Quiver Representations

Javier Orts[1]

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Instituto Superior Técnico
Falcudade de Ciências UL

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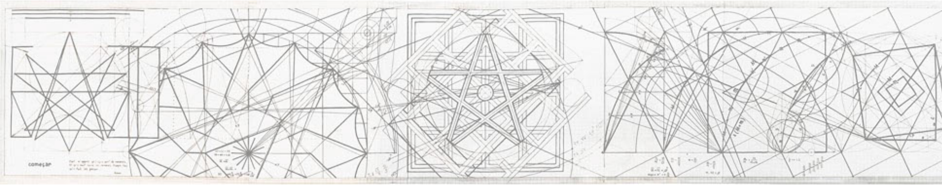

[1] javier.orts@tecnico.ulisboa.pt

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Preliminaries

1.1 Basic definitions

What is a quiver?

A quiver is a **directed graph** where multiple arrows and loops are allowed.

Hence, a quiver Q is specified by the following data: two finite sets A and V and two maps $s, t : A \to V$.

The set A is referred as the set of **arrows**, V is the set of **vertices** and the maps s and t are called the start or **source** and the **target** of an arrow, respectively.

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If a, b are two arrows with $sb = ta$, then we can define the **path** ba.
In general, if $a_1, \dots, a_\ell$ is a sequence of ℓ arrows such that $sa_{k+1} = ta_k$, then

$$p = a_\ell a_{\ell-1} \dots a_1.$$

Remark

For reasons that will be clear later, we consider paths going from right to left:

$$\circ \xleftarrow{a_\ell} \circ \xleftarrow{a_{\ell-1}} \cdots \xleftarrow{a_2} \circ \xleftarrow{a_1} \circ.$$

Acylcic quivers, sinks and sources

Definition (Acyclic quiver)

A quiver Q that has a non-trivial path with $sp = tp$ is call **cyclic**. *Such paths are called* **cycles**. *If the Q has no cycles then it is an* **acyclic** *quiver.*

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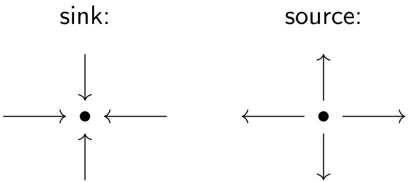
Definition (Sinks and sources)

1. *A vertex i is called a* **sink** *if it is connected to another vertex and $sa \neq i$ for any arrow a.*
2. *A vertex i is called a* **source** *if it is connected to another vertex and $ta \neq i$ for any arrow a.*



Quivers: definition and first example

Alternatively, a quiver can be define using the language of category theory:

Definition (Quiver)

A **quiver** *is a free category with finitely many objects and finite* Hom *sets. Given a quiver (regarded as a graph)* ΓQ*, the objects of* Q *are the vertices of* ΓQ *and the morphisms all the paths. (Given a vertex* i*, we need to define* **trivial path** *of length* 0*:* $se_i = te_i = i$*).*

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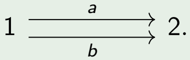
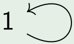
Example

1. The **Kronecker quiver**



2. The **Dynkin quiver**



Quiver representations

Let Q be a quiver and let K be a field.

A ***K*-representation of *Q*** consists of an assignment of a finite-dimensional K-vector space X_i to each vertex $i \in V$, and of a linear map L_a to each arrow $a \in A$.

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If X is a K-representation of Q and $p = a_\ell \dots a_1$ a path on Q of length ℓ, we define the image of p under X to be the composition

$$X(p) = X(a_\ell)X(a_2)\dots X(a_1). \tag{1.1}$$

Finally, if i is a vertex of Q, then $X(e_i) = \mathrm{id}_{X(i)}$.

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Finally, if i is a vertex of Q, then $X(e_i) = \mathrm{id}_{X(i)}$.

Recall that quiver Q is a category with objects the vertices of ΓQ and with morphisms all the paths in Q. This fact, together with relation (1.1) allows one to give a categorical definition for a quiver representation.

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2. *A quiver K-representation Y is said to be a* **subrepresentation** *of another K-representation X if $Y(i) \subset X(i)$ for every vertex i and $X(a)$ restricts to $Y(a)$ for every arrow a.*
3. *A quiver K-representation is called a* **framed K-representation** *if each vector space $X(i)$ is K^{n_i}, for some $n_i \geq 0$.*
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The **dimension vector** *of a quiver representation X is the function $\overrightarrow{\dim} X : V \to \mathbf{N}^V$ given by*

$$\overrightarrow{\dim} X(i) = \dim_K X(i).$$

If nothing is said, any representation will be assumed to be non-trivial.

The category $\mathbf{Rep}_K(Q)$

Since any quiver is a small category, we can consider the functor category $Q^{\mathbf{Vectf}_K}$. This category will be denoted by $\mathbf{Rep}_K(Q)$ and consists of representations of Q. It is an Abelian category.

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Definition (Morphism)

Let Q be a quiver and let X and Y be objects in $\mathbf{Rep}(Q)$. A **morphism** $\phi : X \to Y$ *is an element of* $\mathrm{Hom}_{\mathbf{Rep}(Q)}(X, Y)$.

A morphism $\phi : X \to Y$ is a natural transformation from X to Y, i.e. an assignment $i \in V \overset{\phi}{\mapsto} \phi_i \in \mathrm{Hom}_K(X(i), Y(i))$ making commutative the following diagram:

$$\begin{array}{ccc} X(sa) & \xrightarrow{\phi_{sa}} & Y(sa) \\ {\scriptstyle X(a)}\downarrow & & \downarrow{\scriptstyle Y(a)} \\ X(ta) & \xrightarrow[\phi_{ta}]{} & Y(ta). \end{array}$$

Examples of quiver representations

Example (Labelled representations)

Given a vertex i, there is a special case of frame K-representation given by the assignment

$$S_i(j) = \begin{cases} K, & \text{if } i = j \\ \{0\}, & \text{otherwise.} \end{cases}$$

Sometimes, we will refer to these representations as the **labelled representation**.

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Remark

Most of the operations you know for modules and vector spaces can be translated to quivers. Most of the them are simply defined vertexwise and the one checks it actually works.

1.2 Indecomposable representations and the Krull–Remak–Schmidt theorem

Indecomposable and irreducible representations

Definition (Indecomposable and irreducible representations)

1. *A quiver representations is called* **decomposable** *if it is isomorphic to the product of two quiver representations. A quiver representation that is not decomposable is called* **indecomposable**.
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Example (Simple representations of an acyclic quiver)

The unique simple representations of an acyclic quiver are (up to isomorphism) the labelled representations

$$S_i(j) = \begin{cases} \mathbf{C}, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

The Krull–Remak–Schmidt theorem

Theorem 1.1 (Krull–Remak–Schmidt)

Every (finite dimensional) quiver representation is isomorphic to a direct sum of indecomposable representations and this decomposition is unique up to isomorphism and permutation of factors. More precisely, if

$$X_1 \oplus \cdots \oplus X_p \cong Y_1 \oplus \cdots \oplus Y_q,$$

and $X_1, \ldots X_p, Y_1, \ldots Y_q$ are indecomposable, then $p = q$ and there exists a permutation σ of $\{1, 2, \ldots, p\}$ such that $X_k \cong Y_{\sigma(k)}$ for each k.

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The proof of the theorem needs 2 previous lemmas:

Lemma

Let X *be an indecomposable quiver representation. For each morphism* $\phi : X \to X$*, either* ϕ *is invertible or nilpotent.*

Proof of the Krull–Remak–Schmidt theorem I

Lemma

Suppose that X_1, X_2, Y_1 *and* Y_2 *are quiver representations and*

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} : X_1 \oplus X_2 \longrightarrow Y_1 \oplus Y_2$$

is an isomorphism, where $\phi_{ij} : X_j \to Y_i$. *If* ϕ_{11} *is an isomorphism, then so is* ϕ_{22}.

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Proof of Theorem 1.1.

The first part of the theorem is done by induction on the norm of the dimension vector: if X is decomposable, it splits in two terms Y and Z, with $\|\overrightarrow{\dim} Y\|, \|\overrightarrow{\dim} Z\| < \|\overrightarrow{\dim} X\|$. Apply the induction hypothesis and the result follows.

The second part is again by induction on p and the previous lemmas are needed.

Proof of the Krull–Remak–Schmidt theorem III

Proof of Theorem 1.1.

One considers an isomorphism $\phi = (\phi_{lm})$ between the two decompositions. If ψ is the inverse of ϕ, then

$$\sum_l \psi_{pl}\phi_{lp} = \mathrm{id}_{X_p},$$

which implies that some ϕ_{lp} is not nilpotent. Then, it must invertible, by the first lemma.

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which implies that some ϕ_{lp} is not nilpotent. Then, it must invertible, by the first lemma.

Moreover, rearranging the factors we can assume $l = q$ we can separate each decomposition in two blocks:

$$(X_1 \oplus \cdots \oplus X_{p-1}) \oplus X_p \cong (Y_1 \oplus \cdots \oplus Y_{q-1}) \oplus Y_q.$$

The second lemma is used now, providing an isomrphism bewteen the two big blocks and the induction hypothesis completes the proof.

QED

Homology with quivers

2.1 The path algebra

Definition

The path algebra of Q is the free $\mathbf{C}$-vector space generated by all the paths of Q with the concatenation as operation (a semigroup ring). More precisely:

Definition (Path algebra)

Let Q be a quiver and let $\mathbf{C}Q$ denote the free $\mathbf{C}$-vector space with basis all the paths in Q. If $p = a_1 \dots a_l$ and $q = b_1 \dots b_{\ell'}$ are two paths in Q, define their product as

$$pq = \begin{cases} a_\ell \dots a_1 b_{\ell'} \dots b_1, & \text{if } tb_\ell = sa_1 \\ 0, & \text{otherwise.} \end{cases}$$

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*The $\mathbf{C}$-vector space $\mathbf{C}Q$ together with this operation is called the **path algebra**.*

Proposition

The $\mathbf{C}$-vector space $\mathbf{C}Q$ together with the product defined above constitutes an associative $\mathbf{C}$-algebra. If, moreover, Q is an acyclic quiver, then $\mathbf{C}Q$ has identity element $1 = \sum_{i \in V} e_i$.

The equivalence $\mathbf{Modf}_{\mathbb{C}Q}$ and $\mathbf{Rep}(Q)$s

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The path algebra a quiver Q is finite-dimensional if and only if Q is an acyclic quiver.

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Theorem 2.1

Let $\mathbf{Modf}_{\mathbf{C}Q}$ denote the category of finitely generated left modules over the path algebra $\mathbf{C}Q$. The functor follwing functor $F : \mathbf{Modf}_{\mathbf{C}Q} \to \mathbf{Rep}(Q)$ defines an equivalence:

$$F(X)(i) = e_i X$$

2.2 Projective representations and the hereditary property of $\mathbf{C}Q$

Projective objects I

Recall that a ring R is called **hereditary** if, for each projective R-module P, every submodule of P is again projective. Then the category the category $\mathbf{Mod}_R$ is said to be hereditary.

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Let Q be a quiver.

1. *A representation P is called* **projective** *if P is a projective object in $\mathbf{Rep}(Q)$.*
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Because injective and projective objects are dual to each other, it is enough to study one of them. We will focus on projective quiver representations.
For $i \in V$, define the left $\mathbf{C}Q$-module generated by e_i:

$$P_i = \mathbf{C}Qe_i.$$

We want to show that each P_i is a projective $\mathbf{C}Q$-module.

Projective objects II

Proposition

Let Q be a quiver and let V denote the set of vertices of Q. The path algebra of Q admits the following decompositions:

$$\mathbf{C}Q = \bigoplus_{i \in V} P_i \tag{2.1}$$

Corollary

Each P_i is a projective $\mathbf{C}Q$-module.

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The P_i can be interpreted as quiver representations by virtue of Theorem 2.1:

$$P_i(j) = e_j \mathbf{C}Q e_i$$

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Remark

Geometrically, each P_i is generated by the paths (from right to left) starting at the vertex i.

Preliminary results

Theorem

The quiver representations P_i are projective representations. Moreover, they are the unique indecomposable projective representations, up to isomorphism.

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Let Q be an acyclic quiver and let $R = \mathbf{C}Q$ denote the path algebra of Q. Given X a finitely generated R-module, it is possible to achieve a sequence

$$0 \longrightarrow \bigoplus_{a \in A} P_{ta} \otimes_{\mathbf{C}} e_{sa}X \xrightarrow{f_X} \bigoplus_{i \in V} P_i \otimes_{\mathbf{C}} e_i X \xrightarrow{g_X} X \longrightarrow 0.$$

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Theorem 2.2

If Q is an acyclcic quiver, then the category $\mathbf{Modf}_{\mathbf{C}Q}$ is hereditary ($\mathbf{C}Q$ is semi-hereditary).

2.3 The Euler form

Euler form

Definition (Euler form)

Let Q be a quiver with vertices V and arrows A, and let α and β be two dimension vectors. The **Euler form of Q** *acting on α and β is given by the formula*

$$\langle \alpha, \beta \rangle = \sum_{i \in V} \alpha(i)\beta(i) - \sum_{a \in A} \alpha(sa)\beta(ta).$$

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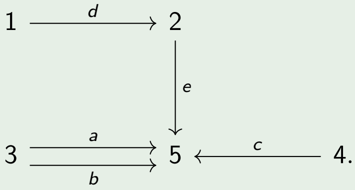
Corollary

The rows of E^{-1} are the dimension vectors of the indecomposable projective representations. Similarly, its columns are the dimension vectors of the indecomposable injective representations.

An example

Example

Let Q be given by the graph



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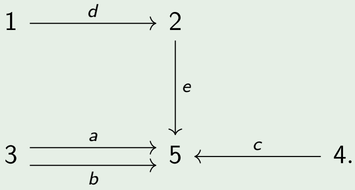

The Euler form of Q is

$$E = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Gabriel's Theorem

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A quiver Q is of finite representation type if and only if ΓQ is a union of Dynkin diagrams of type ADE. In addition, if Q is of finite representation type, then the assignment $X \mapsto \overrightarrow{\dim} X$ establishes a bijection between the isomorphism classes of indecomposable representations and the set of positive roots.

3.1 Dynkin diagrams

ADE Dynkin diagrams

Definition (ADE Dynkin diagrams)

With a **Dynkin diagram of type ADE** *we mean one of the following graphs:*

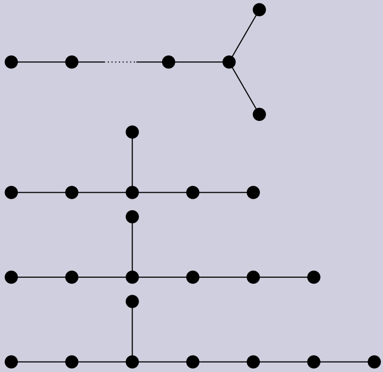
A_n

D_n

E_6

E_7

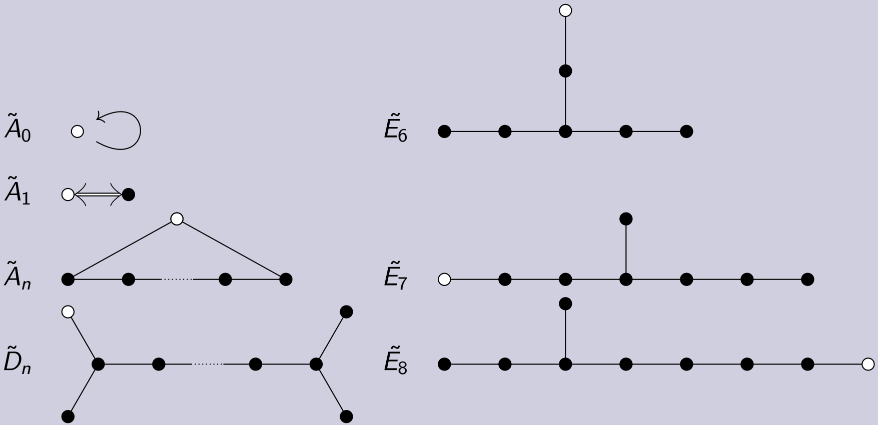
E_8



Extended ADE Dynkin diagrams

Definition (Extended Dynkin diagrams)

By an **extended ADE Dynkin diagram** *we mean one of the following graphs:*



3.2 Quivers of finite representation type and the Tits form

Quivers of finite representation type and the Tits form I

Definition (Finite representation type)

A quiver Q has **finite representation type** *or is said to be of* **finite representation type** *if the category* $\mathbf{Rep}(Q)$ *contains finitely many isomorphism classes of indecomposable objects.*

Remark

For a fixed dimension vector α, a quiver of finite representation type has finitely many isomorphism classes of objects, by the Krull-Remak-Schmidt theorem.

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Definition (Tits form)

Let Q be a quiver. The **Tits form** *of associated to Q on the space of dimension vectors is*

$$B_Q(\alpha) = \sum_{i \in V} \alpha(i)^2 - \sum_{a \in A} \alpha(sa)\alpha(ta).$$

Properties of the Tits form

Some properties/remarks of the Tits form:

1. It does **NOT** depend on the orientation of Q.
2. Then, one cannot expect different quivers to have different Tits forms. B_Q it is not a characteristic element of Q.
3. The Tits form is actually the quadratic form associated to the Euler form: for general α, β the following holds:

$$B_Q(\alpha + \beta) = B_Q(\alpha) + B_Q(\beta) - \langle \alpha, \beta \rangle - \langle \beta, \alpha \rangle.$$

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Lemma

If Q' is a subquiver of Q and $B_Q(\alpha) \geq 1$ for every non-zero dimension vector $\alpha \in \mathbf{N}^V$, then $B_{Q'}(\alpha') \geq 1$ for every nonzero dimension vector $\alpha' \in \mathbf{N}^{V'}$.

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Proposition

Let Q be a quiver and let ΓQ denote its graph. If ΓQ is an extended Dynkin diagram of type ADE, then there exists a non-zero dimension vector α such that $B_Q(\alpha) = 0$.

Quivers of finite representation type and the Tits form II

Proposition

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Quivers whose underlaying graph is an extended Dynkin diagram of type ADE are not of finite representation type.

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Theorem 3.1

If Q is a quiver of finite representation type, then ΓQ is a union of Dynkin diagrams of type ADE.

The main argument used in the proof is the result of the corollary.

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If Q is a quiver of finite representation type, then ΓQ *is a union of Dynkin diagrams of type ADE.*

The main argument used in the proof is the result of the corollary.

Corollary

Quivers of finite representation type are acyclic.

3.3 The reflection functors and the Coxeter functors

Reflections I

Definition (Reflected quiver)

Let Q a quiver with vertices V and arrows A. For each $i \in V$, the **reflected quiver with respect to i** *is the quiver $s_i(Q)$ with the same vertices that Q but where the arrows connecting the vertex i are reversed.*

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The reflections s_i are also defined on dimension vectors:

$$s_i(\alpha)(j) = \begin{cases} \sum_{\substack{a \in A \\ ta = i}} \alpha(sa) + \sum_{\substack{a \in A \\ sa = i}} \alpha(ta) - \alpha(i), & \text{if } i = j \\ \alpha(j), & \text{otherwise.} \end{cases}$$

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Lemma

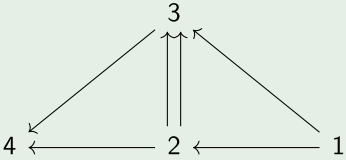
For any quiver Q, vertex i and dimension vector α,

$$B_{s_i(Q)}(s_i(\alpha)) = B_Q(\alpha).$$

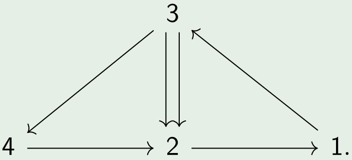
Reflections II

Example

Suppose that Q is given by the graph



then the graph of $s_2(Q)$ is



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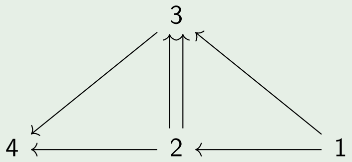

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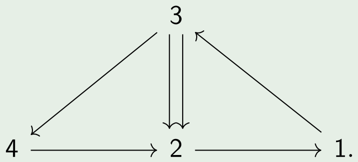

Reflection functors and the Bernstein–Gelfand–Ponomarev theorem

When i is a sink or a source, we can define the **reflection functors**

$$C_i^+, C_i^- : \mathbf{Rep}(Q) \longrightarrow \mathbf{Rep}(s_i(Q)).$$

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$$C_i^+, C_i^- : \mathbf{Rep}(Q) \longrightarrow \mathbf{Rep}(s_i(Q)).$$

They satisfy:

Theorem 3.2 (Bernstein–Gelfand–Ponomarev)

Let Q be a quiver and let X be a non-trivial indecomposable representation of Q.

1. *Suppose that $i \in V$ is a sink. Then:*
    1. *$X \cong S_i$ if and only if $C_i^+(X)$ is trivial*
    2. *If $X \ncong S_i$, then $C_i^+(X)$ is also indecomposable with dimension vector $s_i\left(\overrightarrow{\dim} X\right)$ and $C_i^-(C_i^+(X)) \cong X$.*
2. *Suppose that $i \in V$ is a source. Then:*
    1. *$X \cong S_i$ if and only if $C_i^-(X)$ is trivial*
    2. *If $X \ncong S_i$, then $C_i^-(X)$ is also indecomposable with dimension vector $s_i\left(\overrightarrow{\dim} X\right)$ and $C_i^+(C_i^-(X)) \cong X$.*

Sequences of sinks and socurces

In the theory of Lie algebras, a Coxeter transformation is a distinguished element in the Weyl group defined by concatenating all the reflections associated to a base of the root system.

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Things here are more subtle, since the reflection functors are not defined for arbitrary vertices.

1. We say that a sequence of vertices $i_1, i_2, \ldots, i_p$ is an **admissible sequence of sinks** if for each $1 \leq k \leq p$, the vertex i_k is a sink in $s_{i_{k-1}} s_{i_{k-2}} \cdots s_{i_1}(Q)$.
2. We say that a sequence of vertices $i_1, i_2, \ldots, i_p$ is an **admissible sequence of sources** if for each $1 \leq k \leq p$, the vertex i_k is a source in $s_{i_{k-1}} s_{i_{k-2}} \cdots s_{i_1}(Q)$.

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Lemma

Let Q be an acyclic quiver with n vertices. There exists a labelling for V such that $V = \{1, \dots, n\}$ and $1, \dots, n$ is an admissible sequence of sinks.

The Coxeter functors

Definition (The Coxeter functors)

Let Q be an acyclic quiver with n vertices. The functors

$$C^+ = C_n^+ C_{n-1}^+ \cdots C_1^+ : \mathbf{Rep}(Q) \longrightarrow \mathbf{Rep}(Q)$$

and

$$C^- = C_1^- C_2^- \cdots C_n^- : \mathbf{Rep}(Q) \longrightarrow \mathbf{Rep}(Q)$$

are called the **Coxeter functors** *of Q.*

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are called the **Coxeter functors** *of Q.*

Remark

1. It can be proved that different orderings of V produce naturally isomorphic Coxeter functors.
2. The Coxeter functors play an important role in the proof of Gabriel's theorem. However, most of their properties are given by the well behaviour of the reflection functors showed in the Bernstein–Gelfand–Ponomarev theorem.

3.4 Gabriel's theorem

Terminology

Suppose that Q is a quiver whose underlying graph is a union of Dynkin diagrams of type ADE.

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Suppose that Q is a quiver whose underlying graph is a union of Dynkin diagrams of type ADE.

1. We will write $\overrightarrow{\epsilon_i}$ to denote the the dimension vectors of the labelled representations S_i.
2. Recall that the reflections s_i were defined also on dimension vectors, so the s_i are linear transformations of $\mathbf{Z}^V$ (or of $\mathbf{R}^V$).
3. Let $W(Q)$ denote the subgroup of $\mathrm{End}_{\mathbf{Z}}(\mathbf{Z}^v)$ generated by these elements; we will refer to this group as the **Weyl group** of Q.
4. If $V = \{i_1, \dots, i_n\}$, then the element $c = s_{i_n} \cdots s_{i_1}$ is called a **Coxeter transformation** of Q.
5. We also say that a dimension vector α is a **root** if it lies on the orbit of $\overrightarrow{\epsilon_i}$, for some i.
6. Finally, we say that a root is positive (resp. negative) and write $\alpha \geq 0$ (resp. $\alpha \leq 0$), if $\alpha(i) \geq 0$ (resp. $\alpha(i) \leq 0$) for every vertex i.

Preliminary results

With the above notation, we have:

Proposition 3.3

Suppose that Q is a quiver whose underlying graph is a union of Dynkin diagrams of type ADE. The following statements hold:

1. *The Tits form B_Q is positive definite.*
2. *If α is a root, then $B_Q(\alpha) = 1$.*
3. *There are only finitely many roots.*
4. *$W(R)$ is finite.*
5. *For each root α, either $\alpha \geq 0$ or $\alpha \leq 0$.*
6. *If $v = \{1, \dots, n\}$ with $1, \dots, n$ an admissible sequence of sinks and if X is a quiver representation without projective summands, then*

$$\overrightarrow{\dim} C^{+}(X) = c(\overrightarrow{\dim} X).$$

Gabriel's theorem

Lemma

Let Q be a quiver whose underlying graph is the union of Dynkin diagrams of type ADE. Then, for every nonzero $v \in \mathbf{R}^n$ there exists $l \in \mathbf{N}$ such that $c^l(v)$ has a negative coordinate.

Gabriel's theorem

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Lemma

Suppose that the underlying graph of Q is a union of Dynkin diagrams of type ADE and that α is a dimension vector on Q such that $\alpha \geq 0$ and $s_i(\alpha)$ has a negative component. Then $\alpha = \overrightarrow{\epsilon_i}$.

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Theorem 3.4 (Gabriel's theorem)

A quiver Q is of finite representation type if and only if ΓQ is a union of Dynkin diagrams of type ADE. In addition, if Q is of finite representation type, then the assignment $X \mapsto \overrightarrow{\dim} X$ establishes a bijection between the isomorphism classes of indecomposable representations and the set of positive roots.

Proof of Gabriel's theorem

- Start assuming that the underlying graph of Q is a union of Dynkin diagrams of type ADE and let $c = s_n \cdots s_1$ be the corresponding Coxeter transformation. Let X be an indecomposable representation of Q.

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- There exists $l \in \mathbf{N}$ such that $c^l(\overrightarrow{\dim} X)$ has a negative coordinate. Choose l to be minimal and let k be the first natural such that

$$s_k \cdots s_1 \left(c^{l-1} \left(\overrightarrow{\dim} X \right) \right)$$

has a negative coordinate, $1 \leq k \leq l$.

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has a negative coordinate, $1 \leq k \leq l$.
- We have

$$s_{k-1} \cdots s_1 \left(c^{l-1} \left(\overrightarrow{\dim} X \right) \right) \geq 0$$

and

$$s_k \cdots s_1 \left(c^{l-1} \left(\overrightarrow{\dim} X \right) \right)$$

has a negative component. Then

$$s_{k-1} \cdots s_1 \left(c^{l-1} \left(\overrightarrow{\dim} X \right) \right) = \overrightarrow{\epsilon_k}.$$

Proof of Gabriel's theorem (cont.)

- Similarly

$$C_k^+ \cdots C_1^+((C^+)^{l-1}(X)) = 0$$

while

$$C_{k-1}^+ \cdots C_1^+((C^+)^{l-1}(X)) \neq 0.$$

By Theorem 3.2 we conclude that

$$C_{k-1}^+ \cdots C_1^+((C^+)^{l-1}(X)) \cong S_k.$$

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By Theorem 3.2 we conclude that
$$C_{k-1}^+ \cdots C_1^+((C^+)^{l-1}(X)) \cong S_k.$$
- Because of the properties the functors $C_i^{\pm}$ satisfy, it follows that
$$X \cong (C^-)^{l-1} C_1^- \cdots C_{m-1}^-(S_k).$$
and also
$$\overrightarrow{\dim} X = c^{l-1} s_1 \cdots s_{k-1}(\overrightarrow{\epsilon_m}).$$

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