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Frank-Olme Speck

Técnico, Universidade de Lisboa

Operator Factorization and Boundary Value Problems

Contents

1. Operators associated with BVPs

This paper aims at showing the ease of operator factorization (OF) in the treatise of a boundary value problem (BVP). To be specific we confine ourselves to some standard situation, namely to

- $T \in \mathcal{L}(X, Y)$ a bounded linear operator in Banach spaces (given)
- $T = ESF$ factorization into bounded linear operator in Banach spaces with certain properties (to be determined)

where T is "closely related" to an elliptic linear BVP in the sense of, e.g., Eskin 73/81, Wloka 82, Hsiao-Wendland 08, written in the form

$$\begin{array}{lllll} Au & = & f & \texttt{in} & \Omega \qquad \texttt{(PDE in nice domain)} \\ Bu & = & g & \texttt{on} & \Gamma = \partial\Omega \qquad \texttt{(boundary condition)} \end{array}$$

Associated operator Boutet de Monvel 66, ... Wloka 82, ...

$$L = \begin{pmatrix} A \\ B \end{pmatrix} : \mathcal{X} \to Y = Y_1 \times Y_2 \qquad \texttt{(data space)}$$

$$\mathcal{X} = \texttt{(solution space)}$$

Well-posed linear BVPs

Well-posed problem (Hadamard 1902 : *for* ***some*** *given* f, g)

- is solvable in $\mathcal{X}$ *for* ***all*** $f, g \in Y = Y_1 \times Y_2$
- solution is unique,
- solution depends continuously on data.

This formulation in linear BVPs needs (at least) the assumption that $\mathcal{X}$ and Y are topological vector spaces. In this case the BVP is well-posed iff the associated operator is a linear homeomorphism (invertible and bi-continuous) as a mapping

$$L = \begin{pmatrix} A \\ B \end{pmatrix} : \mathcal{X} \to Y = Y_1 \times Y_2$$

What may be the ease of OF? Some examples

- Discovery and proof of properties - such as well-posedness etc
- explicit solution of "canonical" BVPs - e.g. by WH factorization
- reduction to "simpler" problems - BIEs, semi-homogeneous BVPs
- better understanding of "equivalence" and "reduction"
- regularity of solutions - considering operators acting on scales of Banach spaces
- minimal normalization of ill-posed problems - by "natural" change of topologies in the given spaces
- singularities and asymptotic results - via "factor properties" and "intermediate spaces"

2. A first glance at OF: Full rank factorization

Theorem If $T = LR$ is a *full rank factorization* of a matrix

$$\begin{pmatrix} t_{11} & . & . \\ . & . & . \\ . & . & . \\ . & . & t_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} & . \\ . & . \\ . & . \\ . & a_{mr} \end{pmatrix} \begin{pmatrix} b_{11} & . & . \\ . & . & b_{rn} \end{pmatrix}$$

or a *full range factorization* of a bdd. lin. op. in Banach spaces

$$L^{-}L = I_X \quad , \quad RR^{-} = I_Y$$

then the *reverse order law* holds:

$$T^{-} = R^{-}L^{-}$$

is a generalized inverse (GI) of T, i.e.,

$$TT^{-}T = T.$$

References:
Nashed 76, Nashed-Rall 76, Ben-Israel and Greville 1974/2003.

A generalized reverse order law

Remarks All this holds for bounded linear operators in Banach spaces, as well. But in general the reverse order law fails

- if L and R are only generalized invertible, or
- if L and R are exchanged (i.e., $T = RL$ with above properties). We shall give an example later where $\operatorname{ind} R = 1$, $\operatorname{ind} L = -1$, but it helps to construct T^{-1} by splitting operators of rank 1.

Theorem If $T = L\,C\,R$ where L, R are left resp. right invertible and C is generalized invertible, then the *reverse order law* holds:

$$T^{-} = R^{-}C^{-}L^{-} \qquad \texttt{is a GI of} \quad T.$$

Examples Wiener-Hopf factorization : classical and abstract, Toeplitz operators, Riemann problems, ... In many situations the case appears where L, R are two-sided invertible.

Generalized inverses (1-inverses)

- originated from matrix theory (similar ideas by Fredholm 1903) Moore $\sim$ 1906, 1920, Penrose 1950s, Nashed, Nashed-Rall 1976, Ben-Israel and Greville 1974, 2003 (2nd ed.)
- many results are valid for linear operators or ring elements Nashed 1976, Nashed-Votruba 1976

Theorem Let $T \in \mathcal{L}(X, Y)$ be a bdd. lin. op. in Banach spaces. The following assertions are equivalent:
(i) $T\,T^-\,T = T$ for some $T^- \in \mathcal{L}(Y, X)$;
(ii) $\ker T$ and $\operatorname{im} T$ are complemented (alg. and top.) ;
(iii) There is an operator $T^- \in \mathcal{L}(Y, X)$ such that

$$Tf = g \texttt{ is solvable iff } TT^-g = g$$

and then, the general solution is given by

$$f = T^-g + (I - T^-T)h \ , \ h \in Y.$$

Regularity classes of bounded linear operators in Banach spaces (with closed image)

	$\alpha(T) = 0$	$\alpha(T) < \infty$	$\ker T$ complem.	$\ker T$ closed
$\beta(T) = 0$	bdd. invertible	right inv. Fredholm	right invertible	surjective
$\beta(T) < \infty$	left inv. Fredholm	Fredholm	right regulariz.	semi-Fred. $\mathcal{F}_-$
$\operatorname{im} T$ complem.	left invertible	left regulariz.	generalized invertible	no name
$\operatorname{im} T$ closed	injective	semi-Fred. $\mathcal{F}_+$	no name	normally solvable

Fredholm vs. generalized invertible operators

Theorem If $T \in \mathcal{L}(X,Y)$ is a bounded linear operator in Banach spaces, the following assertions are equivalent (see Mikhlin-Prössdorf 80/86, e.g.):

(a) $T \in \mathcal{F}(X,Y)$ is Fredholm

(b) $\exists_{R_1,R_2 \in \mathcal{L}(Y,X)}\ R_1 T = I_X + V_1\,,\ T R_2 = I_Y + V_2\,,\ V_j$ compact

(c) $\exists_{R_1,R_2 \in \mathcal{L}(Y,X)}$ ... such that V_j have finite rank

(d) $\exists_{R_1,R_2 \in \mathcal{L}(Y,X)}$ ... such that V_j are finite rank projectors

(e) $\exists_{T^- \in \mathcal{L}(Y,X)}\ T\,T^-\,T = T$ i.e. T is generalized invertible

and $T^-T - I_X\,,\ TT^- - I_Y$ are finite rank projectors (onto the kernel and along the image of T, respectively)

Remark The construction of T^- yields an explicit solution.

3. Operator factorization in BVP - how it appears

We shall consider the following examples:

- Potential methods, idea of BIEs (what is "equivalent reduction")
- Reduction to semi-homogeneous systems
  (leading to equivalence after extension relations)
- Cross factorization of invertible operators in Banach spaces
  (abstract and classical WHOs in Sommerfeld problems)
- Normalization (of WHOs as a prototype)
- Regularity of solutions (in scales of Sobolev spaces)
- Asymptotic behavior of solutions (from factor properties)
- Asymmetric factorization of scalar and matrix functions
  (for WH±HOs in diffraction from rectangular wedges)
- Structured matrix operators in wedge diffraction problems

Formulation of BVPs - a closer look at syntax

We look for the (general*) solution (in a certain form**) of

$$\begin{aligned} Au &= f \quad \texttt{in} \quad \Omega \\ Bu &= g \quad \texttt{on} \quad \Gamma = \partial\Omega \end{aligned}$$

where the following are given: Ω is a Lipschitz domain (e.g.) in $\mathbb{R}^n$. $A \in \mathcal{L}(\mathcal{X}, Y_1), B \in \mathcal{L}(\mathcal{X}, Y_2)$ are bounded linear operators in Banach spaces of function(al)s living on Ω or Γ. (f, g) are arbitrarily given in the data space $Y = Y_1 \times Y_2$ (the topological product).

* all solutions for any data given in certain given sets

** explicit, closed analytic, series expansion, numerical (plenty choices), with error estimate, ..., or just leave it open !?

OT formulation: Find (in a certain form**) a generalized inverse of the associated operator $L = \begin{pmatrix} A \\ B \end{pmatrix} : \mathcal{X} \to Y = Y_1 \times Y_2$.

Sceneries of elliptic BVPs 1

Wloka 1982/87 "Semi-classical formulation" (scalar PDE)

$$\Omega \subset \mathbb{R}^n \texttt{ bounded with } (2m+k,\kappa)-\texttt{smooth boundary}$$

$$m \in \mathbb{N}\,,\; k+\kappa \geq 1$$

$$\mathcal{X} = W_2^{2m+l}(\Omega)\,,\; Y_1 = W_2^l(\Omega)\,,\; Y_2 = \prod_{j=1}^{m} W_2^{2m+l-m_j-1/2}(\partial\Omega)$$

$$A = \sum_{|s|\leq 2m} a_s(x) D^s \texttt{ uniformly elliptic}$$

$$\texttt{with } 2m-\texttt{smooth coefficients}$$

$$B_j = \sum_{|s|\leq m_j} b_{j,s}(x) T_0 D^s \texttt{ with Lopatinskii-Shapiro condition}$$

$$\operatorname{ord} B_j \leq 2m-1 \texttt{ and } 2m-\texttt{smooth coefficients}$$

Main Theorem about equivalence of (a) BVP is elliptic, (b) L is smoothable, (c) L is Fredholm, (d) an apriori estimate holds.

Sceneries of elliptic BVPs 2

Hsiao and Wendland 2008 Variational (weak) formulation for

$$\Omega \in \mathbb{R}^n \qquad \text{strong Lipschitz domain}$$

$$Au = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}(a_{jk}(x)\frac{\partial u}{\partial x_k}) + \sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x_j} + c(x)u = f \quad \text{in}\Omega$$

with elliptic symbol and $f \in \tilde{H}_0^{-1}(\Omega) = \tilde{H}^{-1}(\Omega) \ominus \tilde{H}_\Gamma^{-1}(\Omega)$.
Sesquilinear form:

$$a_\Omega(u,v) = \int_\Omega \{ \sum_{j,k=1}^{n} (a_{jk}(x)\frac{\partial u}{\partial x_k})^\top \frac{\partial \bar{v}}{\partial x_j} + \sum_{j=1}^{n} (b_j(x)\frac{\partial u}{\partial x_j})^\top \bar{v} + (c(x)u)^\top \bar{v}\} d$$

Weak solution of the Dirichlet problem (e.g.):

$$\begin{aligned} a_\Omega(u,v) &= < f, \bar{v} >_\Omega \qquad \text{for all} \quad v \in H_0^1(\Omega) \\ T_{0,\Gamma} u &= g \in H^{1/2}(\Gamma). \end{aligned}$$

Further sceneries of elliptic BVPs (working with $L = (A, B)^\top$)

Eskin 1973/81 BVPs for elliptic pseudodifferential equations
Agmon, Agranovich, Boutet de Monvel, Lions, Shamir, Shubin ...

Explicit solution of *canonical problems* in diffraction theory by operator factorization methods

Meister and Speck 1985-1991 Sommerfeld diffraction problems

Meister, Penzel, Speck, Teixeira 1992-94
Diffraction from rectangular wedges

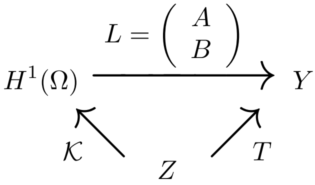
Castro, Duduchava, Speck, Teixeira 2003-05
Unions of finite intervals, quadrants

Ehrhardt, Nolasco, Speck 2010-12 Non-rectangular wedges, rational angles, conical Riemann surfaces

$$\begin{aligned} (\Delta + k^2)u &= 0 \qquad \textit{in } \Omega \\ T_0(\alpha\, u + \beta\, \partial u/\partial x + \gamma\, \partial u/\partial y) &= g \qquad \textit{on } \Gamma = \partial\Omega \end{aligned}$$

Potential methods

The classical idea of potential theory (reduction to a simpler problem/operator) by a suitable potential ansatz



Operator composition $T = L\mathcal{K}$ leads to the study of

- Boundary Integral Equations (BIE) Hsiao and Wendland 08
- Operator Factorization $T = L\mathcal{K}$, $L = T\mathcal{K}^{-1}$ (simplest case),
- Operator Relations (for classes of ops.) in general Castro 98.

Some questions

- In what sense is a BVP "equivalent" or "reduced" to the BIE?

  $\longrightarrow$ Properties of substitution operators $\mathcal{K}$ such as invertible (only algebraically), boundedly invertible (algebraic and topologically), only Fredholm etc - all this can happen, as we shall see.

- What about the reduction of a BVP to a semi-homogeneous problem?

  $\longrightarrow$ Equivalent after extension relations.

- Which kind of operator relations appear in practise?

  $\longrightarrow$ Operator matrix identities, classification of operator relations (application oriented) and their properties.

Operator factorizations/relations that appear frequently

Equivalent operators $S \sim T \quad \Leftrightarrow \quad S = E\,T\,F$
where E, F are boundedly invertible operators in Banach spaces.

Equivalence after extension BGK 80 (→ *minimal factorization*)

$$S \overset{*}{\sim} T \quad \Leftrightarrow \quad \begin{pmatrix} S & 0 \\ 0 & I_{Z_1} \end{pmatrix} = E \begin{pmatrix} T & 0 \\ 0 & I_{Z_2} \end{pmatrix} F$$

Δ - *related operators* (appeared with WHHOs) Castro 98

$$S\ \Delta\ T \quad \Leftrightarrow \quad \begin{pmatrix} S & 0 \\ 0 & S_\Delta \end{pmatrix} = E\,T\,F$$

If E or F are only linear bijections (not necessarily bi-continuous), then S and T are called *algebraically equivalent*, etc, writing

$$S \underset{alg}{\sim} T \quad , \quad S \underset{alg}{\overset{*}{\sim}} T \quad , \quad S \underset{alg}{\Delta} T$$

4. Reduction to semihomogeneous systems

Consider the semihomogeneous (abstract) BVP (in the above syntax)

$$L^0 u = \begin{pmatrix} A \\ B \end{pmatrix} u = \begin{pmatrix} 0 \\ g \end{pmatrix} \in \{0\} \times Y_2 \cong Y_2$$

with associated operator

$$B\,|_{\ker A} : \mathcal{X}_0 = \ker A \longrightarrow Y_2$$

How is this operator related to the full thing

$$L = \begin{pmatrix} A \\ B \end{pmatrix} : \mathcal{X} \longrightarrow Y = Y_1 \times Y_2 \qquad ?$$

In general, they will not be equivalent operators, since Y and Y_2 may not be isomorphic.

But, if A is surjective and $\ker A$ is complemented, i.e., $A : \mathcal{X} \longrightarrow Y_1$ is right invertible, then we have the following relation:

Reduction to semi-homogeneous systems - considered as an operator relation

Lemma Let $L = \begin{pmatrix} A \\ B \end{pmatrix} \in \mathcal{L}(\mathcal{X}, Y_1 \times Y_2)$ be a bounded linear operator in Banach spaces. Further let R be a right inverse of A, i.e.,

$$R \in \mathcal{L}(Y_1, \mathcal{X}) \quad , \quad AR = I \quad .$$

Then the following OF holds

$$L = E\,T\,F = \begin{pmatrix} 0 & A|_{X_1} \\ I|_{Y_2} & B|_{X_1} \end{pmatrix} \begin{pmatrix} B|_{X_0} & 0 \\ 0 & I|_{X_1} \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}$$

where $P = RA$, $Q = I - RA$ are continuous projectors in $\mathcal{X}$, $X_0 = \ker A = \ker P = \operatorname{im} Q$, $X_1 = \operatorname{im} P = \ker Q$. The first and third factor are (boundedly) invertible as

$$\begin{aligned} E &= Y_2 \times X_1 \longrightarrow Y_1 \times Y_2 \\ F &= \mathcal{X} \longrightarrow X_0 \times X_1 \,. \end{aligned}$$

Reduction to semi-homogeneous systems - an example for equivalence after extension

The previous Lemma is proved by verification. A and B are exchangeable. So we arrive at the following result (which seems to be not completely known, cf. HW08):

Theorem Let $L = \begin{pmatrix} A \\ B \end{pmatrix} \in \mathcal{L}(\mathcal{X}, Y_1 \oplus Y_2)$ be a bounded linear operator in Banach spaces. Then

$$\begin{aligned} \exists_{R\in\mathcal{L}(Y_1,\mathcal{X})}\, AR = I \qquad &\Rightarrow \qquad L \overset{*}{\sim} B|_{\ker A}, \\ \exists_{R\in\mathcal{L}(Y_2,\mathcal{X})}\, BR = I \qquad &\Rightarrow \qquad L \overset{*}{\sim} A|_{\ker B}. \end{aligned}$$

There are several interpretations and conclusions, we mention a few.

Well-posedness and reduction to semi-homogeneous systems

Corollary The BVP is well-posed (i.e., L is boundedly invertible) if and only if

1. the two semi-homogeneous problems are well-posed,
2. the solution splits uniquely as $u = u_0 + u^0$ where
$$\begin{aligned} L_0 u_0 &= \begin{pmatrix} f \\ 0 \end{pmatrix} \\ L^0 u^0 &= \begin{pmatrix} 0 \\ g \end{pmatrix}, \end{aligned}$$
3. A and B admit right inverses.

Each of the three conditions for its own is not sufficient for the BVP to be well-posed, but the first two or the last two conditions suffice.

What happens if A or B is not right invertible?

If A is not right invertible, then
(i) A is not surjective, the BVP is not solvable for all data $f \in Y_1$, i.e., Y_1 is chosen too large for a well-posed problem; or
(ii) A is surjective but $\ker A$ not complemented, in which case it may help to change the topology of Y_1 or of $\mathcal{X}$.

Remarks

1. The right inverses R of A or B in applications are often a volume or surface potential (see HW08) or an extension operator, left invertible to a trace operator, see Wloka 82 ...
2. Each of the formulation has advantages in certain situations, see for instance Mikhailov: Boundary-domain integro-differential equations.

5. Equivalence after extension and matrical coupling
- a closer look at equivalence after extension relations

We identified the relation between L and $B|_{\ker A}$ and $A|_{\ker B}$ as $\overset{*}{\sim}$ provided A and B, respectively, are right invertible.

$$S \overset{*}{\sim} T \quad \Leftrightarrow \quad \begin{pmatrix} S & 0 \\ 0 & I_{Z_1} \end{pmatrix} = E \begin{pmatrix} T & 0 \\ 0 & I_{Z_2} \end{pmatrix} F$$

Remark This kind of operator relation (which belongs to the class of so-called *operator matrix identities*) is very important in theory and applications. It appears frequently in "*substitution, factorization, extension and reduction methods*".

We study some of their properties. Of course, it is an equivalence relation (in the genuine sense), i.e., reflexive, symmetric and transitive. However there is much more.

The Theorem of Bart and Tsekanovskii

Theorem BT - part 1 Let $T \in \mathcal{L}(X_1, X_2)$ and $S \in \mathcal{L}(Y_1, Y_2)$ be bounded linear operators in Banach spaces and assume $T \overset{*}{\sim} S$.

Then $\ker T \cong \ker S$. Also $\operatorname{im} T$ is closed if and only if $\operatorname{im} S$ is closed, and in that case $X_2/\operatorname{im} T \cong Y_2/\operatorname{im} S$.

Transfer properties of an EAE relation

Transfer property 1 If $S \overset{*}{\sim} T$, the two operators belong to the same regularity class (of the 16 classes mentioned before on p. 9).

Transfer property 2 If the inverses of the operators E, F in the relation $S \overset{*}{\sim} T$ are known, a generalized inverse of T can be computed from a generalized inverse of S, namely by a "reverse order law"

$$\begin{pmatrix} S & 0 \\ 0 & I_{Z_1} \end{pmatrix} = E \begin{pmatrix} T & 0 \\ 0 & I_{Z_2} \end{pmatrix} F \Rightarrow T^- = R_{11} F^{-1} \begin{pmatrix} S^- & 0 \\ 0 & I_{Z_1} \end{pmatrix} E^{-1}$$

where R_{11} denotes the restriction to the first block of the operator matrix.

More results can be found in the PhD thesis of Castro 1998.

The Theorem of Bart and Tsekanovskii:
- part 2, inverse conclusion

Theorem BT - part 2 Let $T \in \mathcal{L}(X_1, X_2)$ and $S \in \mathcal{L}(Y_1, Y_2)$ be bounded linear operators in Banach spaces and assume that T and S are generalized invertible*.

Then $T \overset{*}{\sim} S$ if and only if $\ker T \cong \ker S$ and $X_2/\operatorname{im} T \cong Y_2/\operatorname{im} S$.

*This assumption is essential (there is an example where sufficiency fails otherwise, see BT 1992).

Matricial coupling Bart, Gohberg, Kaashoek 1984

Theorem BGK 84 Let $S \in \mathcal{L}(X_1, Y_1)$, $T \in \mathcal{L}(X_2, Y_2)$. Then $S \overset{*}{\sim} T$ **if** the two operators are *matricially coupled*, i.e.,

$$\begin{pmatrix} S & * \\ * & * \end{pmatrix} = \begin{pmatrix} * & * \\ * & T \end{pmatrix}^{-1}.$$

In symmetric setting, this is just an interpretation of the formula $PAP + Q \sim P + QA^{-1}Q$, see BGK 85, which is well known.

Theorem Bart-Tsekanovsky 1992
$S \overset{*}{\sim} T$ **iff** S and T are matricially coupled.

Remark As we know already: This implies that

$$\ker S \cong \ker T \quad , \quad \operatorname{coker} S \cong \operatorname{coker} T$$

and the inverse conclusion holds if both operators S and T are generalized invertible.

Examples of BVPs which are matricially coupled
more precisely: whose associated operators are matricially coupled

Diffraction of time-harmonic waves from plane screens in $\mathbb{R}^3$.

Given a 2D special Lipschitz domain Σ (see Stein 70) and $g \in H^{1/2}(\Sigma)$ (the trace of a primary field) we look for the weak solution of the Dirichlet problem

$$\begin{aligned} \Delta + k^2 u &= 0 \quad \texttt{in} \quad \Omega = \mathbb{R}^3 \setminus \Gamma \quad \texttt{where} \quad \Gamma = \overline{\Sigma} \times \{0\} \\ T_0\ u &= g \quad \texttt{on} \quad \Gamma = \partial\Omega \ \texttt{(both banks)} \end{aligned}$$

The Neumann problem for the complementary screen $\Sigma_* = \mathbb{R}^2 \setminus \overline{\Sigma}$ is briefly written as

$$\begin{aligned} \Delta + k^2 u &= 0 \quad \texttt{in} \quad \Omega_* = \mathbb{R}^3 \setminus \Gamma_* \quad \texttt{where} \quad \Gamma_* = \overline{\Sigma_*} \times \{0\} \\ T_1\, u = \partial u / \partial x_3|_{x_3=0} &= h \quad \texttt{on} \quad \overline{\Sigma_*} = \partial\Omega_* \ \texttt{(both banks)} \end{aligned}$$

where $h \in H^{-1/2}(\Sigma_*)$ is given.

Examples of BVPs which are matricially coupled ctd

Theorem The operators associated to the last two BVPs are matricially coupled and (therefore) equivalent after extension:

$$L_{D,\Omega} \quad \overset{*}{\sim} \quad L_{N,\Omega_*}.$$

Proof Putting $\gamma(\xi_1, \xi_2) = \sqrt{\xi_1^2 + \xi_2^2 + k^2}$, we can show (see MS88)

$$L_{D,\Omega} \overset{*}{\sim} W_{\gamma^{-1},\Sigma} = r_\Sigma A_{\gamma^{-1}} = \mathcal{F}^{-1}\gamma^{-1} \cdot \mathcal{F} : H_\Sigma^{-1/2} \to H^{1/2}(\Sigma).$$

By analogy, the Neumann problem for the complementary screen Σ_* yields an associated operator which satisfies

$$L_{N,\Omega_*} \overset{*}{\sim} W_{\gamma,\Sigma_*} = r_{\Sigma_*} A_\gamma = \mathcal{F}^{-1}\gamma \cdot \mathcal{F} : H_{\Sigma_*}^{1/2} \to H^{-1/2}(\Sigma_*).$$

Composing $W_{\gamma^{-1},\Sigma}$ with a continuous extension operator $\ell_2 : H^{1/2}(\Sigma) \to H^{1/2}(\mathbb{R}^2)$ we obtain that $P_2 = \ell_2 r_\Sigma$ projects along $H^{1/2}(\Sigma_*)$ and

$$W_{\gamma^{-1},\Sigma} \overset{*}{\sim} \tilde{W}_{\gamma^{-1},\Sigma} = \ell_2 W_{\gamma^{-1},\Sigma} = P_2 A_{\gamma^{-1}}|_{P_1 H^{-1/2}} : H_\Sigma^{-1/2} \to P_2 H^{1/2}.$$

Similarly

$$W_{\gamma,\Sigma_*} \overset{*}{\sim} \tilde{W}_{\gamma,\Sigma_*} = \ell_1 W_{\gamma,\Sigma*} = Q_1 A_\gamma|_{Q_2 H^{-1/2}} : H^{1/2}_{\Sigma_*} \to Q_1 H^{-1/2}.$$

Now we have

- P_1 is a projector in $H^{-1/2}(\mathbb{R}^2)$ onto $H^{-1/2}_{\Sigma}$
- P_2 is a projector in $H^{1/2}(\mathbb{R}^2)$ along $H^{1/2}_{\Sigma_*}$
- $Q_1 = I - P_1$ is a projector in $H^{-1/2}(\mathbb{R}^2)$ along $H^{-1/2}_{\Sigma}$
- $Q_2 = I - P_2$ is a projector in $H^{1/2}(\mathbb{R}^2)$ onto $H^{1/2}_{\Sigma_*}$.

$$\begin{pmatrix} P_2 A_{\gamma^{-1}} P_1 & P_2 A_{\gamma^{-1}} Q_1 \\ Q_2 A_{\gamma^{-1}} P_1 & Q_2 A_{\gamma^{-1}} Q_1 \end{pmatrix} = \begin{pmatrix} P_1 A_\gamma P_2 & P_1 A_\gamma Q_2 \\ Q_1 A_\gamma P_2 & Q_1 A_\gamma Q_2 \end{pmatrix}^{-1}$$

because of $A_{\gamma^{-1}} = A_\gamma^{-1}$. Hence $\tilde{W}_{\gamma^{-1},\Sigma}$ and $\tilde{W}_{\gamma,\Sigma_*}$ are matricially coupled , thus equivalent after extension to each other and to $L_{D,\Omega}$ and L_{N,Ω_*}, as well, by transitivity.

OF methods for singular operators
related to BVPs such as boundary integral operators

- Rich history: Muskhelishvili, Gakhov, Vekua, Mikhlin, Gohberg-Krein, Simonenko, Prössdorf, Coburn, Douglas, Devinatz-Shinbrot, Spitkovsky, ...
- Many operator classes: Wiener-Hopf, Toeplitz, Riemann problems, abstract settings, ...
- Plenty of constructive methods: Scalar and matrix functions, rational functions, decomposing algebras, generalized factorization, ... LU, polar decomposition, ...

6. General Wiener-Hopf operators (WHOs)

Here we consider some algebraic methods, which become constructive when combined with explicit factorization of matrix functions. The following type of operators are so-called *general WHOs* (Shinbrot 1964), *abstract Wiener-Hopf operators* (Cebotarev 1967), or *projections of operators* (Gohberg-Krupnik 1973/79)

$$W \quad = \quad T_P(A) \quad = \quad PA|_{PX} \quad : \quad PX \to PX\,(= \operatorname{im} P)$$

where X is a Banach (or even Hilbert) space, $P = P^2 \in \mathcal{L}(X)$ a bounded projector, $A \in \mathcal{GL}(X)$ an invertible linear operator.

Variants which partly admit the same results:

$$W \quad = \quad P_2A|_{P_1X} \quad : \quad P_1X \to P_2Y \quad \texttt{asymmetric WHO}$$

where X, Y are a Banach spaces, P_1, P_2 bounded projector in X and Y, respectively and $A \in \mathcal{L}(X, Y)$ an invertible linear operator; further

$$w \quad = \quad pap$$

where $a, p \in \mathcal{R}$, which is a unital algebra, a invertible and $p^2 = p$.

The cross factorization theorem

Theorem S 83 Let $W = T_P(A) = PA|_{PX}$ be a *general WHO* (i.e.) where X is a Banach space, $A \in \mathcal{GL}(X)$, $P^2 = P \in \mathcal{L}(X)$. Then W is generalized invertible if and only if

$$A = A_- \, C \, A_+$$

where $A_\pm \in \mathcal{GL}(X), A_+PX = PX, A_-QX = QX \quad (Q = I - P)$ and C splits the space X twice into four subspaces such that

$$\begin{array}{ccccc} X & = & \overbrace{X_1 \dot{+} X_0}^{PX} & \dot{+} & \overbrace{X_2 \dot{+} X_3}^{QX} \\ & = & \underbrace{Y_1 \dot{+} Y_2}_{PX} & \dot{+} & \underbrace{Y_0 \dot{+} Y_3}_{QX} \end{array}$$

where C maps each X_j onto Y_j, j = 0, 1, 2, 3 , i.e., $X_0 = C^{-1}QCPX$, $X_1 = C^{-1}PCPX$ etc. In this case,

$$W^- = PA_+^{-1}PC^{-1}PA_-^{-1}|_{PX} \quad \texttt{is a GI of} \quad W.$$

The cross factorization theorem - sketch of proof

The sufficient part is quickly done by verification in a few lines. The necessary part works with space decomposition and reduction to a one-sided invertible, restricted operator, it needs one page and a half. Both are done for the asymmetric version, see S 83.

The sufficient part for the algebra setting is as simple as before. Necessity of the factorization consists in guessing a cross factorization from a generalized inverse, i.e. if $wvw = w$, then

$$\begin{aligned} a &= a_- \; c \; a_+ \\ &= [e + qav]\,[a - ava + w + a(p - vw)a^{-1}(p - wv)a] \times \\ &\quad \times [e + vaq - (p - vw)a^{-1}(p - wv)a] \end{aligned}$$

represents a cross factorization of a.

However verification needs almost two pages, see S 85.

The cross factorization theorem - historic remarks

Shinbrot 1964 : Definition and basic properties of general WHOs.

Cebotarev 1967 : One-sided invertibility in the algebraic setting.

Devinatz and Shinbrot 1969 : Invertibility of (symmetric) general WHOs in separable Hilbert spaces.

Gohberg and Krupnik 1973 : Properties of a projection of a bounded linear operator.

Speck 1983 : Symmetric version of the cross factorization theorem.

Speck 1985 : Asymmetric and algebraic versions.

A F dos Santos 1988 : A geometric perspective ...

Associated WHOs $QA^{-1}|_{QX}$

were discussed for Hilbert space operators by Devinatz-Shinbrot 1969

The following observation was made by Speck 1984 during a conference in Oberwolfach, Germany, after the talk of I. Gohberg.

Remark *Associated general WHOs* are equivalent after extension:

$$PA|_{PX} \quad \overset{*}{\sim} \quad QA^{-1}|_{QX}$$

where X is a Banach space, $A \in \mathcal{GL}(X)$, $P^2 = P = I - Q \in \mathcal{L}(X)$. This well-known relation has a certain similarity with matricial coupling (in symmetric space settings $X_j = Y_j$).

Proof

$$\begin{aligned} PAP + Q &= (I - PAQ)(PA + Q) = (I - PAQ)(P + QA^{-1})A \\ &= (I - PAQ)(I + QA^{-1}P)(P + QA^{-1}Q)A. \end{aligned}$$

The remark was published and commented by BGK 1985 in an addendum in IEOT 8 (1985) 890-891.

Variants of the Cross Factorization Theorem

Theorem (asymmetric version of cross factorization) S 85
Let $W = P_2 A|_{P_1 X}$ be an *asymmetric WHO* (i.e.) where X, Y are Banach spaces, $A \in \mathcal{L}(X,Y)$ invertible , $P_1^2 = P_1 \in \mathcal{L}(X)$, $P_2^2 = P_2 \in \mathcal{L}(Y)$.
Then W is generalized invertible if and only if there exists a Banach (intermediate) space Z and a projector $P \in \mathcal{L}(X)$ such that A splits into invertible operators

$$\begin{array}{cccccc} A & = & A_- & C & A_+ \\ & & Y \leftarrow Z & \leftarrow Z & \leftarrow X \end{array}$$

where A_+ maps $P_1 X$ onto PZ , A_- maps QZ onto $Q_2 Y$, and C is a cross factor (in modified obvious setting). In this case,

$$W^- = P_1 A_+^{-1} P C^{-1} P A_-^{-1}|_{P_2 Y} \quad \texttt{is a GI of} \quad W.$$

Remark This includes operators acting between Sobolev spaces of different order and Simonenkos concept of generalized factorization.

7. WHOs in Sobolev spaces - originated from diffraction theory

$$W = r_+ A : L^2_+ \to L^2(\mathbb{R}_+)$$

$$A = \mathcal{F}^{-1}\Phi_A \cdot \mathcal{F} , \; \Phi_A \in C^\mu(\ddot{\mathbb{R}}) \cap \mathcal{G}L^\infty(\mathbb{R})$$

$$W_s = \begin{cases} \operatorname{Rst} W : H^s_+ \to H^s(\mathbb{R}_+) , & s > 0 \\ \operatorname{Ext} W : H^s_+ \to H^s(\mathbb{R}_+) , & s < 0 \end{cases}$$

Then it is well known that (see Eskin 1973, Duduchava 1979, ...)

- $W \sim \ell_0 r_+ A : L^2_+ \to L^2_+$ which has the form of a general WHO,
- a cross factorization $A = A_- C A_+$ exists explicitly provided $\Phi_A(+\infty)/\Phi_A(-\infty) \notin \mathbb{R}_-$, i.e., Φ_A is 2-regular,
- a generalized inverse $W_s^- = ...$ is explicitly obtained provided the lifted symbol $(\xi - i)^s \, \Phi_A(\xi) \, (\xi + i)^{-s}$, is 2-regular.

Explicit formulas for W^-

Instead of the Fourier symbol Φ consider the index-free function (see Duduchava 1979)

$$\Phi_0 = \zeta^{-\omega}\, \Phi^{-1}(+\infty)\, \Phi$$

where

$$\omega = \frac{1}{2\pi i}\int_{\mathbb{R}} d\log\Phi \quad , \quad \Re e\,\omega + \frac{1}{2} \notin \mathbb{Z} \quad , \quad \zeta(\xi) = \frac{\lambda_-(\xi)}{\lambda_+(\xi)} = \frac{\xi - i}{\xi + i}.$$

Φ_0 admits a *canonical Wiener-Hopf factorization* in $C^\mu(\dot{\mathbb{R}})$

$$\Phi_0 = \Phi_{0-}\, \Phi_{0+} \quad , \quad \Phi_{0\pm} = \exp\{P_\pm \log\Phi_0\}$$

where $P_\pm$ denote the Hilbert projections $P_\pm = (I \pm S_{\mathbb{R}})/2$. A generalized factorization (see Simonenko 1968) is given by

$$\begin{aligned} \Phi &= \Phi_- \cdot \zeta^\kappa \cdot \Phi_+ = \lambda_-^{\omega-\kappa}\Phi_{0-} \cdot \zeta^\kappa \cdot \lambda_+^{-\omega+\kappa}\Phi_{0+}\Phi(+\infty) \\ \kappa &= \max\{z \in \mathbb{Z} : z \le \Re e\,\omega + \frac{1}{2}\}. \end{aligned}$$

Now a cross factorization of A is obtained putting

$$A = A_- \, C \, A_+ = \mathcal{F}^{-1}\Phi_- \cdot \mathcal{F} \quad \mathcal{F}^{-1}\zeta^{\kappa} \cdot \mathcal{F} \quad \mathcal{F}^{-1}\Phi_+ \cdot \mathcal{F}.$$

It represents a bounded operator factorization through the Sobolev space $Z = H^{\Re e\omega - \kappa}$.

A generalized inverse of W is given by

$$W^- = A_+^{-1} P_Z C^{-1} P_Z A_-^{-1} \ell_0.$$

where P_Z is the extension/restriction of the Hilbert projection on Z.

Further it enables asymptotic results in terms of an expansion of the generalized inverse in a scale of Sobolev spaces, using the chain of generalized inverses of W_s , see Penzel-S 1993.

8. WH systems and factorization of matrix functions

$$W = r_+ A \ : \ (L^2_+)^{n\times n} \to L^2(\mathbb{R}_+)^{n\times n}$$

$$A = \mathcal{F}^{-1}\Phi_A \cdot \mathcal{F}\ ,\ \Phi_A \in C^\mu(\ddot{\mathbb{R}})^{n\times n} \cap \mathcal{G}L^\infty(\mathbb{R})^{n\times n}$$

lead to the factorization of matrix functions. A few keywords are

- rational matrix functions, see Clancey-Gohberg 81
- decomposing, $\mathcal{R}$-algebras Goh-Feldman 71, Mikhlin-Pröss 80
- generalized factorization Simonenko 67, Litvinchuk-Spit 87
- piecewise continuous matrix functions Duduchava 79
- constructive methods: triangular, Daniele-Khrapkov, paired, Jones, ... Chebotarev 56, Khrapkov 71, ..., A F dos Santos
- rationally reducible to those Spit-Tashbaev 89, Ehrhardt-S 02
- AP, SAP, ... Böttcher-Karlovich 97, B-K-Spit 02, ..., Bastos

9. Factor properties, intermediate spaces and singularities

There is a close relation between properties of the factors, here particularly their increase at infinity, the kind of intermediate space, and the asymptotic behavior of solutions at zero.

Example: Classical, scalar WHO with analytical index zero

$$W = r_+ A \, : \, L^2_+ \to L^2(\mathbb{R}_+)$$

$$A = \mathcal{F}^{-1} \Phi_A \cdot \mathcal{F} \quad , \quad \Phi_A \in C^\mu(\ddot{\mathbb{R}}) \cap \mathcal{G}L^\infty(\mathbb{R})$$

$$A = A_- \quad A_+ \quad , \quad \Phi_A = \Phi_- \quad \Phi_+ \quad \texttt{(generalized fact.)}$$

$$Y \leftarrow Z \leftarrow X \quad , \quad Z = H^{\Re e\, \omega} \quad , \quad \Re e\, \omega \in]-1/2, 1/2[.$$

For $g \in C^\infty(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$, the singular behavior of the solution is

$$W^{-1} g(x) = A_+^{-1} \, P \, A_-^{-1} \ell_0 g(x) \sim |\, x \,|^{-\Re e\, \omega} \quad \texttt{as} \quad |\, x \,| \to 0.$$

In the system's case there appear log terms, as well, see Castro 95.

Minimal normalization of WHOs

Theorem MoST 1998

Let W as before, $s \in \mathbb{R}$ be critical, i.e., W_s not Fredholm. Then $W_{s+\varepsilon}$ is Fredholm for $\varepsilon \in]0, 1/2[$ with generalized inverse $W^-_{s+\varepsilon}$ given by factorization (usual formula) and W_s can be *image normalized* replacing the image space $H^s(\mathbb{R}_+)$ by a proper dense subspace $\overset{<}{H}^s(\mathbb{R}_+) = r_+ \Lambda_-^{-s-1/2} H_+^{-1/2} \subset H^s(\mathbb{R}_+)$ such that the restricted operator

$$\overset{<}{W}^s = \operatorname{Rst} W_s : H^s_+ \to \overset{<}{H}^s(\mathbb{R}_+)$$

is Fredholm and has a GI given by

$$(\overset{<}{W}^s)^- = \operatorname{Ext} W^-_{s+\varepsilon} : \overset{<}{H}^s(\mathbb{R}_+) \to H^s_+ .$$

Remark There is an analog for *domain normalization*, matrix cases, transfer of normalization by operator relations like $\overset{*}{\sim}$ etc.

10. Wiener-Hopf plus/minus Hankel operators (CTOS)

$$T = r_+ A(I \pm J)\ell_0 = r_+ A \ell^c \ : \ L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$$

$$A = \mathcal{F}^{-1}\Phi_A \cdot \mathcal{F} \ , \ \Phi_A \in C^\mu(\ddot{\mathbb{R}}) \cap \mathcal{G}L^\infty(\mathbb{R})$$

$$Jf(x) = f(-x) \, , \, x \in \mathbb{R} \ , \quad \ell^c = \ell^e \text{ or } \ell^o \text{ (even/odd extension)}$$

Then

- $T \sim \ell_0 r_+ A : L^2_{e/o} \to L^2_+$ which is understood as general WHO,
- a cross factorization $A = A_- C A_e$ exists explicitly provided $\Phi_A(+\infty)/\Phi_A(-\infty) \notin e^{i\pi/2}\mathbb{R}_+$ resp. $\notin e^{3i\pi/2}\mathbb{R}_+$,
- and then a generalized inverse $W_s^- = ...$ is explicitly obtained by asymmetric factorization, see CST 2004.

Matrix versions by Castro-S 2005 and Castro-Duduchava-S 2006.

Asymmetric factorization (scalar case): a direct approach to the inversion of CTOS

The idea comes from Basor and Ehrhardt 2004. Instead of the Fourier symbol Φ consider the index-free function (see Duduchava 1979)

$$\Psi = \zeta^{-\omega}\,\Phi^{-1}(+\infty)\,\Phi$$

where

$$\omega = \frac{1}{2\pi i}\int_{\mathbb{R}} d\log\Phi \quad , \quad \Re e(\omega) \pm \frac{1}{4} \notin \mathbb{Z} \quad , \quad \zeta(\xi) = \frac{\lambda_-(\xi)}{\lambda_+(\xi)} = \frac{\xi - i}{\xi + i}.$$

Putting $\widetilde{\Psi} = J\Psi$ define the symmetrized function

$$G = \Psi\widetilde{\Psi}^{-1} \in C^{\mu}(\dot{\mathbb{R}}) \quad \texttt{with} \quad \operatorname{ind} G = 0,$$

which admits an *antisymmetric Wiener-Hopf factorization* in $C^{\mu}(\dot{\mathbb{R}})$

$$G = G_-\,G_+ = G_-\,G_-^{-1}.$$

From antisymmetric to asymmetric factorization (scalar)

Now a cross factorization is obtained putting (see Castro-S 2004)

$$\Phi = \Phi_- \zeta^\kappa \Phi_e \quad , \quad \kappa = \max\{z \in \mathbb{Z} : z \leq \Re e(\omega) \pm \frac{1}{4}\}$$

$$\Phi_- = \lambda_-^{2(\omega-\kappa)} \exp\{P_- \log G\} \quad , \quad \Phi_e = \zeta^{-\kappa} \Phi_-^{-1} \Phi.$$

It can be proved that Φ_e is an even function!

The above sign of $\pm$ depends on which version we consider:

$$T = r_+ A(I \pm J)\ell_0 = r_+ A\ell^c : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$$

$$\ell^c = \ell^e \,\texttt{or}\, \ell^o \,(\texttt{even/odd extension})$$

11. Structured matrix operators (by example): ΨDOs occurring in BVPs for the HE in a quarter-plane

$$\begin{aligned} Q_1 &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\,,\ x_2 > 0\} \\ \Gamma_1 &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\,,\ x_2 = 0\} \\ \Gamma_2 &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\,,\ x_2 > 0\} \end{aligned}$$

Determine (all weak solutions) $u \in H^1(Q_1)$ (explicitly and in closed analytical form) such that

$$Au(x) = (\Delta + k^2)u(x) = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + k^2\right)u(x) = 0 \qquad \text{in} \quad Q_1$$

$$B_1 u(x) = \left(\alpha u + \beta \frac{\partial u}{\partial x_2} + \gamma \frac{\partial u}{\partial x_1}\right)(x) = g_1(x) \qquad \text{on} \quad \Gamma_1$$

$$B_2 u(x) = \left(\alpha' u + \beta' \frac{\partial u}{\partial x_1} + \gamma' \frac{\partial u}{\partial x_2}\right)(x) = g_2(x) \qquad \text{on} \quad \Gamma_2\,.$$

Sometimes it is useful to consider *"small regularity"*: $u \in H^{1+\varepsilon}(Q_1)$.

BVPs for the HE in a quarter-plane ctd

Herein the following data are given: a complex *wave number* k with $\Im m\, k > 0$, constant coefficients $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ as fixed parameters and arbitrary $g_j \in H^{-1/2}(\Gamma_j)$. Note that β and β' are the coefficients of the normal derivatives, whilst γ and γ' are those of the tangential derivatives. In case of a *Dirichlet condition*, i.e., $\beta = \gamma = 0$, we assume $g_1 \in H^{1/2}(\Gamma_1)$.

The space of weak solutions of the HE is denoted by

$$\mathcal{H}^1(Q_1) = \left\{u \in H^1(Q_1) : (\Delta + k^2)u = 0\right\} = \ker A$$

in the previous notation and we shall consider the operator L^0 associated to the semi-homogeneous problem.

The impedance problem

The *impedance problem* shows the following boundary conditions:

$$\Im_1 u(x) = \frac{\partial u(x)}{\partial x_2} + i p_1 u(x) = g_1(x) \qquad \text{on} \quad \Gamma_1$$
$$\Im_2 u(x) = \frac{\partial u(x)}{\partial x_1} + i p_2 u(x) = g_2(x) \qquad \text{on} \quad \Gamma_2$$

where the imaginary part of p_j turns out to be important:

1. $\Im m\, p_j > 0$: physically most reasonable due to positive finite conductance in electromagnetic theory for instance;
2. $p_1 = 0$ or/and $p_2 = 0$: *Neumann condition(s)* allow a much simpler solution, MPST 93, CST 04;
3. if both $\Im m\, p_j$ are negative the potential approach has to be modified in a cumbersome way (in contrast to the mixed case which can be solved like (1));
4. if $p_j \in \mathbb{R} \setminus \{0\}$ for $j = 1, 2$, the problem needs another kind of normalization (LAP) that is not carried out here.

Compatibility conditions

The Dirichlet problem (DD) is only solvable under certain compatibility conditions (see Hsiao-Wendland 2008) for the Dirichlet data

$$g_1 - g_2 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+),$$

i.e., this function is extendible by zero onto the full line $\mathbb{R}$ such that the zero extension $\ell_0(g_1 - g_2)$ belongs to $H^{1/2+\varepsilon}(\mathbb{R})$. The Neumann problem (NN) needs a compatibility condition

$$g_1 + g_2 \in \widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+),$$

if and only if $\varepsilon = 0$. The mixed problems (DN) do not require any additional condition.

Solution of the DN problem and half-line potentials (HLPs)

The solution is amazingly simple in the DN case:

$$u(x_1, x_2) = \mathcal{F}^{-1}_{\xi\mapsto x_1} e^{-t(\xi)x_2}\widehat{\ell^e g_1}(\xi) - \mathcal{F}^{-1}_{\xi\mapsto x_2} e^{-t(\xi)x_1} t^{-1}(\xi)\widehat{\ell^o g_2}(\xi)$$

where ℓ^e and ℓ^o denote even and odd extension.

Theorem MPST 93, CST 04

The following mapping

$$\mathcal{K}_{DN,Q_1} : X = H^{1/2}(\Gamma_1) \times H^{-1/2}(\Gamma_2) \to \mathcal{H}^1(Q_1)$$

$$u = \mathcal{K}_{DN,Q_1}(f, g)^T = \mathcal{K}_{D,Q_{12}}\ell^e f + \mathcal{K}_{N,Q_{14}}\ell^o g$$

$$\mathcal{K}_{D,Q_{12}}\ell^e f(x) = \mathcal{F}^{-1}_{\xi\mapsto x_1} \exp\left[-t(\xi)x_2\right]\widehat{\ell^e f}(\xi)\ , \qquad x \in Q_{12}$$

$$\mathcal{K}_{N,Q_{14}}\ell^o g(x) = -\mathcal{F}^{-1}_{\xi\mapsto x_2} \exp\left[-t(\xi)x_1\right] t^{-1}(\xi)\,\widehat{\ell^o g}(\xi)\ , \quad x \in Q_{14}$$

is a toplinear isomorphism that satisfies

$$(T_{0,\Gamma_1}, T_{1,\Gamma_2})^T \mathcal{K}_{DN,Q_1} = I_X$$

$$\mathcal{K}_{DN,Q_1}(T_{0,\Gamma_1}, T_{1,\Gamma_2})^T = I_{\mathcal{H}^1(Q_1)}\ .$$

Resolvent of the DN problem as potential operator

- Using this representation as a potential operator, it was possible to solve explicitly a great number of BVPs, see CST 2004.
- The reason was that the corresponding boundary ΨDO a matricial 2×2 structured operator that has a triangular form in many cases.
- It also gave the idea to introduce so-called *half-line potentials* (HLPs):

Half-line potentials (HLPs) CST 06

Let $m_j \in \mathbb{N}_0$, $\psi_j : \mathbb{R} \to \mathbb{C}$ be measurable functions such that ψ_j is m_j-regular, i.e., $t^{-m_j}\psi_j \in \mathcal{G}L^\infty$ and let $\ell_j : H^{1/2-m_j}(\mathbb{R}_+) \to H^{1/2-m_j}(\mathbb{R})$ be continuous extension operators for $j = 1, 2$. Then

$$\begin{aligned} u(x) &= \mathcal{F}^{-1}_{\xi\mapsto x_1}\left\{\exp[-t(\xi)x_2]\psi_1^{-1}(\xi)\widehat{\ell_1 f_1}(\xi)\right\} \\ &\quad +\mathcal{F}^{-1}_{\xi\mapsto x_2}\left\{\exp[-t(\xi)x_1]\psi_2^{-1}(\xi)\widehat{\ell_2 f_2}(\xi)\right\} \end{aligned}$$

with $f_j \in H^{1/2-m_j}(\mathbb{R}_+)$ and $x = (x_1, x_2) \in Q_1$ is said to be a *half-line potential* (HLP) in Q_1 with density (f_1, f_2).

We call it *strict* for $\mathcal{H}^1(Q_1)$ if it defines a bijective mapping, writing

$$\begin{aligned} \mathcal{K} &= \mathcal{K}_1 + \mathcal{K}_2 = \mathcal{K}^{\psi_1,\psi_2} \\ &: X = H^{1/2-m_1}(\mathbb{R}_+) \times H^{1/2-m_2}(\mathbb{R}_+) \to \mathcal{H}^1(Q_1) \end{aligned}$$

specifying ℓ_j when necessary. Keeping in mind low regularity properties: $\mathcal{K}^\varepsilon : X^\varepsilon = H^{1/2-m_1+\varepsilon}(\mathbb{R}_+) \times H^{1/2-m_2+\varepsilon}(\mathbb{R}_+) \to \mathcal{H}^{1+\varepsilon}(Q_1)$, we speak about a *strict* HLP for $\mathcal{H}^{1+\varepsilon}(Q_1)$ in the corresponding case.

Half-line potentials (HLPs) CST 06

Proposition Let $L = (B_1, B_2)^T$ and $\mathcal{K}$ be given as before. Then the composed operator $T = L\mathcal{K}$ has the form:

$$T = \begin{pmatrix} r_+ A_{\phi_{11}} \ell_1 & C_0 A_{\phi_{12}} \ell_2 \\ C_0 A_{\phi_{21}} \ell_1 & r_+ A_{\phi_{22}} \ell_2 \end{pmatrix} : X \to Y$$

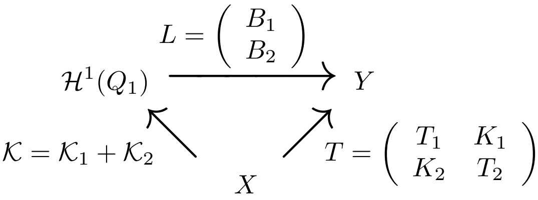
where $Y = H^{-1/2}(\mathbb{R}_+)^2$ identifying Γ_j with $\mathbb{R}_+$ and

$$\phi_{11} = \sigma_1 \psi_1^{-1} = (\alpha - \beta t + \gamma \vartheta)\psi_1^{-1}\ , \quad \phi_{12} = \sigma_{1*}\psi_2^{-1} = (\alpha + \beta\vartheta - \gamma t)\psi_2^{-1}$$
$$\phi_{21} = \sigma_{2*}\psi_1^{-1} = (\alpha' + \beta'\vartheta - \gamma' t)\psi_1^{-1}\ , \ \ \phi_{22} = \sigma_2 \psi_2^{-1} = (\alpha' - \beta' t + \gamma'\vartheta)\psi_2^{-1}$$

The main diagonal contains CTOS if $\ell_j = \ell^{e/o}$. The others are Fourier integral operators (combined with extensions) defined for any $\phi \in L^\infty$ by

$$K^{(s)} = C_0 A_\phi \ell^o : H^s(\mathbb{R}_+) \to H^s(\mathbb{R}_+) \quad , \quad s \in]-3/2, 1/2[,$$
$$K^{(s)} f(x_1) = (2\pi)^{-1} \int_{\mathbb{R}} \exp[-t(\xi)x_1] \phi(\xi) \widehat{\ell^o f}(\xi)\, d\xi\ , \ \ x_1 \in \mathbb{R}_+\ .$$

They are well-defined and bounded if $s \in]-3/2, 1/2[$. In this case, $K^{(s)} = 0$ if and only if ϕ is an even function. Replacing ℓ^o by ℓ^e, we have boundedness of $K^{(s)}$ for $s \in]-1/2, 3/2[$ being zero if and only if ϕ is odd.



Problem solvable by the DN ansatz

Theorem CST 04 The following classes of interior wedge problems can be explicitly solved by generalized inversion of T provided $T$is of normal type:

$\Gamma_1 \setminus \Gamma_2$	D	N	$\mathcal{I}$	$\mathcal{T}$	O	G
D	**II**	**O**	**I**	$\mathrm{II}^*_\pm$	III	**III**
N	O*	II	II*	I*	III*	III*
$\mathcal{I}$	I*	II		$\mathrm{I}^*_\pm$		
$\mathcal{T}$	$\mathbf{II}_\pm$	I	$\mathbf{I}_\pm$	IV	IV	**IV**
O	III*	III		IV*		
G	III*	**III**		IV*		

Legend *:* (referring to CST 04)

bold	–	*reference problems discussed in detail.*
*	–	*belongs to the corresponding class where the two variables are exchanged, see (2.11).*
O	–	*invertible by representation formulas (2.11).*
I	–	*direct inversion by Theorem 3.2 (even pre-symbol), Example 3.3.*
I_-	–	*dito (minus type), Example 3.5.*
I_+	–	*right inversion via AFIS, see Proposition 5.3, cf. Corollary 5.6.*
II, $\mathrm{II}_\pm$	–	*like* I, $\mathrm{I}_\pm$ *after image normalization, Example 3.4.*
III	–	*Fredholm and one-sided invertible via AFIS, eventually after normalization, see Theorem 5.4, Theorem 5.5, only one scalar factorization needed.*
IV	–	*similar with two scalar operators, see Corollary 5.6*
empty spaces	–	*correspond with open problems, not decomposing (triangular) in the preceding sense.*

The general problem (of normal type) CST 06

In case of the remainder BVPs in the above diagram, one needs a different ansatz in order to end up with a triangular matrix operator. Namely, given (as before)

$$Au(x) = (\Delta + k^2)u(x) = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + k^2\right) u(x) = 0 \qquad \text{in} \quad Q_1$$

$$B_1 u(x) = \left(\alpha u + \beta \frac{\partial u}{\partial x_2} + \gamma \frac{\partial u}{\partial x_1}\right)(x) = g_1(x) \qquad \text{on} \quad \Gamma_1$$

$$B_2 u(x) = \left(\alpha' u + \beta' \frac{\partial u}{\partial x_1} + \gamma' \frac{\partial u}{\partial x_2}\right)(x) = g_2(x) \qquad \text{on} \quad \Gamma_2 \, .$$

with Fourier symbols

$$\sigma_1 = \alpha - \beta t + \gamma \vartheta \quad , \quad \sigma_2 = \alpha' - \beta' t + \gamma' \vartheta$$

where $t(\xi) = (\xi^2 - k^2)^{1/2}$, $\vartheta(\xi) = -i\xi$. Both symbols are assumed to be 1-regular, i.e., $t^{-1}\sigma_j \in \mathcal{G}L^\infty$.

The companion operator trick CST 06

Now the trick consists in using a special ansatz (see page 52) where either $\psi_1 = \sigma_2^*$ or $\psi_2 = \sigma_1^*$. This guarantees that T is triangular!

In order to have a strict ansatz, we need that the corresponding BVP with Fourier symbols ψ_1, ψ_2 is well-posed. Therefore we consider a companion BVP with

$$B^* = \begin{pmatrix} B_2^* \\ B_2 \end{pmatrix} \qquad \text{or} \qquad B^* = \begin{pmatrix} B_1 \\ B_1^* \end{pmatrix}.$$

In case of the **impedance problem** it turns out that these problems are explicitly solvable by the DN ansatz, but the resulting potential operator $\mathcal{K}$ (generalized inverse to B^*) is

- only right invertible with defect number $\beta(\mathcal{K}) = 1$, if both $\Im m\, p_j$ are negative,
- is strict if at least one $\Im m\, p_j$ is positive (corresponding choice),
- needs normalization otherwise or different idea.

Example of a "bad factorization" CST 06

Thus we found an example (impedance problem with $\beta(\mathcal{K}) = 1$) for

$$\begin{array}{ccc} \mathcal{H}^1(Q_1) & \xrightarrow{L = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}} & Y \\ \mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2 \nwarrow & & \nearrow T = \begin{pmatrix} T_1 & K_1 \\ K_2 & T_2 \end{pmatrix} \\ & X & \end{array}$$

where both $\mathcal{K}, T$ are left invertible with defect number 1,

$$T = L\mathcal{K} \quad , \quad L = T\mathcal{K}^- + LP_1.$$

The first is **not a full range factorization**, but $T\mathcal{K}^-$ is, and LP_1 has rank 1. So it could be proved that L is invertible and the impedance problem also well-posed for these parameters.

12. Conclusion

- Operator factorization and, moreover, operator matrix identities are a very convenient vehicle for the description of equivalence or reduction of problems.
- They are a powerful tool for the inversion of singular operators related to canonical BVPs.
- They enable a precise description of properties of these operators, their normalization and asymptotics.

Many thanks for your attention !