Indefiniteness in semi-intuitionistic set theories: On a conjecture of Feferman

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The Continuum Hypothesis

The Continuum Hypothesis, CH, is the statement that *every infinite set of reals is either in one-one correspondence with* $\mathbb{N}$ *or with* $\mathbb{R}$.

The Continuum Hypothesis problem

Is the Continuum Hypothesis true or false?

The main results about this problem are:

- The consistency of CH with $\mathbf{ZFC}$ (Gödel, 1940),
- The independence of CH from $\mathbf{ZFC}$ (Cohen, 1964).

For some this settles the question.

For others it means that we should add axioms to $\mathbf{ZFC}$ that decide the CH problem.

Feferman's perspective on CH

"[...] I believe that the Continuum Hypothesis (CH) is not a definite mathematical problem. My reason for that is that the concept of arbitrary set essential to its formulation is vague or underdetermined and there is no way to sharpen it without violating what it is supposed to be about. In addition, there is considerable circumstantial evidence to support the view that CH is not definite" (Feferman, 2011)

Feferman's framework for definiteness

"One way of saying of a statement φ that it is definite is that it is true or false; on a deflationary account of truth that's the same as saying that the Law of Excluded Middle (LEM) holds of φ, i.e. one has $\varphi \vee \neg\varphi$. Since LEM is rejected in intuitionistic logic as a basic principle, that suggests the slogan, *'What's definite is the domain of classical logic, what's not is that of intuitionistic logic.'*
[...] And in the case of set theory, where every set is conceived to be a definite totality, we would have classical logic for bounded quantification while intuitionistic logic is to be used for unbounded quantification" (Feferman, 2011)

Semi-Constructive Set theory

Language: $\{\in\}$;

Formulas:

Atomic formulas: $x \in y$, $x = y$;

Logical connectives: $\wedge, \vee, \rightarrow, \neg$;

Quantifiers: $\forall, \exists$;

Quantifiers of the forms $\forall x \in u$, $\exists x \in u$ are called bounded.

Bounded or Δ_0-formulas are formulas where all quantifiers are bounded.

SCS is based on intuitionistic logic.

Axioms of SCS

Extensionality: $\forall u \forall v \, [\forall x \in u \, (x \in v) \wedge \forall y \in v \, (y \in u) \rightarrow u = v]$;

Pair: $\forall u \forall v \exists w \, (u \in w \wedge v \in w)$;

Union: $\forall u \exists w \forall x \in u \, (x \subseteq w)$;

Infinity: There is a smallest inductive set;

Δ_0-Separation: $\forall u \exists w \forall x \, [x \in w \leftrightarrow x \in u \wedge \varphi(x)]$
for all bounded formulas $\varphi(x)$;

ϵ-Induction: $\forall x \, [(\forall y \in x \, \theta(y)) \rightarrow \theta(x)] \rightarrow \forall x \, \theta(x)$
for all formulas $\theta(x)$;

Δ_0-LEM: $\varphi \vee \neg\varphi$ for all bounded formulas φ;

$\mathrm{AC}_{\mathrm{Set}}$: $\forall x \in u \, \exists y \, \psi(x, y) \rightarrow$
$\exists f \, [Fun(f) \wedge dom(f) = u \wedge \forall x \in u \, \psi(x, f(x))]$
for all formulas $\psi(x, y)$;

MP: $\neg\neg\exists x \varphi(x) \rightarrow \exists x \varphi(x)$ for all bounded formulas φ.

Semi-Constructive Set theory

Remarks

- $\mathbf{SCS}$ proves Strong Collection i.e.

$$\forall x \in u \, \exists y \, \psi(x,y) \rightarrow$$
$$\exists z \, [\forall x \in u \, \exists y \in z \, \psi(x,y) \wedge \forall y \in z \exists x \in u \, \psi(x,y)]$$

  for all formulas $\psi(x,y)$;

- $\mathbf{SCS}$ proves the Bounded Omniscience Scheme, i.e.

$$\forall x \in u \, [\varphi(x) \vee \neg\varphi(x)] \rightarrow [\forall x \in u \, \varphi(x) \vee \exists x \in u \, \neg\varphi(x)]$$

  for all formulas $\varphi(x)$.

- $\mathbf{SCS}$ does not have the Power Set axiom.

Semi-intuitionistic set theory $\mathbf{T}$

$\mathbf{T}$ is the theory $\mathbf{SCS}$ + '$\mathbb{R}$ is a set'

Any of the following equivalent statements can be used to formalize the existence of $\mathbb{R}$ as a set:

- The collection of all functions form $\mathbb{N}$ to $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, is a set;
- The collection of all subsets of $\mathbb{N}$ is a set.

Main Theorem

In $\mathbf{T}$ we can formulate the Continuum Hypothesis in this manner:

$$\forall x \subseteq \mathbb{R}\,[x \neq \emptyset \to (\exists f\, f : \mathbb{N} \twoheadrightarrow x \vee \exists f\, f : x \twoheadrightarrow \mathbb{R})]$$

Theorem (Rathjen)

$\mathbf{T}$ *does not prove* $CH \vee \neg CH$.

The relativized constructible hierarchy $L[A]$

Definition (Language of $L[A]$)

The language of $L[A]$, $\mathscr{L}_\in(P)$, consists of the language of set theory augmented by a unary predicate symbol P.
Any two sets A, X give rise to a structure $\langle X, \in, A \cap X \rangle$ for $\mathscr{L}_\in(P)$.

Definition (Definable set)

A set $Y \subseteq X$ is said to be definable in $\langle X, \in, A \cap X \rangle$ if there is a formula $\varphi(x, y_1, \ldots, y_r)$ of $\mathscr{L}_\in(P)$ and $b_1, \ldots, b_r \in X$ such that

$$Y = \{a \in X \mid \langle X, \in, A \cap X \rangle \models \varphi(a, b_1, \ldots, b_r)\}$$

$Def^A(X)$ denotes the class of the sets definable in $\langle X, \in, A \cap X \rangle$.

The relativized constructible hierarchy $L[A]$

The sets $L_\alpha[A]$ are defined by recursion on α as follows:

$$
\begin{aligned}
L_0[A] &= \emptyset, \\
L_{\alpha+1}[A] &= Def^A(L_\alpha[A]), \\
L_\gamma[A] &= \bigcup_{\alpha<\gamma} L_\alpha[A] \quad \text{for limit } \gamma, \\
L[A] &= \bigcup_{\alpha\in Ord} L_\alpha[A].
\end{aligned}
$$

The relativized constructible hierarchy $L[A]$

Proposition

1. $\alpha \leqslant \beta \Rightarrow L_\alpha[A] \subseteq L_\beta[A]$.
2. $\alpha < \beta \Rightarrow L_\alpha[A] \in L_\beta[A]$.
3. $L_\alpha[A]$ *is transitive.*
4. $L[A] \cap \alpha = L_\alpha[A] \cap \alpha = \alpha$.
5. *For* $\alpha \geqslant \omega$, $|L_\alpha[A]| = |\alpha|$.
6. $L[A] \models \mathbf{ZF}$.

The relativized constructible hierarchy $L[A]$

Proposition (Cont.)

7. *There is a* Σ_1 *formula* $wo(x,y,z)$ *such that*

$$\mathbf{ZF} \vdash \text{``}\{\langle x,y\rangle \mid wo(x,y,A)\} \text{ is a wellordering of } L[A]\text{''}$$

   and if $<_{L[A]}$ *denotes the wellordering of* $L[A]$ *determined by* wo*, then for any limit* $\gamma > \omega$*,*

$$<_{L[A]} \cap L_\gamma[A] \times L_\gamma[A] \text{ is } \Sigma_1^{L_\gamma[A]}.$$

8. $L[A]$ *is a model of* AC*.*
9. $B = A \cap L[A] \Rightarrow L[A] = L[B] \wedge (V = L[B])^{L[A]}$.
10. $\gamma > \omega$ *limit* $\wedge B = A \cap L_\gamma[A] \Rightarrow L_\gamma[A] = L_\gamma[B]$.

Ordered pair, projection, n-tuples

- $\langle a, b\rangle$ denotes the ordered pair of two sets a and b.
- If $c = \langle a, b\rangle$ let $(c)_0 = a$ and $(c)_1 = b$.
- Ordered n-uples are defined via
  - $\langle a_1\rangle := a_1$,
  - $\langle a_1, \ldots, a_{n+1}\rangle := \langle\langle a_1, \ldots, a_n\rangle, a_{n+1}\rangle$.

Satisfiability

To each formula ψ of $\mathscr{L}_{\in}(P)$ we assign a Gödel number $\ulcorner\psi\urcorner$.

There is a formula $Sat(v,w)$ of $\mathscr{L}_{\in}(P)$ such that for all Δ_0 formulas $\varphi(x_1,\ldots,x_n)$ of $\mathscr{L}_{\in}(P)$ the following holds for any limit $\gamma > \omega$ and all $\vec{a} = a_1,\ldots,a_n \in L_\gamma[A]$:

$$L_\gamma[A] \models \varphi(\vec{a}) \quad \text{iff} \quad L_\gamma[A] \models Sat(\ulcorner\varphi\urcorner, \langle\vec{a}\rangle).$$

Sat is uniformly $\Delta_1^{L_\gamma[A]}$ for limits $\gamma > \omega$.

Indexes for partial $\Sigma_1^{L[A]}$ functions

Let $\gamma > \omega$ be a limit.
For $e, a_1, \ldots, a_n \in L_\gamma[A]$ define

$$[e]_n^{L_\gamma[A]}(a_1, \ldots, a_n) \simeq b$$

if e is an ordered pair $\langle \ulcorner \psi \urcorner, c \rangle$ with ψ a Δ_0 formula of $\mathscr{L}_\in(P)$, not involving free variables other than $x_1, \ldots, x_{n+2}$, such that

$$L_\gamma[A] \models Sat(\ulcorner \psi \urcorner, \langle a_1, \ldots, a_n, c, d \rangle) \tag{1}$$

and $(d)_0 = b$ where d is the $<_{L[A]}$-least pair satisfying (1).

Likewise, $[e]_n^{L[A]}(a_1, \ldots, a_n) \simeq b$ is defined replacing $L_\gamma[A]$ by $L[A]$ in the definition above.

Indexes for partial $\Sigma_1^{L[A]}$ functions

Lemma

Let $\tau > \omega$ be a limit of limits.

1. *For $e \in L_\tau[A]$, the partial function f on $L_\tau[A]$ given by*

$$f(a_1,\ldots,a_n) = b \quad \textit{iff} \quad [e]_n^{L_\tau[A]}(a_1,\ldots,a_n) \simeq b$$

   is $\Sigma_1^{L_\tau[A]}$.

2. *For every n-ary partial $\Sigma_1^{L_\tau[A]}$ function f there exists an index $e \in L_\tau[A]$ such that, for all $a_1,\ldots,a_n \in L_\tau[A]$,*

$$f(a_1,\ldots,a_n) = b \quad \textit{iff} \quad [e]_n^{L_\tau[A]}(a_1,\ldots,a_n) \simeq b$$

3. *1 and 2 hold with $L[A]$ in place of $L_\tau[A]$.*
4. *$[e]_n^{L_\tau[A]}(a_1,\ldots,a_n) \simeq b$ implies $[e]_n^{L[A]}(a_1,\ldots,a_n) \simeq b$ and $[e]_n^{L_\lambda[A]}(a_1,\ldots,a_n) \simeq b$ for all limits $\lambda > \tau$.*
5. *If $[e]_n^{L[A]}(a_1,\ldots,a_n) \simeq b$ then $[e]_n^{L_\lambda[A]}(a_1,\ldots,a_n) \simeq b$ for some limit λ.*

Realizability

For $d \in L[A]$ and set-theoretic sentences ψ with parameters in $L[A]$ we define the realizability relation $d \Vdash_A \psi$ as:

$$
\begin{array}{rll}
e \Vdash_A c \in d & \text{iff} & c \in d \\
e \Vdash_A c = d & \text{iff} & c = d \\
e \Vdash_A \varphi \wedge \psi & \text{iff} & (e)_0 \Vdash_A \varphi \text{ and } (e)_1 \Vdash_A \psi \\
e \Vdash_A \varphi \vee \psi & \text{iff} & [(e)_0 = 0 \wedge (e)_1 \Vdash_A \varphi] \text{ or } [(e)_0 = 1 \wedge (e)_1 \Vdash_A \psi] \\
e \Vdash_A \varphi \rightarrow \psi & \text{iff} & \forall a \, [a \Vdash_A \varphi \Rightarrow [e]^{L[A]}(a) \Vdash_A \psi] \\
e \Vdash_A \exists x \theta(x) & \text{iff} & (e)_1 \Vdash_A \theta((e)_0) \\
e \Vdash_A \forall x \theta(x) & \text{iff} & \forall a \, [e]^{L[A]}(a) \Vdash_A \theta(a).
\end{array}
$$

Realizability Theorem

Let $\mathbb{R}^{L[A]}$ be the set of real numbers in the sense of $L[A]$.
If $\psi(x_1,\ldots,x_n)$ is a formula of set theory and $\mathscr{D}$ is a proof of $\psi(x_1,\ldots,x_n)$ in $\mathbf{T}$, then we can effectively construct a hereditarily finite set $e_{\mathscr{D}}$, which does not depend on A, such that for all $a_1,\ldots,a_n \in L[A]$

$$[e_{\mathscr{D}}]^{L[A]}(a_1,\ldots,a_n,\mathbb{R}^{L[A]}) \Vdash_A \psi(a_1,\ldots,a_n).$$

Sketch of the Main Theorem's proof

1. Towards a contradiction, assume $\mathbf{T} \vdash CH \vee \neg CH$.
2. Then exists $e \in L[A]$ such that, for all A,

$$[e]^{L[A]}(\mathbb{R}^{L[A]}) \Vdash_A CH \vee \neg CH.$$

3. $[e]^{L[A]}(\mathbb{R}^{L[A]})$ is either $\langle 0, e_{CH} \rangle$, where e_{CH} realizes CH, or $\langle 1, e_{\neg CH} \rangle$, where $e_{\neg CH}$ realizes $\neg CH$.

Sketch of the Main Theorem's proof

4. Using forcing we construct C such that, in $L[C]$, $\mathbb{R}^{L[C]}$ has cardinality $\aleph_2$.
5. Then $L[C] \nvDash CH$ and there is no $d \in L[C]$ such that $d \Vdash_C CH$.
6. Hence, $[e]^{L[C]}(\mathbb{R}^{L[C]}) \simeq b$ with $(b)_0 = 1$.

Sketch of the Main Theorem's proof

7. By forcing again, we construct E such that
   - $[e]^{L[C\cup E]}(\mathbb{R}^{L[C]}) \simeq b$,
   - $\mathbb{R}^{L[C]} = \mathbb{R}^{L[C\cup E]}$,
   - $L[C\cup E] \models CH$.
8. Using the order $<_{L[C\cup E]}$ we can construct a $\Sigma_1^{L[C\cup E]}$ function that realizes CH in $L[C\cup E]$.
9. Then $(b)_0 = 0$. Contradiction!

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