Bundle gerbes on supermanifolds

John Huerta

http://math.ucr.edu/~huerta

Instituto Superior Técnico

TQFT Club
5 June 2020

Wess–Zumino–Witten

The Wess–Zumino–Witten model is a theory of maps from a Riemann surface to a Lie group:

$$g\colon \Sigma \to G.$$

Its equations are given by an action functional with two terms:

$$S_{\mathrm{WZW}}[g] = S_{\mathrm{kin}}[g] + S_{\mathrm{top}}[g].$$

We are interested in the topological term, S_{top}.

Wess–Zumino–Witten

The topological term is defined by extending Σ^2 to be the boundary of a 3-manifold, and extending the map g to this 3-manifold:

$$
\begin{array}{ccc}
\Sigma^2 & \xrightarrow{\;g\;} & G \\
{\scriptstyle\partial}\big\downarrow & \nearrow_{\tilde{g}} & \\
M^3 & &
\end{array}
$$

With this extension, define:

$$S_{\mathrm{top}}[g] = k \int_{M^3} \tilde{g}^* H$$

where $H = (\omega, [\omega, \omega])$ is the canonical 3-form on G.

Wess–Zumino–Witten

Problem S_{top} depends on $\tilde{g}$ rather than just g.

Solution $\mathcal{A}_{\text{top}}[g] = \exp\left(iS_{\text{top}}[\tilde{g}]\right)$ depends only on g, because H is **integral**:

$[H] \in H^3(M, \mathbb{R})$ is in the image of $H^3(M, 2\pi\mathbb{Z}) \to H^3(M, \mathbb{R})$.

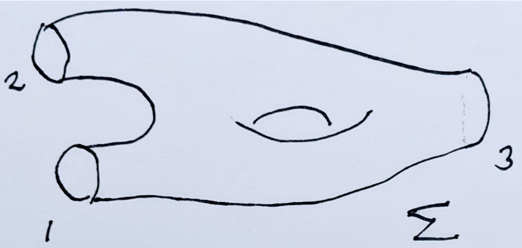
Equivalently:

$$\int_{Z^3} H \in 2\pi\mathbb{Z}$$

for any cycle Z^3 in G.

Wess–Zumino–Witten

Problem $\mathcal{A}_{\text{top}}$ is undefined when Σ has a boundary.



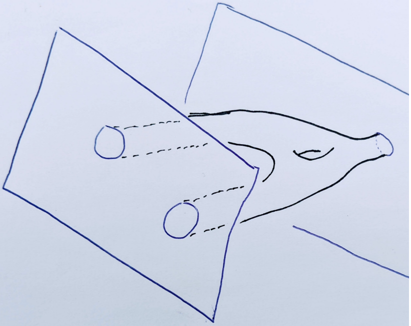
Solution Define $\mathcal{A}_{\text{top}} = \text{Hol}(g)$, the 'surface holonomy' of a 'gerbe' on G. This is a map of lines canonically associated to the boundary of Σ:

$$\text{Hol}(g)\colon L_1 \otimes L_2 \to L_3.$$

Wess–Zumino–Witten

Problem $\mathcal{A}_{\text{top}}$ is a map of lines, not an element of $\mathrm{U}(1)$.

Solution Restrict $g\colon \Sigma \to G$ to have suitable boundary conditions:



These boundary conditions are called **D-branes.** They are submanifolds of G equipped with a trivialization of the gerbe, which in turn trivializes the lines L_i:

$$\mathrm{Hol}(g) \in \mathrm{U}(1).$$

Gerbes and line bundles

Gerbes generalize complex line bundles:

- A complex line bundle on a manifold X gives a class in $H^2(X, \mathbb{Z})$, its **characteristic class** or **Chern class**.
- The characteristic class determines the line bundle up to isomorphism.
- Choosing a connection, the curvature is a 2-form that represents the characteristic class.
- Using the connection, we can construct the holonomy along curves in X.

Gerbes and line bundles

Gerbes generalize complex line bundles:

- A gerbe on a manifold X gives a class in $H^3(X, \mathbb{Z})$, its **characteristic class** or **Dixmier–Douady class**.
- The characteristic class determines the gerbe up to equivalence.
- Choosing a connection, the curvature is a 3-form that represents the characteristic class.
- Using the connection, we can construct the holonomy along surfaces in X.

Gerbes and line bundles

Gerbes generalize complex line bundles:

- Given any closed, integral 2-form F on a manifold X, there is a line bundle with curvature F.
- Given any closed, integral 3-form H on a manifold X, there is a gerbe with curvature H.

The basic gerbe

To show that a gerbe exists on a manifold X, we just need to identify a closed, integral 3-form.

- When $X = G$ is a compact, simple Lie group, we can take the canonical 3-form:

  $$H = \mathrm{tr}(\omega, [\omega, \omega]).$$

  There is a bundle gerbe $\mathcal{G}$ with this curvature, used to define the WZW model.

- When G is simply connected, and H is scaled to generate the integral cohomology, we call $\mathcal{G}$ the **basic gerbe.** Every gerbe $\mathcal{H}$ on G is a tensor power of $\mathcal{G}$:

  $$\mathcal{H} \simeq \mathcal{G}^{k\otimes}.$$

Supermanifolds

We would like to generalize this story to when G is a super Lie group, so we need some supergeometry.

Definition

A **smooth supermanifold** M is a pair $(|M|, \mathcal{O}_M)$, where:

- $|M|$ is a topological manifold;
- $\mathcal{O}_M$ is a sheaf of supercommutative superalgebras;
- $|M|$ admits an atlas $\{(U_\alpha, \varphi_\alpha \colon U_\alpha \to \mathbb{R}^m)\}$ of coordinate charts which extend to local isomorphisms of sheaves:

$$\mathcal{O}_M|_{U_\alpha} \cong \mathcal{C}^\infty_{U_\alpha} \otimes \Lambda(\theta_1, \ldots, \theta_n).$$

We say M has dimension $m|n$.

Supermanifolds: Examples

- A **super Cartesian space**, $\mathbb{R}^{m|n} = (\mathbb{R}^m, \mathcal{C}^\infty_{\mathbb{R}^p} \otimes \Lambda(\theta_1, \ldots, \theta_n))$.
- A **super domain**, $U^{m|n} = (U, \mathcal{C}^\infty_U \otimes \Lambda(\theta_1, \ldots, \theta_n))$, for any open set $U \subseteq \mathbb{R}^m$.
- For any vector bundle $E \to X$ over a manifold X, $M = (X, \Gamma(\Lambda E^*))$.
- In particular for $TX \to X$, we get the **odd tangent bundle** $\Pi TX = (X, \Omega_X)$.

Supermanifolds: Properties

Given a supermanifold M, let $\mathcal{J} \subseteq \mathcal{O}_M$ be the nilpotent ideal. Then

$$M_{\mathrm{rd}} = (|M|, \mathcal{O}_M/\mathcal{J})$$

is an ordinary manifold: the **body** of M. This is because, locally, $\mathcal{J}$ is generated by the odd coordinates $\theta_1, \ldots, \theta_n$. We have a canonical inclusion:

$$i\colon M_{\mathrm{rd}} \to M.$$

Question

How different is M from M_{rd}? For instance, how do gerbes on M and M_{rd} compare?

Supermanifolds: Last definitions

- A map $Y \to M$ of supermanifolds is a **surjective submersion** if it is locally isomorphic to a projection $U \times V \to V$, for superdomains U and V.
- A **vector bundle** on M is a sheaf of $\mathcal{O}_M$-modules, locally free of finite rank $p|q$.
  - The **tangent bundle** $\mathcal{T}M$ is the sheaf of derivations of $\mathcal{O}_M$.
  - The **cotangent bundle** is the dual, $\mathcal{T}^*M = \mathrm{Hom}(\mathcal{T}M, \mathcal{O}_M)$.
  - The **de Rham complex** is $\Omega_M = \Lambda(\mathcal{T}^*M)$.
- A **line bundle** on M is a sheaf of $\mathbb{C} \otimes \mathcal{O}_M$-modules, locally free of rank $1|0$.

Now we are ready to discuss gerbes. We'll use 'bundle gerbes', a notion due to Michael Murray.

Bundle gerbes

A **bundle gerbe** $\mathcal{G}$ on the supermanifold M consists of the following data:

- a surjective submersion, $\pi \colon Y \to M$.
- an even complex line bundle over the fiber square of Y, $Y^{[2]} = Y \times_M Y$:

$$\begin{array}{ccc} L & & \\ \downarrow & & \\ Y^{[2]} & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & Y \\ & & \downarrow{\scriptstyle \pi} \\ & & M. \end{array}$$

- a line bundle isomorphism $\mu \colon \pi_3^* L \otimes \pi_1^* L \to \pi_2^* L$ over $Y^{[3]}$. The map μ is called the bundle gerbe **multiplication**, which is associative.

Bundle gerbes

A **gerbe connection** on a bundle gerbe $\mathcal{G}$ over M consists of the following data:

- a connection ∇ on the line bundle L, compatible with multiplication;
- a 2-form B on Y;
- satisfying the descent condition:

$$F_\nabla = \pi_1^* B - \pi_2^* B.$$

The **curvature** of $\mathcal{G}$ is then the unique closed 3-form H on M such that:

$$\pi^* H = dB.$$

Bundle gerbes

The **trivial bundle gerbe** $\mathcal{I}_B$:

$$\begin{array}{ccc} M \times \mathbb{C} & & \\ \downarrow & & \\ M & \overset{1}{\underset{1}{\rightrightarrows}} & M \\ & & \downarrow{\scriptstyle 1} \\ & & M. \end{array}$$

with trivial multiplication, trivial connection on $M \times \mathbb{C}$, and B an arbitrary 2-form.

Bundle gerbes

The usual operations on line bundles generalize to bundle gerbes:

- Given a bundle gerbe $\mathcal{G}$, we can construct its dual $\mathcal{G}^*$.
- Given two bundle gerbes $\mathcal{G}$ and $\mathcal{H}$ on M, we can construct their tensor product $\mathcal{G} \otimes \mathcal{H}$.
- Given a bundle gerbe $\mathcal{G}$ on N and a map $f\colon M \to N$, we can construct its **pullback** $f^*\mathcal{G}$, a gerbe on M.

Classification of bundle gerbes

- The collection of equivalence classes of bundle gerbes $\mathbb{G}(M)$ on M forms a group, with multiplication given by tensoring, and inverse given by the dual.
- Likewise for $\mathbb{G}^{\nabla}(M)$, the bundle gerbes with connection.

Classification of bundle gerbes

Let $\mathcal{O}_{\mathbb{C}}^{\times} = (\mathbb{C} \otimes \mathcal{O}_0)^{\times}$, the even, invertible elements in the complexified structure sheaf of M.

Theorem

There is a group isomorphism

$$DD\colon \mathbb{G}(M) \to H^2(|M|, \mathcal{O}_{\mathbb{C}}^{\times})$$

between the group of bundle gerbes and second Čech cohomology.

Theorem

There is a group isomorphism

$$\mathrm{Del}\colon \mathbb{G}^{\nabla}(M) \to H^2(|M|, \mathcal{O}_{\mathbb{C}}^{\times} \to \Omega_{\mathbb{C}}^1 \to \Omega_{\mathbb{C}}^2)$$

between the group of bundle gerbes with connection and second Deligne cohomology.

Proof sketch for $\mathbb{G}(M) \cong H^2(|M|, \mathcal{O}_{\mathbb{C}}^{\times})$.

Choose a good open cover $\{U_\alpha\}$ of $|M|$, and local sections $s_\alpha \colon U_\alpha \to Y$. Then we get maps $(s_\alpha, s_\beta) \colon U_{\alpha\beta} \to Y^{[2]}$, and we can pull the line bundle L back to line bundles $L_{\alpha\beta}$ on each $U_{\alpha\beta}$, trivial with nonvanishing section $\sigma_{\alpha\beta}$. Over $U_{\alpha\beta\gamma}$, the bundle gerbe multiplication gives us:

$$\mu(\sigma_{\alpha\beta} \otimes \sigma_{\beta\gamma}) = g_{\alpha\beta\gamma}\sigma_{\alpha\gamma}$$

for some $g_{\alpha\beta\gamma} \in \mathcal{O}_{\mathbb{C}}(U_{\alpha\beta\gamma})$. This is our Čech cocycle.

□

Classification of bundle gerbes

The same results for manifolds are due to Murray and Danny Stevenson, and this generalization is straightforward.

The crucial observation, due to Kostant, is that we have a short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} (\mathcal{O}_{\mathbb{C}})_0 \xrightarrow{\exp} \mathcal{O}_{\mathbb{C}}^{\times} \longrightarrow 0$$

Corollary

$$\mathbb{G}(M) \cong H^3(|M|, \mathbb{Z})$$

Corollary

The curvature of a gerbe with connection is a closed, integral 3-form. Conversely, any integral 3-form on M is the curvature of some bundle gerbe.

The basic gerbe on super Lie groups

For G compact, simple, simply connected, the basic gerbe is the unique gerbe with curvature the closed, integral 3-form:

$$H = \mathrm{tr}(\omega, [\omega, \omega])$$

scaled so that H represents the generator of $H^3(G, \mathbb{Z}) \cong \mathbb{Z}$.

Question

Can we find a similar 3-form for super Lie groups?

Super Lie groups

As with Lie groups, it helps to work at the super Lie algebra level:

Theorem (V. Kac)

A simple super Lie algebra over $\mathbb{C}$ admits an ad-invariant inner product if and only if it is one of:

$$\mathfrak{sl}(m|n), \quad \mathfrak{psl}(n|n), \quad \mathfrak{osp}(m|n), \quad D(2,1;\alpha), \quad G(3), \quad F(4).$$

The inner product is unique up to rescaling.

Here, for the complex super vector space $\mathbb{C}^{m|n} = \mathbb{C}^m \oplus \Pi\mathbb{C}^n$:

- $\mathfrak{sl}(m|n) = \left\{ A \colon \mathbb{C}^{m|n} \to \mathbb{C}^{m|n} \,:\, \mathrm{str}(A) = 0 \right\}$.
- $\mathfrak{psl}(n|n) = \mathfrak{sl}(n|n)/\mathbb{C}I$.
- $\mathfrak{osp}(m|n) = \left\{ A \colon \mathbb{C}^{m|n} \to \mathbb{C}^{m|n} \,:\, g(Ax, y) = (-1)^{|A||x|} g(x, Ay) \right\}$.
- $D(2,1;\alpha), \quad G(3), \quad F(4)$ are exceptional.

Super Lie groups

A linear map $A\colon C^{m|n} \to \mathbb{C}^{m|n}$ corresponds to a block matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Diagonal is even:

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

Off diagonal is odd:

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

The **supertrace** is simply:

$$\mathrm{str}(A) = \mathrm{tr}(a) - \mathrm{tr}(d).$$

Proposition

On $\mathfrak{sl}(m|n)$, $\mathfrak{psl}(n|n)$, $\mathfrak{osp}(m|n)$, the supertrace inner product is ad-invariant:

$$\mathrm{str}(AB).$$

Super Lie groups

Thus on super Lie groups $\mathrm{SL}(m|n)$, $\mathrm{PSL}(n|n)$, $\mathrm{OSp}(m|n)$ or their real forms, a candidate 3-form is:

$$H = \mathrm{str}(\omega, [\omega, \omega])$$

where ω is the Maurer–Cartan form.

Proposition

H is closed, and can be rescaled to be integral.

The basic gerbe on super Lie groups

Definition

A **basic gerbe** for $G = \mathrm{SL}(m|n)$, $\mathrm{PSL}(n|n)$, $\mathrm{OSp}(m|n)$ or a real form is a bundle gerbe with curvature H, with H scaled as small as possible to make it integral.

Towards an explicit model

For any super Lie group G, we can choose a diffeomorphism:

$$G = G_{\mathrm{rd}} \times \mathfrak{g}_1.$$

Hence, along with the canonical inclusion, we have a projection:

$$i\colon G_{\mathrm{rd}} \hookrightarrow G, \quad p\colon G \to G_{\mathrm{rd}}.$$

These give maps between the de Rham complexes on G_{rd} and G:

$$i^*\colon \Omega^\bullet(G) \to \Omega^\bullet(G_{\mathrm{rd}}), \quad p^*\colon \Omega^\bullet(G_{\mathrm{rd}}) \to \Omega^\bullet(G).$$

Proposition

The map p^* splits the short exact sequence:

$$0 \longrightarrow K^\bullet \longrightarrow \Omega^\bullet(G) \xrightarrow{i^*} \Omega^\bullet(G_{\mathrm{rd}}) \longrightarrow 0$$

and $K^\bullet$ is acyclic.

Towards an explicit model

This allows us to decompose

$$H = H_{\mathrm{rd}} + d\beta.$$

where H_{rd} is a closed, integral 3-form on G_{rd}.

Choosing a gerbe $\mathcal{G}_{\mathrm{rd}}$ with curvature H_{rd}, we have:

$$\mathcal{G} = p^*\mathcal{G}_{\mathrm{rd}} \otimes \mathcal{I}_\beta$$

is a basic gerbe on G, having curvature H.

An explicit model on $\mathrm{PSU}(n|n)$

For $G = \mathrm{PSU}(n|n)$, a real form of $\mathrm{PSL}(n|n)$, we have:

- $G_{\mathrm{rd}} = \mathrm{SU}(n) \times \mathrm{SU}(n)$;
- $H_{\mathrm{rd}} = i^* H = H_1 - H_2$, since $\mathrm{str}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \mathrm{tr}(a) - \mathrm{tr}(d)$.
- $\mathcal{G}_{\mathrm{rd}} = \mathcal{G}_1 \otimes \mathcal{G}_2^*$.
- $\beta = \mathrm{str}(\omega_1, \omega_1)$, where ω_1 is the $\mathfrak{g}_1$-valued part of ω.

Theorem

The basic gerbe $\mathcal{G}$ on $\mathrm{PSU}(n|n)$ is unique up to equivalence, and has the form:

$$\mathcal{G} = p^*\mathcal{G}_1 \otimes p^*\mathcal{G}_2^* \otimes \mathcal{I}_\beta .$$

Further questions

- Can $\mathcal{G}$ be made equivariant with respect to the adjoint action of G on itself?
- What are the obstructions to lifting an equivariant gerbe from an ordinary manifold to a supermanifold?
- What is the moduli theory of gerbes?