Integrability of Liouville Conformal Field Theory

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Two faces of Quantum Field Theory

(1) Axiomatic

- Wightman, Haag-Kastler, Osterwalder-Schrader, Belavin-Polyakov-Zamolodchicov, Segal,...
- Bootstrap and OPE for Conformal Field Theory
- Algebraic, explicit formuli

(2) Constructive

- Find examples satisfying axioms (QED, ϕ^4_4, QCD...)
- Action functionals, path integrals, renormalization group
- Analytic, approximative, often perturbative

This talk: a path from (2) to (1) in **Liouville CFT**

Quantum fields$\to$ Random fields

- Hilbert space of physical states $\mathcal{H}$, "vacuum" state $\psi_0 \in \mathcal{H}$
- Space-time $(\mathbf{x}, t) \in \mathbb{R}^{d+1}$
- Fields $\hat{V}_\alpha(\mathbf{x}, t)$ linear operators on $\mathcal{H}$
- Physical content encoded in Wightman functions

$$(\psi_0, \prod_{k=1}^{N} \hat{V}_{\alpha_k}(\mathbf{x_k}, t_k)\psi_0)$$

and axioms on their symmetries and regularity

- **Positivity of energy** $\implies$ analytic continuation $t \to i\tau$

$$(\psi_0, \prod_{k=1}^{N} \hat{V}_{\alpha_k}(\mathbf{x_k}, i\tau_k)\psi_0) = \langle \prod_{k=1}^{N} V_{\alpha_k}(x_k) \rangle$$

$V_\alpha(x)$ **random** functions on $x = (\mathbf{x}, \tau) \in \mathbb{R}^{d+1}$.

Random fields $\rightarrow$ Quantum fields

- Probability space Ω, expectation $\langle \cdot \rangle$
- Random (generalized) functions $V_\alpha(x, \omega)$, $x \in \mathbb{R}^n$, $\omega \in \Omega$
- **Correlation functions**

$$\langle \prod_{k=1}^{N} V_{\alpha_k}(x_k) \rangle$$

  and axioms on their symmetries and regularity.
- **Reflection Positivity** $\implies$ analytic continuation $x \rightarrow (\mathbf{x}, -it)$,
  $\implies$ reconstruction of $\mathcal{H}$, $\hat{V}_\alpha(\mathbf{x}, t)$. (Osterwalder, Schrader 1972)

Conformal Field Theory

Random fields model **statistical physics**

At **critical temperature** such systems have **conformal symmetry** and the QFT is **conformal field theory**

This extra symmetry gives rise to strong constraints on correlation functions via **conformal bootstrap**

In 2 dimensions bootstrap was used by Belavin, Polyakov and Zamoldchicov (1984) to classify CFT's and find explicit expressions for the correlation functions in several cases

In more than 2 dimensions bootstrap has led to spectacular numerical predictions (e.g. 3d Ising model)

2d Conformal Field Theory (CFT)

Data

- 2d surface Σ , Riemannian metric g
- Expectation $\langle \cdot \rangle_{\Sigma,g}$
- **Primary fields** $V_\alpha(z)$, $z \in \Sigma$, conformal weights Δ_α

Axioms (1): Diffeomorphism and Weyl covariance

$$\langle \prod_i V_{\alpha_i}(\psi(x_i)) \rangle_{\Sigma,g} = \langle \prod_i V_{\alpha_i}(x_i) \rangle_{\Sigma,\psi^* g}$$

$$\langle \prod_i V_{\alpha_i}(x_i) \rangle_{\Sigma,e^\varphi g} = e^{cA(\varphi,g)} \prod_i e^{-\Delta_{\alpha_i}\varphi(x_i)} \langle \prod_i V_{\alpha_i}(x_i) \rangle_{\Sigma,g}$$

$$A(\varphi,g) = \frac{1}{96\pi} \int_\Sigma (|\nabla_g \varphi|^2 + 2R_g\varphi) dv_g$$

c is the **central charge** that classifies the CFT's.

Structure Constants

For $\Sigma = S^2$ moduli space is one point:

- Every smooth metric can be written:

$$g = \psi^*(e^\varphi \hat{g})$$

- Conformal automorphisms of $\hat{g} \cong PSL_2(\mathbb{C})$

Hence 3-point functions

$$\langle V_{\alpha_1}(x_1) V_{\alpha_2}(x_2) V_{\alpha_3}(x_3) \rangle_{S^2, \hat{g}}$$

are determined up to constants $C(\alpha_1, \alpha_2, \alpha_3)$, the **structure constants** of the CFT.

Bootstrap

Axioms (2) **Operator Product Expansion**:

$$V_{\alpha_1}(z_1)V_{\alpha_2}(z_2) = \sum_{\alpha\in\mathcal{S}} C^{\alpha}_{\alpha_1\alpha_2}(z_1, z_2, \partial_{z_2})V_{\alpha}(z_2)$$

Holds when inserted to expectation:

$$\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)\ldots\rangle_{\Sigma} = \sum_{\alpha\in\mathcal{S}} C^{\alpha}_{\alpha_1\alpha_2}(z_1, z_2, \partial_{z_2})\langle V_{\alpha}(z_2)V_{\alpha_3}(z_3)\ldots\rangle_{\Sigma}$$

- $C^{\alpha}_{\alpha_1\alpha_2}$ are **determined** by the structure constants
- $\mathcal{S}$ is called the **spectrum** of the CFT

Iterating OPE:

- Correlations are determined by $C(\alpha_1, \alpha_2, \alpha_3)$ and $\langle V_{\alpha}(z)\rangle_{\Sigma}$
- $\Sigma = S^2 \implies$ only $C(\alpha_1, \alpha_2, \alpha_3)$ enter

Upshot: to “solve a CFT“ need to find its spectrum and structure constants.

Bootstrap for structure constants

Compute 4-point function on the sphere S^2 in two ways:

$$\langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle_{S^2} = \sum_{\alpha \in \mathcal{S}} C^{\alpha}_{\alpha_1 \alpha_2} \langle V_{\alpha} V_{\alpha_3} V_{\alpha_4} \rangle_{S^2} = \sum_{\alpha \in \mathcal{S}} C^{\alpha}_{\alpha_1 \alpha_3} \langle V_{\alpha} V_{\alpha_2} V_{\alpha_4} \rangle_{S^2}$$

This becomes a **quadratic equation** for structure constants.

It has proven to be a very constraining condition c.f. 3d Ising model.

Solutions

Compare w. harmonic analysis on compact/noncompact groups:

1. **Compact CFT's**

(a) $\mathcal{S}$ is **finite**: minimal models (e.g. Ising model)
Belavin, Polyakov, Zamolodchicov (1983)

(b) $\mathcal{S}$ is **countable**: compact G WZW models, G/H coset theories

Explicit formuli for $C(\alpha_1, \alpha_2, \alpha_3)$ in terms of Coulomb gas integrals (Dotsenko,Fateev, ...)

2. **Non-compact CFT's**

$\mathcal{S}$ is **continuous**: WZW with noncompact group, **Liouville model**, Toda CFT's

Explicit formula for $C(\alpha_1, \alpha_2, \alpha_3)$ conjectured by Dorn, Otto, Zamolodchicov, Zamolodchicov (1995) (the **DOZZ formula**).

Constructive CFT

Try to find examples satisfying the Axioms from **functional integrals** over fields $X:\Sigma\to M$

$$\langle\prod_\alpha V_\alpha\rangle_\Sigma=\int\prod_\alpha V_\alpha(X)e^{-S(X)}DX$$

Minimal models $M=\mathbb{R}$ and S is (scaling limit of)

$$S(X)=\int_\Sigma((\nabla_g X)^2+P(X))dv_g$$

with P, V_α polynomials in X with unknown coefficients.

WZW models $M=G$ Lie Group, S explicit

Direct analysis from functional integral hard.

Liouville model

Classical Liouville action functional for $X : \Sigma \to \mathbb{R}$

$$S_L(X) = \int_\Sigma ((\nabla_g X)^2 + QR_g X + \mu e^{\gamma X}) dv_g$$

If $Q = \frac{2}{\gamma}$ the minimiser of S_L solves the **Liouville equation**

$$\Delta_g X = QR_g + \mu\gamma e^{\gamma X} \Leftrightarrow R_{e^{\gamma X} g} = -\frac{1}{2}\mu\gamma^2.$$

Solution defines a metric $e^{\gamma X} g$ with constant negative curvature $\implies$ uniformising map $f : \mathbb{D} \to \Sigma$ (Picard, Poincare).

Polyakov (81): natural probability law for Riemannian metrics:

$$\mathbb{P}(e^{\gamma X} g) \propto e^{-S_L(X)}$$

"Quantum uniformisation"

Quantum Liouville model

$$\langle F \rangle_\Sigma = \int F(X) e^{-\int_\Sigma ((\nabla_g X)^2 + QR_g X + \mu e^{\gamma X}) dv_g} DX$$

- $Q = Q_{quantum} = \frac{2}{\gamma} + \frac{\gamma}{2}$
- $\mu > 0$ dependence explicit (KPZ scaling).
- γ only parameter

-Building block of noncritical string theory

-Kniznik-Polyakov-Zamolodchikov (86): scaling limit statistical physics models on of random surfaces parametrized by γ.

-E.g. $\gamma = \sqrt{3}$ describes Ising model on a planar map

-Alday-Gaiotto-Tachicawa: related to SuSy Yang-Mills at $d = 4$

Conformal Field Theory

Curtright, Thorn (82) conjectured: **spectrum** of LCFT is **continous** and primary fields are **vertex operators**

$$V_\alpha = e^{\alpha X}, \quad \alpha \in Q + i\mathbb{R}$$

What are the **structure constants**?

Polyakov: BPZ conformal field theory "unsuccessful attempt to solve Liouville theory"

In 1995 Zorn and Otto and Zamolodchicov and Zamolodchicov proposed a remarkable formula for the Liouville structure constants

$$C(\alpha_1, \alpha_2, \alpha_3) = \langle e^{\alpha_1 X(0)} e^{\alpha_2 X(1)} e^{\alpha_3 X(\infty)} \rangle$$

DOZZ formula

$$C_{DOZZ}(\alpha_1,\alpha_2,\alpha_3) = \hat{\mu}^{-s} \frac{\Upsilon'(0)\Upsilon(\alpha_1)\Upsilon(\alpha_2)\Upsilon(\alpha_3)}{\Upsilon(\frac{\alpha_1+\alpha_2+\alpha_3-2Q}{2})\Upsilon(\frac{\alpha_2+\alpha_3}{2})\Upsilon(\frac{\alpha_1+\alpha_3}{2})\Upsilon(\frac{\alpha_1+\alpha_2}{2})}$$

- $\hat{\mu} = \frac{\pi\Gamma(\frac{\gamma^2}{4})(\frac{\gamma}{2})^{\frac{4-\gamma^2}{2}}}{\Gamma(1-\frac{\gamma^2}{4})}\mu$
- Υ is an entire function on $\mathbb{C}$ defined by

$$\Upsilon(\alpha)^{-1} = \Gamma_2(\alpha|\tfrac{\gamma}{2},\tfrac{2}{\gamma})\Gamma_2(2Q-\alpha|\tfrac{\gamma}{2},\tfrac{2}{\gamma})$$

$C_{DOZZ}(\alpha_1,\alpha_2,\alpha_3)$ has simple poles in α_i on

$$\{-\tfrac{\gamma}{2}\mathbb{N}-\tfrac{2}{\gamma}\mathbb{N}\}\cup\{Q+\tfrac{\gamma}{2}\mathbb{N}+\tfrac{2}{\gamma}\mathbb{N}\}$$

Liouville Bootstrap

C_{DOZZ} solves the quadratic bootstrap equations numerically and seems to be the only solution for $c > 1$ with primaries of bounded spins.

This would imply the bootstrap formula

$$\langle e^{\alpha_1 X(0)} e^{\alpha_2 X(z)} e^{\alpha_3 X(1)} e^{\alpha_4 X(\infty)} \rangle_{S^2} =$$

$$= \int_{\mathbb{R}_+} C_{DOZZ}(\alpha_1, \alpha_2, Q - ip) C_{DOZZ}(\alpha_3, \alpha_4, Q + ip) |\mathcal{F}(\alpha, p, z)|^2 dp$$

$\mathcal{F}(\alpha, p, z)$ purely representation theoretic **spherical conformal blocks** determined by c, α_i, p.

Constructive LCFT

1. Give a mathematical meaning to the functional integral

$$\langle \prod_i e^{\alpha_i X(z_i)} \rangle_{S^2} = \int \prod_i e^{\alpha_i X(z_i)} e^{-S_L(X)} DX$$

2. Prove

$$\langle e^{\alpha_1 X(0)} e^{\alpha_2 X(1)} e^{\alpha_3 X(\infty)} \rangle_{S^2} = C_{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$$

3. Prove the bootstrap formula for the four point function

Probabilistic Liouville model

What is the mathematical meaning of the integral

$$\int e^{-\int(|\nabla_g X|^2 + QR_g X + \mu e^{\gamma X})dv_g} DX \quad ?$$

We define it in terms of the **Gaussian Free Field** (GFF) on Σ:

$$\phi_g(z) = \sum_{n=1}^{\infty} \frac{x_n}{\sqrt{\lambda_n}} e_n(z)$$

- e_n are eigenfunctions of Laplacean Δ_g on Σ:

$$-\Delta_g e_n = \lambda_n e_n, \quad n \geq 0$$

- x_n i.i.d. normal random variables variance 1

We set

$$X = ce_0 + \phi_g$$

$e_0(z) =$ constant (zero mode of Δ_g) , $c \in \mathbb{R}$ and

$$e^{-\int |\nabla_g X|^2 dv_g} DX := dc \times \prod_{n=1}^{\infty} e^{-\frac{1}{2}x_n^2} \frac{dx}{\sqrt{2\pi}}$$

Gaussian Multiplicative Chaos (GMC)

The GFF ϕ_g is not a function but a **distribution**:

$$\mathbb{E}\phi_g(x)\phi_g(y) = \log|x-y|^{-1} + \text{bounded}$$

so to define $e^{\gamma X}$ we need to **regularize**

$$\phi_{g,N} = \sum_{n=1}^{N} \frac{x_n}{\sqrt{\lambda_n}} e_n$$

and renormalize (Kahane '86):

$$\lim_{N\to\infty} e^{\gamma\phi_{g,N}(z) - \frac{\gamma^2}{2}\mathbb{E}\phi_{g,N}(z)^2} dz = M_g(dz) \quad \textit{almost surely}$$

M_g is called **Gaussian Multiplicative Chaos** measure on Σ.

M_g is a **random multifractal measure**

In particular $M_g(\Sigma) < \infty$ almost surely.

Probabilistic Liouville Theory

By Gauss-Bonnet and $X = c + \phi_g$

$$Q \int_\Sigma R_g X dv_g = Q\chi(\Sigma)c + Q \int_\Sigma R_g \phi_g dv_g$$

so we define the Liouville theory by

$$\langle F(X) \rangle_{\Sigma,g} := \int_\mathbb{R} e^{-Q\chi(\Sigma)c} \mathbb{E}\left[F(c+\phi_g) e^{-Q\int \phi_g R_g dv_g} e^{-\mu e^{\gamma c} M_g(\Sigma)} \right] dc$$

Primary field correlation functions

$$\langle \prod_{i=1}^n e^{\alpha_i X(z_i)} \rangle_{\Sigma,g} = \int_\mathbb{R} e^{(\sum \alpha_i - Q\chi(\Sigma))c} \mathbb{E}\left[\prod_{i=1}^n e^{\alpha_i \phi_g(z_i)} e^{-Q\int \phi_g R_g dv_g} e^{-\mu e^{\gamma c} M_g(\Sigma)} \right] dc$$

are defined by similar renormalisation (Wick ordering) as well.

Axioms (1)

Theorem (David, K, Rhodes, Vargas, 2015) *The Liouville correlation functions exist and are nontrivial if the **Seiberg bounds** hold:*

$$(1)\quad \alpha_i < Q \ \ \forall i, \quad \text{and} \quad (2)\ \sum_{i=1}^{n} \alpha_i > Q\chi(\Sigma)$$

They satisfy Diff and Weyl Axioms with central charge

$$c = 1 + 6Q^2$$

- (2): convergence of c-integral
- (1): regularity of GMC
- For $\Sigma = S^2$: $\chi(S^2) = 2 \implies \sum_{i=1}^{n} \alpha_i > 2Q$ and $\alpha_i < Q$. Hence **only $n \geq 3$** are finite!
- Probabilistic theory: $\alpha \in \mathbb{R}$, **not** in spectrum.

Structure constants

In particular the structure constants exist and are given by

$$C(\alpha_1,\alpha_2,\alpha_3) := \langle e^{\alpha_1 X(0)} e^{\alpha_2 X(1)} e^{\alpha_3 X(\infty)} \rangle_{S^2}$$

$$= \frac{2}{\gamma}\mu^{-s}\Gamma(s) \lim_{u\to\infty} |u|^{4\Delta_{\alpha_3}} \mathbb{E}\left(\int \frac{|z\vee 1|^{\gamma(\alpha_1+\alpha_2+\alpha_3}}{|z|^{\gamma\alpha_1}|z-1|^{\gamma\alpha_2}|z-u|^{\gamma\alpha_3}} M_g(dz) \right)^{-s}$$

in the region

$$s := \frac{\alpha_1+\alpha_2+\alpha_3-2Q}{\gamma} > 0, \quad \alpha_i < Q$$

Similar expressions for *n*-point functions.

Integrability

Does the probabilistic expression satisfy the DOZZ formula?

Theorem (K, Rhodes, Vargas, Annals of Mathematics **191**, 81) Let α_i satisfy the Seiberg bounds. Then

$$C(\alpha_1, \alpha_2, \alpha_3) = C_{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$$

Proof combines **probabilistic** analysis of GMC to derive **algebraic** identities for the structure constants that determine them uniquely.

DOZZ is an **integrability** result for **multiplicative chaos**.

It is analogous to Fyodorov-Bouchaud conjecture on $\mathbb{S}^1$:

$$\mathbb{E}\Big(\int_{\mathbb{S}^1} M(dz)\Big)^p = \frac{\Gamma(1-p\frac{\gamma^2}{2})}{\Gamma(1-\frac{\gamma^2}{2})^p}$$

Indeed, this follows from Liouville on the **unit disk** (Remy 2018)

Bootstrap

To complete integrability of Liouville prove the bootstrap conjecture: express the 4-point function

$$\langle e^{\alpha_1 X(0)} e^{\alpha_2 X(z)} e^{\alpha_3 X(1)} e^{\alpha_4 X(\infty)} \rangle_{S^2}$$

in terms of 3-point functions

$$\sum_{\alpha \in \mathcal{S}} C(\alpha_1, \alpha_2, \alpha) C(\alpha_3, \alpha_4, \alpha) f(\{\alpha\}, z)$$

Idea:

1. Express correlation functions as **scalar products**
2. $\mathcal{S} =$ **spectrum** of the **Hamiltonian** of the QFT
3. z-dependence from **conformal Ward identities**

Reflection positivity

Setup: Euclidean QFT

$$\langle F \rangle = \int F(X) e^{-S(X)} DX$$

for fields $X(\mathbf{x}, \tau)$, $\mathbf{x} \in \mathcal{M}$, $\tau \in \mathbb{R}$.

$\mathcal{F}_\pm$: functionals $F(X)$, depend on $X|_{\tau \geq 0}$ ($X|_{\tau \leq 0}$ resp.)

Reflection $\theta : \tau \to -\tau$, extends to $\Theta : \mathcal{F}_+ \to \mathcal{F}_-$

Definition. $\langle \cdot \rangle$ is **reflection positive** if

$$\langle F \Theta F \rangle \geq 0 \quad \forall F \in \mathcal{F}_+$$

Scalar product $F, G \in \mathcal{F}_+ \to (F, G) := \langle F \Theta G \rangle$

QFT Hilbert space $\mathcal{H} = \overline{\mathcal{F}_+ / \{(F, F) = 0\}}$

Time translation $\tau \to \tau + t$, $t \geq 0$, extends to $T_t : \mathcal{H} \to \mathcal{H}$.

T_t is a semigroup with positive generator H: $T_t = e^{-tH}$.

H is the **Hamiltonian** of the QFT and quantum fields are

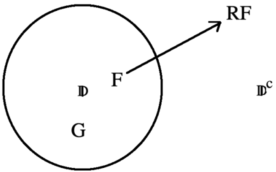
$$\hat{V}_\alpha(\mathbf{x}, t) = e^{-itH} V_\alpha(\mathbf{x}, 0) e^{itH}$$

Hilbert space for LCFT

Consider LCFT on $S^2 = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Map $\mathbb{R} \times S^1 \to \hat{\mathbb{C}}$ by $z = e^{-t+i\theta}$. Then

Hilbert space $\mathcal{F}_{\mathbb{D}}$ = functionals $F(X)$ that depend on $X|_{\mathbb{D}}$

Reflection $t \to -t$ becomes $z \to \bar{z}^{-1}$



Hamiltonian H = generator of dilations $z \to e^{-t} z$.

Proposition (GKRV 2020) H is a positive self adjoint operator on $\mathcal{H}$ for all $\gamma < 2$.

4-point function

By $PSL_2(\mathbb{C})$ the four-point function can be reduced to

$$G_4(z) := \langle e^{\alpha_1 X(0)} e^{\alpha_2 X(z)} e^{\alpha_3 X(1)} e^{\alpha_4 X(\infty)} \rangle_{S^2}$$

with $|z| < 1$. By reflection positivity

$$G_4(z) = (\Psi_{\alpha_1 \alpha_2}(z), \Psi_{\alpha_3 \alpha_4}(1)) \quad (*)$$

with

$$\Psi_{\alpha\beta}(z) = e^{\alpha X(0)} e^{\beta X(z)} \in \mathcal{F}_{\mathbb{D}}.$$

Bootstrap is obtained by factorising $(*)$ using the **spectral resolution** of H.

H as a Schrödinger operator

Reduce the functional integral

$$(F, G) = \langle F \Theta G \rangle = \int F(X) \Theta G(X) e^{-S_L(X)} DX$$

to an integral over $X|_{\partial \mathbb{D}}$. This can be done probabilistically: Recall $X(z) = c + \phi(z)$. Let $\varphi(\theta) = \phi(e^{i\theta})$. Then

$$\varphi(\theta) = \sum_{n \neq 0} \frac{\varphi_n}{\sqrt{n}} e^{in\theta}, \quad \Re \varphi_n, \Im \varphi_n \stackrel{law}{=} N(0, 1)$$

Proposition. There is a **unitary** map

$$U : \mathcal{F}_{\mathbb{D}} \to L^2(dc \times \prod_{n>0} e^{-\frac{1}{2}|\varphi_n|^2} \frac{d\varphi_n d\bar{\varphi}_n}{\pi}) := \mathcal{H}$$

s.t.

$$(F, G) = (UF, UG)_{\mathcal{H}}$$

H as a Schrödinger operator

Furthermore for $\gamma < \sqrt{2}$.

$$UHU^{-1} = H_0 + \mu V$$
$$H_0 = -\frac{1}{2}\frac{d^2}{dc^2} - \Delta_\varphi + \frac{Q^2}{2}$$
$$V(c,\varphi) = e^{\gamma c}\int_0^{2\pi} e^{\gamma\varphi(\theta) - \frac{\gamma^2}{2}\mathbb{E}\varphi(\theta)^2} d\theta$$

where

$$\Delta_\varphi = \sum_{n>0} \partial_{\varphi_n}\partial_{\varphi_{-n}} + \dots$$

Find eigenfunctions $\psi(c,\varphi)$ of H:

$$(H_0 + \mu V)\psi = E\psi$$

Toy Liouville

Keep only c variable:

$$H = \tfrac{1}{2}(-\frac{d^2}{dc^2} + Q^2) + \mu e^{\gamma c}$$

Schrödinger operator on $L^2(\mathbb{R})$ with a wall potential

$$V(c) = e^{\gamma c} \to \begin{cases} 0 & \text{if } c \to -\infty \\ \infty & \text{if } c \to \infty \end{cases}$$

Scattering theory: Generalized eigenfunctions

$$\psi_p(c) \sim \begin{cases} e^{ipc} + R(p)e^{-ipc} & c \to -\infty \\ 0 & c \to \infty \end{cases}$$

with $p \in \mathbb{R}_+$ and eigenvalue $\frac{1}{2}(Q^2 + p^2) = 2\Delta_{Q+ip}$.

Spectrum of H_0

$\mathcal{H}$ carries a unitary representation of two commuting **Virasoro algebras** with generators L_n and $\tilde{L}_n$:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}$$

where the central charge is $c = 1 + 6Q^2$ and $H_0 = L_0 + \tilde{L}_0$.
The generalized eigenfunctions of H_0 are

$$\psi^0_{p,\nu,\tilde{\nu}} = L_{\nu_1} \dots L_{\nu_n} \tilde{L}_{\tilde{\nu}_1} \dots \tilde{L}_{\tilde{\nu}_n} \psi^0_p$$

where $\nu_n \le \dots \le \nu_1 < 0$ and $\psi^0_p = e^{ipc}$ is the **highest weight state**

$$L_n \psi^0_p = 0, n > 0, \quad L_0 \psi^0_p = \Delta_{Q+ip} \psi^0_p$$

and (let $|\nu| := \sum \nu_i$)

$$H\psi^0_{p,\nu,\tilde{\nu}} = (2\Delta_{Q+ip} + |\nu| + |\tilde{\nu}|)\psi^0_p := E(p,\nu,\tilde{\nu})\psi^0_p$$

Spectrum of LCFT

Theorem (GKRV 2020). H has a complete set of generalized eigenfunctions indexed by $p \in \mathbb{R}_+$ and $\nu, \tilde{\nu}$

$$\psi_{p,\nu,\tilde{\nu}} \sim \psi^0_{p,\nu,\tilde{\nu}} + \text{reflected waves} \quad c \to -\infty$$

and

$$(\psi_{p,\nu,\tilde{\nu}}, \psi_{p',\nu',\tilde{\nu}'}) = \delta(p-p')\mathcal{F}(p)_{\nu,\nu'}\mathcal{F}(p)_{\tilde{\nu},\tilde{\nu}'}$$

with $\mathcal{F}(p)$ a Gram matrix (Shapovalov form).

Corollary. Let $\Psi_{p,\nu,\tilde{\nu}} = U^{-1}\psi_{p,\nu,\tilde{\nu}}$. Plancharel identity holds

$$G_4(z) = \int_{\mathbb{R}_+} \sum_{\nu,\nu',\tilde{\nu},\tilde{\nu}'} (\Psi_{\alpha_1\alpha_2}(z), \Psi_{p,\nu,\tilde{\nu}})(\Psi_{p,\nu',\tilde{\nu}'}, \Psi_{\alpha_3\alpha_4}(1))\mathcal{F}(p)^{-1}_{\nu,\nu'}\mathcal{F}(p)^{-1}_{\tilde{\nu},\tilde{\nu}'}dp$$

Remains to connect $(\Psi_{\alpha_1\alpha_2}(z), \Psi_{p,\nu,\tilde{\nu}}) = (V_{\alpha_1}(0)V_{\alpha_2}(z), \Psi_{p,\nu,\tilde{\nu}})$ to structure constants.

Ward identity

Theorem. (GKRV 2020) For an explicit function $\mathcal{T}(\alpha,\beta,p,\nu)$

$$(V_{\alpha_1}(0)V_{\alpha_2}(z),\Psi_{p,\nu,\tilde\nu}) = \mathcal{T}(\alpha_1,\alpha_2,p,\nu)\mathcal{T}(\alpha_1,\alpha_2,p,\tilde\nu)C_{DOZZ}(\alpha_1,\alpha_2,Q+ip) \tag{1}$$

Heuristic explanation: $\Psi_{p,\nu,\tilde\nu} = L_\nu \tilde L_{\tilde\nu} V_{Q+ip}(0)$ and

$$\begin{aligned}(V_{\alpha_1}(0)V_{\alpha_2}(z),\Psi_{p,0,0}) &= (V_{\alpha_1}(0)V_{\alpha_2}(z),V_{Q+ip}(0)) \\ &= \langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{Q+ip}(\infty)\rangle_{S^2} = C_{DOZZ}(\alpha_1,\alpha_2,Q+ip)\end{aligned}$$

$\mathcal{T}$ produced by $L_\nu \tilde L_{\tilde\nu}$ via **conformal Ward identities**.

Actual proof:

- Analytic continuation of $\psi_{p,\nu,\tilde\nu}$: $Q+ip\to\alpha\in\mathbb{R}$
- Probabilistic proof of the Ward identity (1)

Remarks:

- $\psi_{p,\nu,\tilde\nu}$ are **macroscopic states**, **not** created by local fields.
- For $\alpha\in\mathbb{R}$ $V_\alpha(z)$ is local field but creates a state **not** in the spectrum.

Bootstrap

Corollary. (GKRV) Bootstrap formula holds:

$$\langle e^{\alpha_1 X(0)} e^{\alpha_2 X(z)} e^{\alpha_3 X(1)} e^{\alpha_4 X(\infty)} \rangle =$$

$$= \int_{\mathbb{R}_+} C_{DOZZ}(\alpha_1, \alpha_2, Q + ip) C_{DOZZ}(\alpha_3, \alpha_4, Q + ip) |\mathcal{F}(\alpha, p, z)|^2 dp$$

where $\mathcal{F}$ are spherical holomorphic conformal blocks given by

$$\mathcal{F}(\alpha, p, z) := \sum_{n=0}^{\infty} \beta_n z^n$$

The sum converges for almost all p and

$$\beta_n := \sum_{|\nu|,|\nu'|=n} \mathcal{T}(\alpha_1, \alpha_2, p, \nu) \mathcal{F}(p)^{-1}_{\nu,\nu'} \mathcal{T}(\alpha_3, \alpha_4, p, \nu).$$

Prospects

- Similar formuli for *n*-point functions
- Bootstrap for LCFT on Torus (in progress)
- Extension to $\gamma \in [\sqrt{2}, 2)$ (in progress)

Summary

Compare to harmonic analysis on $SU(2)$ vs. $SL(2,\mathbb{R})$:

Compact CFT's: algebra

Non-compact CFT's: analysis and probabilty

Thank you!

Proof ideas

1. **Analyticity.** $C(\alpha_1, \alpha_2, \alpha_3)$ are analytic in a neighborhood of $\alpha_1 + \alpha_2 + \alpha_3 > 2Q$, $\alpha_i < Q$.

2. **Reflection.** $C(\alpha_1, \alpha_2, \alpha_3)$ has analytic continuation beyond $\alpha_i \in (0, Q)$ which satisfies

$$C(\alpha_1, \alpha_2, \alpha_3) = R(\alpha_1) C(2Q - \alpha_1, \alpha_2, \alpha_3)$$

3. **Periodicity.** Let $\alpha = \frac{\gamma}{2}$ or $\alpha = \frac{2}{\gamma}$. Then for all $\alpha_1 \in \mathbb{R}$:

$$C(\alpha_1 - \alpha, \alpha_2, \alpha_3) = D(\alpha, \alpha_1, \alpha_2, \alpha_3) C(\alpha_1 + \alpha, \alpha_2, \alpha_3)$$

For $\gamma^2 \notin \mathbb{Q}$ this determines $C = C_{DOZZ}$. Continuity in $\gamma \implies \square$.

Reflection and Periodicity

DOZZ formula satisfies reflection and periodicity with

$$D(\alpha, \alpha_1, \alpha_2, \alpha_3) = -\frac{1}{\pi\mu} \frac{\Gamma(-\alpha^2)\Gamma(-\alpha\alpha_1)\Gamma(-\alpha\alpha_1 - \alpha^2)\Gamma(\frac{\alpha}{2}(2\alpha_1 - \bar{\alpha}))}{\Gamma(\frac{\alpha}{2}(2Q - \bar{\alpha}))\Gamma(\frac{\alpha}{2}(2\alpha_3 - \bar{\alpha}))\Gamma(\frac{\alpha}{2}(2\alpha_2 - \bar{\alpha}))}$$
$$\times \frac{\Gamma(1 + \frac{\alpha}{2}(\bar{\alpha} - 2Q))\Gamma(1 + \frac{\alpha}{2}(\bar{\alpha} - 2\alpha_3))\Gamma(1 + \frac{\alpha}{2}(\bar{\alpha} - 2\alpha_2))}{\Gamma(1 + \alpha^2)\Gamma(1 + \alpha\alpha_1)\Gamma(1 + \alpha\alpha_1 + \alpha^2)\Gamma(1 + \frac{\alpha}{2}(\bar{\alpha} - 2\alpha_1))}$$
$$R(\alpha) = -((\frac{\gamma}{2})^{\frac{\gamma^2}{2} - 2}\tilde{\mu})^{\frac{2(Q-\alpha)}{\gamma}} \frac{\Gamma(\frac{\gamma}{2}(\alpha - Q))\Gamma(\frac{2}{\gamma}(\alpha - Q))}{\Gamma(\frac{\gamma}{2}(Q - \alpha))\Gamma(\frac{2}{\gamma}(Q - \alpha))}.$$

In particular the **reflection relation** has been a mystery:

$$e^{\alpha\phi} = R(\alpha)e^{(2Q-\alpha)\phi}$$

In our proof

- Coefficients R and D follow from asymptotic analysis of multiplicative chaos integrals
- The **reflection coefficient** $R(\alpha)$ has a probabilistic origin in tail behaviour of multiplicative chaos.