Nonabelian Cohomology

Arber Selimi

Institut Superior Tecnico

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Outline

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For a topological space X, and an abelian group G, there is a bijection $H^n(X,G) = [X, K(G,n)]$, using this result we will get the intuition to define the nonabelian cohomology.
We start the talk by introducing the notion of Crossed modules, and give the first examples. We continue by introducing Crossed complexes and giving the constructions that we need.
We finish the talk by defining the nonabelian cohomology using the intuition coming from the bijection given above, and we interpret cohomology classes via extensions.

1. A crossed module over a group $\mathcal{M} = (\mu : M \to P)$ is a morphism of groups $\mu : M \to P$, called the boundary of $\mathcal{M}$ together with an action $(m, p) \to m^p$ of P on m satisfying the following two axioms
   1. $\mu(m^p) = p^{-1}\mu(m)p$
   2. $n^{-1}mn = m^{\mu(n)}$

   for all $m, n \in M$ and $p \in P$. We call $\mathcal{M}$ a crossed P-module.
2. A morphism between two crossed modules is defined to be a pair of homomorphism (f, g), such that they commute with the action.
3. We denote by $XMod/Grp$ the category of crossed modules
4. $ker\mu$ is abelian, and $im\mu$ is normal subgroup of P.

1. If $M \trianglelefteq P$, the boundary map given by the inclusion $i : M \to P$, and the action via conjugation, then M is a P crossed module.
2. Central Extensions: If $\mu : M \to P$ is a surjective boundary, such that $\ker \mu$ is contained in $Z(G)$, and the action of P on M is given by conjugation, them M is a P crossed module.
3. Consider the n-th dihedral group $D_{2n} = \langle x, y | x^n, y^2, xyxy \rangle$ and another copy $\tilde{D}_{2n} = \langle u, v | v^n, v^2, uvuv \rangle$. We can consider $\tilde{D}_{2n}$ as a crossed module over D_{2n}, with boundary operator $\mu : \tilde{D}_{2n} \to D_{2n}$, given in generators by $u \mapsto x^2$ and $v \mapsto y$, and the action of D_{2n} on $\tilde{D}_{2n}$ given by $u^y = vuv^{-1}$, $v^y = v, u = u, v = vu$.
   Note that μ is isomorphism if n is odd, and $coker\mu \simeq ker\mu \simeq \mathbb{Z}/2\mathbb{Z}$

Let (X, A, x) be a based pair of spaces. The long exact sequence of homotopy groups gives $\partial : \pi_2(X, A, x) \to \pi_1(A, x)$, and the action given by composition

$$I^2 \cong I^2 \times \{1\} \to I^3 \to I^2 \times \{0\} \cup J^{n-1} \times I \to X.$$

It can be checked that the axioms of crossed module are satisfied.

(Classifying space) If $\mathcal{M}(\mu : M \to P)$ is a crossed module, we denote by $B\mathcal{M}$ a model which satisfies $\pi_1(B\mathcal{M}) = P/Im\mu$, and $\pi_2(B\mathcal{M}) = ker\mu$, and higher homotopy groups trivial.

A $cat^1 - group$ is a triple $\mathcal{G} = (G, s, t)$ such that G is a group, and s, t are homomorphism such that

1. $st = t$ and $ts = s$
2. $[kers, kert] = 1$

A morphism betweem (G, s, t) and (G', s', t') is a group homomorphism $f : G \to G'$ such that $s'f = fs$ and $t'f = ft$.

Theorem: There is an equivalence of categories $XMod/Grp$ and $Cat^1 - group$ given by

$$\lambda : \mathcal{M}(\mu : M \to P) \longmapsto (P \ltimes M, s, t)$$

where $s(g, m) = (g, 1)$ and $t(g, m) = (g\mu(m), 1)$, and

$$\gamma : (G, s, t) \longmapsto (t_| : Kers \to Ims)$$

where Ims acts on $Kers$ by conjugation.

A crossed module over the groupoid $P = (P_1, P_0)$ is given by thr groupoid $M = (M_2, P_0)$ and a morphism of groupoids $\mu : M \to P$ which is the identity on objects, and satisfy

1. M is a family of groups $\{M_2(p)\}_{p \in P_0}$.
2. P acts on the right, and satisfies $x^1 = x$, $x^{ab} = (x^a)^b$, and $(x + y)^a = x^a + y^a$.
3. μ preserves the action, and $x^{\mu c} = -c + x + c$, for all $x \in M_2(p)$ and any c.

The category of Crossed modules over Groupoids is denoted by $XMod$.

Example

For a pair (X, A) and $C \subset A$, we consider $P = \pi_1(A, C)$ the fundamental groupoid of (A, C).The fundamental crossed module $\Pi_2(X, A, C)$ includes a family of groups $\{\pi_2(X, A, x)\}_{x \in C}$ and action is similar with the one for homotopy groups.

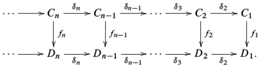
Definition: A crossed complex C over C_1 is a sequence

$$C : \cdots \to C_n \to C_{n-1} \to \cdots \to C_2 \to C_1$$

such that

1. For $n \geq 2$ C_n is a totally disconnected groupoid with $\{C_n(X)\}_{x \in C_0}$ are abelian groups for $n \geq 3$.
2. There is a right action such that, if $c \in C_n(x)$ and $c_1 \in C_1(x, y)$ then $c^{c_1} \in C_n(y)$.
3. For $n \geq 2$, the maps $\delta_n : C_n \to C_{n-1}$ preserves the action, and $\delta_{n-1} \circ \delta_n = 0$, and $Im\delta_2$ acts by conjugation on C_2 and trivially on C_n for $n \geq 3$.

A morphism of crossed comples is a family of morphisms of groupoids $f_n : C_n \to D_n$ such that



commutes and induces the same map $C_0 \to D_0$.

Definition For any crossed complex C and for $n \geq 2$ there are totally disconnected groupoid $H_n(C)$ given by a family of abelian groups

$$H_n(C, x) = ker\delta_n(x)/Im\delta_{n+1}(x)$$

for all $x \in C_0$. Called the family of n-homology groups of the crossed complex C.

If we denote by $\pi_1(C) = Coker\delta_2 = C_1/Im\delta_2$ then $\pi_1(C)$ acts on H_n, making $H_n(C)$ a $\pi_1(C)$-module.

Remarks: A morphism of crossed complexes induces a morphism on homology. There is a notion of weak equivalence etc...

Eilenberg-Mac Lane Crossed Complex: For $n = 1$ the crossed complex $\mathbb{K}(G, 1) := \cdots \to 0 \to \cdots \to 0 \to G$ is usually denoted only by G. For $n \geq 2$ the crossed complex $\mathbb{K}(M, G)_n := \cdots 0 \to M \to 0 \cdots \to 0 \to G$.

Augmentation module of G: Let G be a groupoid. Consider $\mathbb{Z}G$ as a right $G-$module, consists of $\mathbb{Z}(p) = \mathbb{Z}$, for $p \in G_0$ with trivial action of G. The augmenttion map

$$\epsilon : \mathbb{Z}G \to \mathbb{Z}$$

given by $\sum n_i g_i \to \sum n_i$ is a morphism of G modules ,and the kernel is called the augmentation module, and we denote it by $I(G)$.

Definition: Let k, l:$C \to D$ be morphisms between two crossed complexes. A homotopy from k to l is $h : k \simeq l$, given by maps $h_n : C_n \to D_{n+1}$, satisfying the following properties

1. For $n \geq 0$
$$h_n(c) \in \begin{cases} D_1(lc, kc) & \text{if } n = 0 \\ D_{n+1}(tlc) & \text{if } n > 0 \end{cases}$$
2. If $c + c'$ and $c \cdot c'$ are defined, then $h_1(cc') = h_1(c)^{lc'} h_1(c')$, and $h_n(c + c') = h_n(c) + h_n(c')$.
3. For $n \geq 2$, h_n preserves the action over l, i.e $h_n(c^{c_1}) = (h_n c)^{l_1 c_1}$.
4. k is completely determined from l and h.

1. Let $\mathcal{M}(\mu : M \to G$ and $\mathcal{A}(\alpha : A \to P)$ crossed modules, and $k = (k^2, k^1)$ and $l = (l^2, l^1)$ morphisms $\mathcal{M} \to \mathcal{A}$. A homotopy $h : k \simeq l$ is a function $h : P \to A$ such that
   1. $h(p_1 p) = h(p_1)^{l^1 p}(hp)$
   2. $k^1(p) = l^1(p)(\alpha(h(p)))$
   3. $k^2(m) = l^2(m)(h(\mu(m)))$

   for all $m \in M$ and $p \in P$.
2. Contraction: For crossed complex C, $h : id_c \simeq 0$ satisfies
   1. $h_0 c \in C_1(c, 0)$ if $c \in C_0$.
   2. $h_1(cc') = h(c)h(c')$ if $c, c' \in C_1$ and cc' is defined.
   3. $h_n(c_1^c) = h_n(c_1)$ if $c_1 \in C_n$ $c \in C_1$ and c_1^c is defined.
   4. $\delta_2 h_1 c = (h_0 \delta c)^{-1} c(h_0 tc)$ if $c \in C_1$.
   5. $\delta_{n+1} h_n c = -h_{n-1}\delta_n c + c^{h_0 tc}$ if $c \in C_n,\ n \geq 0$

A crossed complex C is called aspherical if $H_n(C,x)=0$ for all $n\geq 2$ and $x\in C_0$. If $\pi_1(C,x)=0$ for all $x\in C_0$ it is called acyclic.

A crossed complex C together with a morphism $\phi : C_1 \to G$, to a groupoid G, such that $G_0 = C_0$ is called augmented crossed complex, denoted by (C,ϕ).

A crossed resolution of a groupoid G is a aspherical crossed complex C augmented over G. Denote

$$C := \cdots \xrightarrow{\delta_{n+1}} C_n \to \cdots \to C_2 \xrightarrow{\delta_1} C_1 \overset{\phi}{\Rightarrow} G$$

Any groupoid admits a free crossed resolution.

Crossed resolution of $\mathbb{Z}/a\mathbb{Z}$

If $\mathbb{Z}/a\mathbb{Z}$ is the cyclic group of order a generated by c, then we have a free crossed resolution

$$F := \cdots \xrightarrow{\delta_{n+1}} \mathbb{Z}[\mathbb{Z}/a\mathbb{Z}] \xrightarrow{\delta_n} \cdots \to \mathbb{Z}[\mathbb{Z}/a\mathbb{Z}] \xrightarrow{\delta_1} \mathbb{Z} \overset{\phi}{\Rightarrow} \mathbb{Z}/a\mathbb{Z}$$

where $\phi(x_1) = c$, and for $n \geq 2$, $\mathbb{Z}[\mathbb{Z}/a\mathbb{Z}]$ is the $\mathbb{Z}/a\mathbb{Z}$-module on a generator x_n, and boundary maps are given by

$$\delta_n(x) = \begin{cases} x_{n-1}(1-c) & \text{if } n\text{-odd} \\ x_{n-1}(1+c+\cdots+c^{a-1}) & \text{if } n\text{-even} \end{cases}$$

Let G be a groupoid. We form $N^{\Delta}(C)_n = \{[a_1, ..., a_n] | a_i \in G\}$ of composable sequences of elements of G, where the basepoint of $t[a_1, ..., a_n]$ is the final point ta_n of a_n. And we denote by $F_n^{st}(G)$ th free groupoid on this set. For $n \geq 2$ we consider $\delta_n : F_n^{st}(G) \to F_{n-1}^{st}(G)$ given by

$$\delta_2[a, b] = [ab]^{-1}[a][b],$$

$$\delta_3[a, b, c] = [a, b]^c[b, c]^{-1}[a, b, c]^{-1}[ab, c],$$

and for $n \geq 4$

$$\delta_n[a_1, ..., a_n] = [a_1, ..., a_{n-1}]^{a_n} + (-1)^n[a_2, ..., a_n] +$$

$$\sum_{i=1}^{n-1}(-1)^{n-i}[a_1, ..., a_{i-1}, a_i a_{i+1}, ..., a_n]$$

This gives the resolution

$$F_*^{st} := \cdots \xrightarrow{\delta_{n+1}} F_*^{st}(G)_n \xrightarrow{\delta_n} F_*^{st}(G)_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} F_*^{st}(G)_1 \overset{\phi}{\Rightarrow} G$$

Definition of Nonabelian Cohomology

Let G be a groupoid, and M be a $G-$module. The n-th cohomology of G with coefficients in M is defined to be

$$H^n(G, M) = [F_*^{st}, \mathbb{K}_n(M, G); \phi]$$

where

1. F_*^{st} is the standard free crossed resolution of the groupoid G.
2. ϕ is the standard morphism $\phi : F_1^{st}(G) \to G$.
3. $\mathbb{K}_n(M, G) := \cdots \to 0 \to M \to 0 \to \cdots \to G \to 0$.

Dimension 2 cohomology of a group

Definition Let $\mathcal{M}(\mu : M \to P)$ be a crossed module. An extension (i, p, σ) of type $\mathcal{M}$ of the group M by the group G is

1. An exact sequence $1 \to M \xrightarrow{i} E \xrightarrow{p} G \to 1$ such that E acts on M by conjugation. ($(i : M \to E)$ is a crossed module)
2. There is σ such that $\mu = \sigma \circ i$

Equivalent extensions: Two extensions (i, p, σ) and (i', p', σ') are called equivalent if there is $\phi : E \to E'$, such that

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \phi} & & \| & & \\ 1 & \longrightarrow & M & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1, \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{\sigma} & P \\ {\scriptstyle \phi}\downarrow & & \| \\ E' & \xrightarrow{\sigma'} & P. \end{array}$$

Five lemma implies that ϕ is an isomorphism.

We denote by $OpExt_{\mathcal{M}}$ the set of equivalence classes of all extensions of type $\mathcal{M}$ of M by G.

Theorem: Suppose

$$0 \to \pi \xrightarrow{i} F_2 \xrightarrow{\delta} F_1 \xrightarrow{\phi} G \to 1$$

is a crossed exact sequence, and $\mathcal{M}(\mu : M \to P)$ a crossed module. If $\mathcal{F}(\delta : F_2 \to F_1$ is a crossed module, and $[\mathcal{F}, \mathcal{M}]^0$ the homotopy class of morphisms $k = (k^2, k^1) : \mathcal{F} \to \mathcal{M}$, such that $(k^2 \circ i)(\pi) = 1$. There is a natural injection

$$E : [\mathcal{F}, \mathcal{M}]^0 \to OpExt_{\mathcal{M}}(G, M)$$

such that the class of k is sent to the extension

$$1 \to M \to E(k) \to G \to 1$$

where $E(k)$ is the quotient of $F_1 \ltimes M$ by the elements $(\mu m, (k^2(m))^{-1})$, and F_1 acts on M via P. If F_1 is free then E is bijection.

Definition: Let G be a group, and M crossed module over G. A crossed n-fold extension of M by G is a crossed resolution E of G, such that $E_{n+1} = M$ as a G module, and $E_i = 0$ for $i > n + 1$.
A morphism $f : E \to E'$ of n-fold extensions (for the same crossed module $\mathcal{M} = (\mu : M \to G)$) is a morphism of crossed resolutions which induce identity on M and G.

Remark: If n=1 we have an abelian extension, because of axiom 2 of crossed module.

Two crossed n-fold extensions E and E' are called similar if we can find crossed n-fold extensions $E = E^1, ..., E^k = E'$, and morphisms $f^i : E^i \to E^{i+1}$ or $f^i : E^{i+1} \to E^i$, such that E and E' are connected.

The above relation is an equivalence relation, and we denote the set of equivalence classes of crossed $n-$-fold by $OpExt^n(G, M)$.

Theorem: For a group G and a G-module M, a crossed n-fold extension E of M determines a cohomology class $k_E \in H^{n+1}(G, M)$. Conversly any such class determines a crossed n-fold extension of M by G.

In other words there is a bijection

$$OpExt^n(G, M) \cong H^{n+1}(G, M).$$

The abelian group structure on the right hand side is called Baer Sum.

Sketch of the proof

Let E be crossed n fold extension of M by G, and F a free resolution of G. Since E is aspherical, there is a morphism $f : E \to F$, and the morphism $f_{n+1} : M \to F_{n+1}$ induces the class $k_E \in H^{n+1}(G, M)$.

Conversly, suppose that we have a morphism of $G-$modules $f_{n+1} : M \to F_{n+1}$, such that $f_{n+1} \circ \delta_{n+2} = 0$.

For $n \geq 2$, we form the n-fold extension by

$$\delta_n(x) = \begin{cases} F_i & \text{if } i < 1 \\ (F_n \times M)/D & \text{if } i = n \end{cases}$$

where D is the submodule of the product generated by the elements $(\delta_{n+1}, c, f_{n+1}c)$ for all $c \in F_{n+1}$.

For $n = 1$, we take the semidirect product $F_1 \ltimes M$.

Definition Let $M \subset P$ be normal subgroups of Q, so that Q acts on M and P by conjugation, and $\mu : M \to P$ and $i : P \to Q$ inclusions. Then the crossed module $\zeta : M \times (M^{ab} \otimes I(Q/P) \to Q$ where for $m, n \in M$ and $x \in I(Q/P)$ we have $\zeta(m, [n] \otimes x) = m$, and the action given by $(m, [n] \otimes x)^q = (m^q, [m^q] \otimes ([q] - 1) + [n^q] \otimes x[q])$. This is called the induced crossed module.

Example: If P is normal subgroup of Q, such that Q/P is isomorphic to $\mathbb{Z}/a\mathbb{Z}$ Let u be a generator of Q/P,then the first postnikov invariant k^3 of the crossed module induced by the inclusion $P \to Q$ lies in the third Cohomology group

$$H^3(\mathbb{Z}/a\mathbb{Z}, P^{ab} \otimes I(\mathbb{Z}/a\mathbb{Z})) \cong P^{ab} \otimes \frac{I(\mathbb{Z}/a\mathbb{Z})}{I(\mathbb{Z}/a\mathbb{Z})(1-t)}$$

where

$$k^3 \longmapsto [u^a] \otimes (1-t).$$

Note that k^3 generally is not trivial.

If A is $\mathbb{Z}/a\mathbb{Z}$ we form $\mathrm{Hom}_{\mathbb{Z}/a\mathbb{Z}}(-, A)$ for the free crossed resolution

$$F := \cdots \xrightarrow{\delta_{n+1}} \mathbb{Z}[\mathbb{Z}/a\mathbb{Z}] \xrightarrow{\delta_n} \cdots \to \mathbb{Z}[\mathbb{Z}/a\mathbb{Z}] \xrightarrow{\delta_1} \mathbb{Z} \overset{\phi}{\Rightarrow} \mathbb{Z}/a\mathbb{Z}$$

where each term has a free single generator, so $H^3(\mathbb{Z}/a\mathbb{Z}, A)$ can be computet from the sequence

$$A \xrightarrow{M} A \xrightarrow{N} A$$

where M and N are given by $(1-t)$ and $1 + t + ... + t^{a-1}$ respectively. In our case, for $A := P^{ab} \otimes I(\mathbb{Z}/a\mathbb{Z})$, since $I(\mathbb{Z}/a\mathbb{Z})$ is generated by elements $1 - t^r$, for $1 \leq r < a$, we have that $N(I(\mathbb{Z}/a\mathbb{Z})) = 0$. So

$$H^3(\mathbb{Z}/a\mathbb{Z}, P^{ab} \otimes I(\mathbb{Z}/a\mathbb{Z})) \cong P^{ab} \otimes \frac{I(\mathbb{Z}/a\mathbb{Z})}{I(\mathbb{Z}/a\mathbb{Z})(1-t)}$$

Sketch of the solution

We consider

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & \mathbb{Z}[\mathsf{C}_a] & \xrightarrow{\delta_4} & \mathbb{Z}[\mathsf{C}_a] & \xrightarrow{\delta_3} & \mathbb{Z}[\mathsf{C}_a] & \xrightarrow{\delta_2} & \mathsf{C}_\infty & \overset{\phi}{\Longrightarrow} & \mathsf{C}_a & \longrightarrow & 0 \\
 & & & & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow 1 & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow[i]{} & P \times A & \xrightarrow[\xi]{} & Q & \underset{\psi}{\Longrightarrow} & \mathsf{C}_a & \longrightarrow & 0.
\end{array}$$

Let $u \in Q$, such that $\phi(u)$ is the generator of $\mathbb{Z}/a\mathbb{Z}$. Define $f_1(x_1) = u$, and $f_2(x_2) = (u^a, 0)$, so we get the commutativity of the first two sqares. Finally for $A = P^{ab} \otimes I(\mathbb{Z}/a\mathbb{Z}$ from

$$f_2\delta_3(x_3) = (0, [u^a] \otimes (1 - t))$$

so we get the statement.

Special case: If $Q = \mathbb{Z}/4\mathbb{Z}$ generated by u, and P the subgroup generated by u^2, then the exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

from the solution of the main example represents the nontrivial class of $H^3(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

Let $\mathcal{U} = \{U_\lambda\}_{\lambda \in A}$ be an open cover of X. Define

$$p : EU = \bigsqcup U_\lambda \to X$$

sending $(x, \lambda) \to x$, $x \in U_\lambda$. We say $(x, \lambda) \sim (x, \mu)$ if $x \in U_\lambda \cap U_\mu$. And we form the groupoid of equivalence classes $Equ(\mathcal{U})$. And using the constructions before we have $F_\star(\mathcal{U})$, so we can define $H^0(\mathcal{U}, C) := [F_\star(\mathcal{U}), C]$.

If $\mathcal{V}$ is a refinement of $\mathcal{U}$, and ϕ, π two refinements. Ten they induce the same map $[F_\star(\mathcal{U}), C] \to [F_\star(\mathcal{V}), C]$

This property gives the starting point of Cech Cohomology.

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