as arrows the irreducible maps.

Gabriel's Theorem

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Knitting algorithm

1. Projective modules: $i \to j$ in $Q \implies P_j \to P_i$ in the AR-quiver.
2. Meshes: $\mathbf{\underline{dim}}\, M + \mathbf{\underline{dim}}\, N = \sum_{i=1}^{k} \mathbf{\underline{dim}}\, L_i$, where

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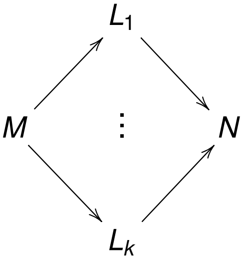

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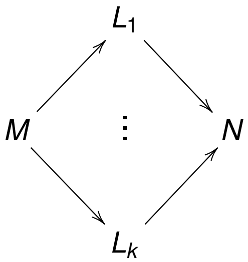

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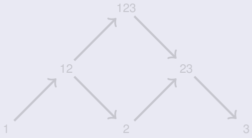

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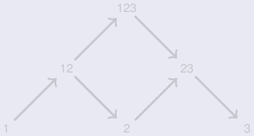

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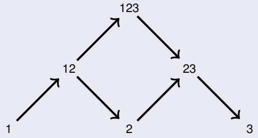

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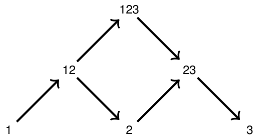

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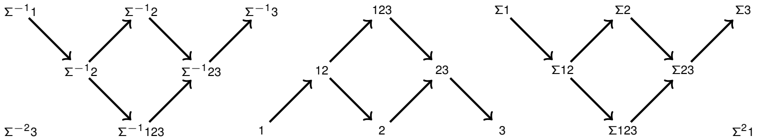

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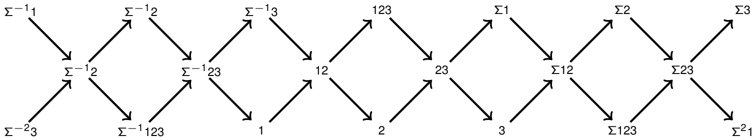

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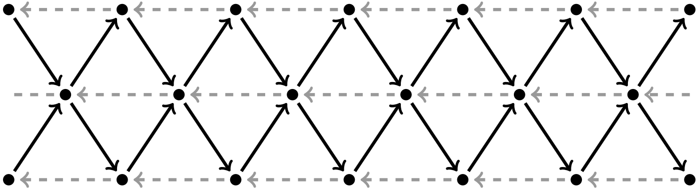

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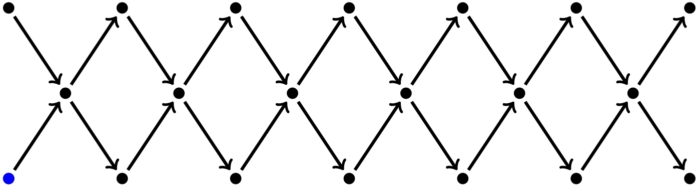

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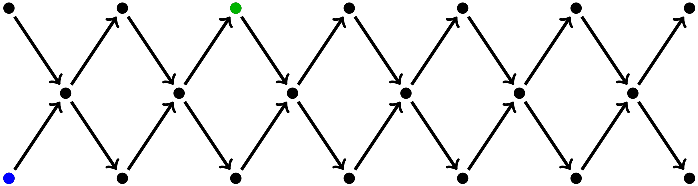

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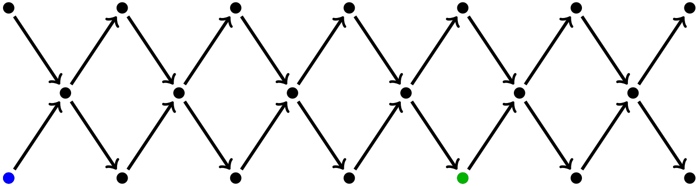

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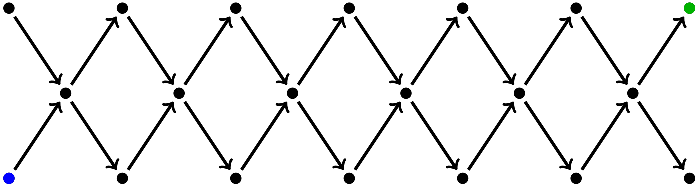

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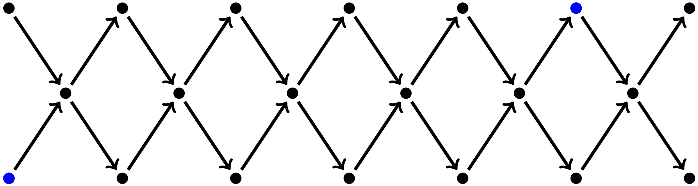

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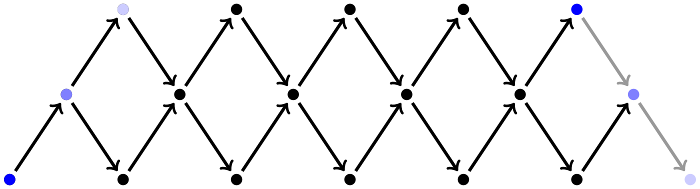

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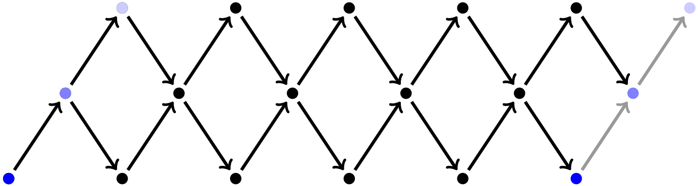

Tools to understand the structure of triangulated categories

One can consider:

- generators: tilting/silting objects, cluster-tilting objects, simple-minded collections, etc.
- particular subcategories: **torsion pairs**, t-structures, co-t-structures.

In tilting theory: torsion pairs give ways of comparing different module categories.
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Definition

A pair of subcategories (X, Y) is a *torsion pair* if:

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2. $\forall t \in \mathsf{T}, \exists\, x \to t \to y \to \Sigma x$ with $x \in \mathsf{X}, y \in \mathsf{Y}$.

Torsion pairs

Definition

Let X be a subcategory of T.

1. X is *extension closed* if given $x' \to x \to x'' \to \Sigma x'$ in T with $x', x'' \in$ X, then $x \in$ X.
2. X is *contravariantly finite* if $\forall t \in$ T, $\exists f : x \to t$, with $x \in$ X such that $\mathrm{Hom}_{\mathrm{T}}(x', f) : \mathrm{Hom}_{\mathrm{T}}(x', x) \to \mathrm{Hom}_{\mathrm{T}}(x', t)$ is surjective.
3. $\mathrm{X}^{\perp} := \{t \in \mathrm{T} \mid \mathrm{Hom}_{\mathrm{T}}(\mathrm{X}, t) = 0\}$.

Proposition (Iyama-Yoshino)

The following statements are equivalent.

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Combinatorial model of $B_m(\mathbf{k}A_n)$

P polygon with $N = \begin{cases} (n+1)|m| - 2 & \text{if } m \geqslant 2 \\ (n+1)|m| + 2 & \text{if } m \leqslant -1 \end{cases}$ vertices.

Definition

Let $i, j (i < j)$ be two vertices of P. The pair $\{i, j\}$ is a *m-diagonal* if $\exists k \in \{1, \ldots, n\}$ such that:

$$1 - km = \begin{cases} i - j & \text{if } m \geqslant 2 \\ j - i & \text{if } m \leqslant -1. \end{cases}$$

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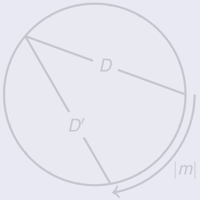
- $\Gamma_0 =$ the set of m-diagonals of P.
- $\Gamma_1 : D \to D'$ if:



- $\tau(\{i,j\}) = \{i - |m|, j - |m|\}$.

Proposition (Caldero-Chapoton-Schiffler, Baur-Marsh, CS)

The AR-quiver of $B_m(A_n)$ *and* $\Gamma(m,n)$ *are isomorphic as stable translation quivers.*

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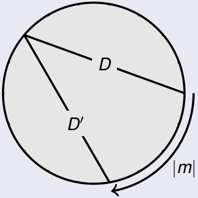

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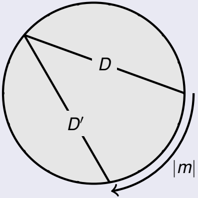

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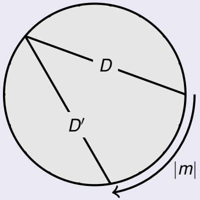

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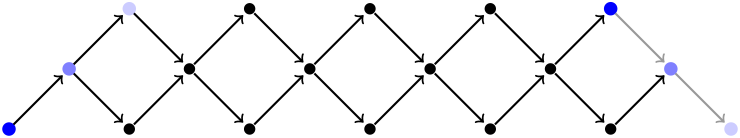
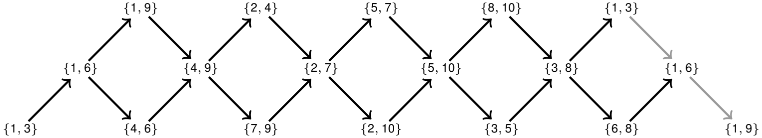
$B_3(\mathbf{k}A_3)$:



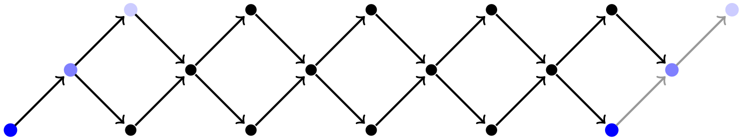
Example in negative CY case

$\Gamma(3,3)$:



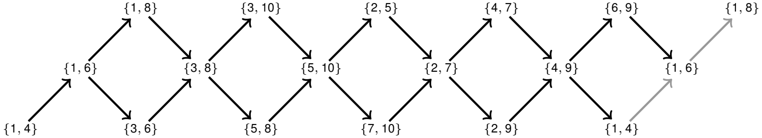
Example in positive CY case

$\mathrm{B}_{-2}(\mathbf{k}A_3)$:



Example in positive CY case

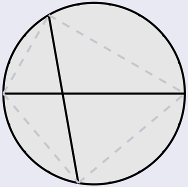
$\Gamma(-2,3)$:



Ptolemy arcs

Definition

- Ptolemy arcs of class I:



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Definition

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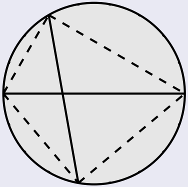

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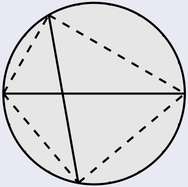

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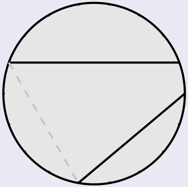

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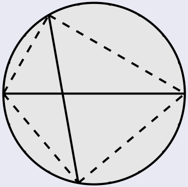

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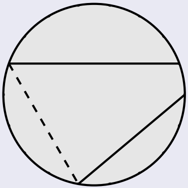

Classification of torsion pairs in $\mathsf{B}_m(A_n)$

Theorem (CS-Pauksztello, Holm-Jørgensen-Rubey for $m = -1$)

- X *subcategory of* $\mathsf{B}_m(A_n)$
- $\mathscr{X}$ *corresponding set of m-diagonals*

Then $(\mathsf{X}, \mathsf{X}^{\perp})$ *is a torsion pair if and only if:*

	$m \leqslant -1$	$m \geqslant 2$
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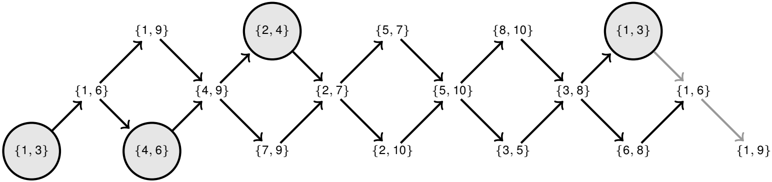

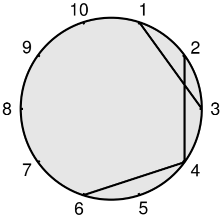

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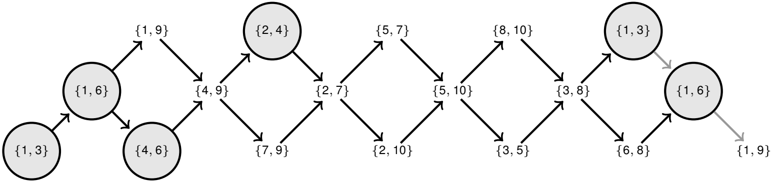

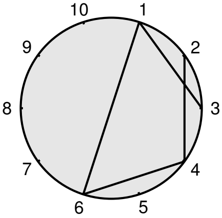

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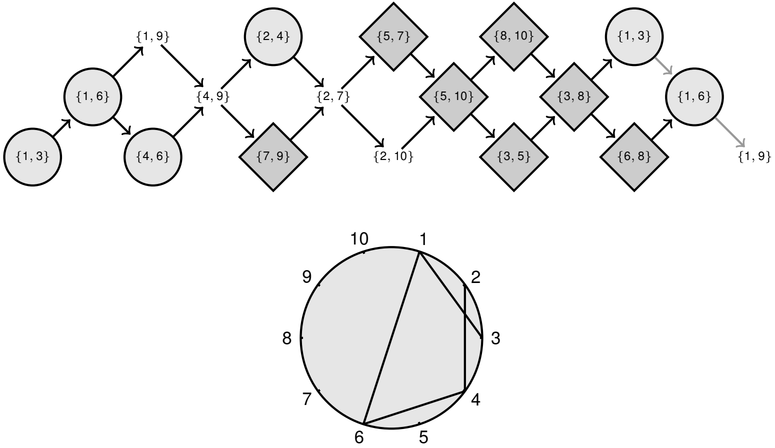

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