Flows of vector fields: are integral curves unique?

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Summary

Outline of the talk

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The flow of a vector field

Given a vector field $\mathbf{b} : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$, consider the flow $\mathbf{X}$ of $\mathbf{b}$

$$\begin{cases} \frac{d}{dt}\mathbf{X}(t,x) = \mathbf{b}_t(\mathbf{X}(t,x)) & \forall t \in [0,\infty) \\ \mathbf{X}(0,x) = x. \end{cases}$$

It can be seen

- as a collection of trajectories $\mathbf{X}(\cdot, x)$ labelled by $x \in \mathbb{R}^d$;
- as a collection of diffeomorphisms $\mathbf{X}(t, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$.

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Continuity/transport equation

Consider the related PDE, named continuity equation

$$\begin{cases} \partial_t \mu_t + \operatorname{div}(\mathbf{b}_t \mu_t) = 0 & \text{in } (0,\infty) \times \mathbb{R}^d \\ \mu_0 \text{ given.} \end{cases}$$

When $\mathbf{b}_t$ is sufficiently smooth and $\mu_t : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ is a smooth function, all derivatives can be computed.

Even if $\mathbf{b}_t$ is only a bounded vector field and $\{\mu_t\}_{t\in[0,\infty)}$ is a 1-parameter family of finite measures, the PDE makes sense distributionally.

When

$$\operatorname{div}\mathbf{b}_t \equiv 0,$$

the continuity equation is equivalent to the transport equation

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Connection between continuity equation and flows

Solutions of the CE flow along characteristic curves of **b**

Given **b**, its flow **X** an initial distribution of mass $\mu_0 \in \mathscr{P}(\mathbb{R}^d)$, a solution of the CE is

$$\mu_t := \mathbf{X}(t,\cdot)_\# \mu_0.$$

Recall that the measure $\mathbf{X}(t,\cdot)_\#\mu_0$ is defined by

$$\int_{\mathbb{R}^d} \varphi(x)\, d[\mathbf{X}(t,\cdot)_\#\mu_0](x) = \int_{\mathbb{R}^d} \varphi(\mathbf{X}(t,x))\, d\mu_0(x) \quad \forall \varphi : \mathbb{R}^d \to \mathbb{R}.$$

Indeed, for any test function $\varphi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi\, d\mu_t = \frac{d}{dt}\int_{\mathbb{R}^d} \varphi(\mathbf{X}(t,x))\, d\mu_0(x) = \int_{\mathbb{R}^d} \nabla\varphi(\mathbf{X}) \cdot \partial_t \mathbf{X}\, d\mu_0$$
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This is the distributional formulation of the continuity equation.

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This is the distributional formulation of the continuity equation.

Regularity of $\mathbf{b}$ matters

Is the solution of the continuity equation starting from μ_0 unique?

YES if $\nabla \mathbf{b}$ is bounded

Given a solution ν_t to CE, set $\widetilde{\nu}_t = \mathbf{X}(t,\cdot)^{-1}_{\#}\nu_t$. An analogous computation shows that

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi \, d\widetilde{\nu}_t = 0,$$

so

$$\mathbf{X}(t,\cdot)^{-1}_{\#}\nu_t = \widetilde{\nu}_t = \nu_0 = \mu_0 \quad \Rightarrow \quad \nu_t := \mathbf{X}(t,\cdot)_{\#}\mu_0.$$

NO if $\mathbf{b}$ is less regular

As soon as uniqueness for the ODE fails.

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Theory for smooth vector fields

Cauchy-Lipschitz Theorem

Let $\mathbf{b}_t$ a vector field with $\nabla \mathbf{b}_t$ locally bounded. Then for every $x \in \mathbb{R}^d$ there exists a unique maximal solution $\mathbf{X}(\cdot, x) : [0, T_{\mathbf{X}}(x)) \to \mathbb{R}^d$ of the ODE.

For every $x \in \mathbb{R}^d$ such that $T_{\mathbf{X}}(x) < \infty$ the trajectory $\mathbf{X}(\cdot, x)$ blows up properly

$$\lim_{t \to T_{\mathbf{X}}(x)} |\mathbf{X}(t, x)| = \infty.$$

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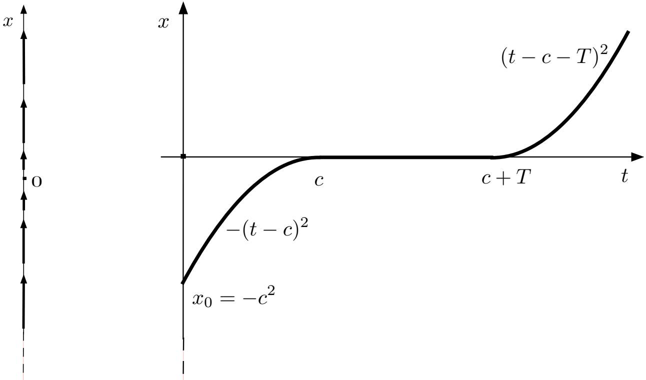
Nonsmooth theory: lack of uniqueness

One-dimensional autonomous vector field with lack of uniqueness

$$\mathbf{b}(x) = 2\sqrt{|x|}, \qquad x \in \mathbb{R}$$

Given $x_0 = -c^2 < 0$, the 1-parameter family of curves that stop at the origin for an arbitrary time $T \geq 0$, solve the ODE.



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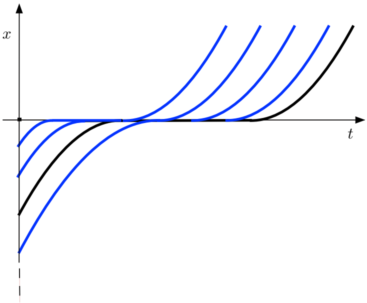

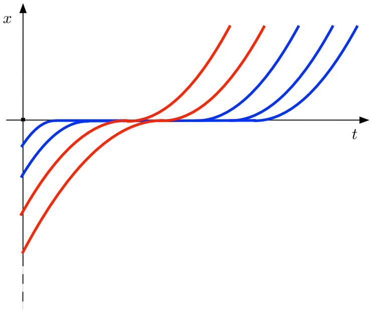

Between all the possible integral curves, a "better selection" could be made by the ones that do not stop in 0. In other words, we wish to select a collection of integral curves that "do not concentrate".

Selection of a flow

Regular lagrangian flows

Given a vector field $\mathbf{b} : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$, the map $\mathbf{X} : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is a regular Lagrangian flow of $\mathbf{b}$ if:

(i) for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$, $\mathbf{X}(\cdot, x)$ solves the ODE $\dot{x}(t) = \mathbf{b}_t(x(t))$ starting from x;

(ii) $\mathbf{X}(t, \cdot)_\# \mathscr{L}^d \leq C \mathscr{L}^d$ for every $t \in [0, T]$ and for some $C > 0$.

Theorem ([Di Perna-Lions '89)

, [Ambrosio '04]] Let us assume that $|\nabla \mathbf{b}_t| \in L^1_{loc}(\mathbb{R}^d)$, $\operatorname{div} \mathbf{b}_t \in L^\infty(\mathbb{R}^d)$ *and*

$$\frac{|\mathbf{b}_t(x)|}{1 + |x|} \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d).$$

Then there exists a unique regular Lagrangian flow $\mathbf{X}$ *of* $\mathbf{b}$.

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Remarks on the DiPerna-Lions theorem

The regularity assumption $|\nabla \mathbf{b}_t| \in L^1_{loc}(\mathbb{R}^d)$ can be replaced by

Assumption

For every compactly supported $\mu_0 \in L^\infty(\mathbb{R}^d)$ there exists a unique bounded, compactly supported solution of the CE starting from μ_0.

This is satisfied by several classes of vector fields:

- when locally $\nabla \mathbf{b}_t$ is a matrix-valued finite measure (namely, $\mathbf{b}_t$ is a function of bounded variation $BV_{loc}(\mathbb{R}^d;\mathbb{R}^d)$ function), [Ambrosio 04];
- by singular integrals of L^1 functions, for instance convolutions of the form $h * \frac{x}{|x|^d}$ with $h \in L^1(\mathbb{R}^d)$, [Bouschut, Crippa 13] and of measures, with some additional structure [Bohun, Bouschut, Crippa 13].

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Remarks on the DiPerna-Lions theorem

- a different approach to this result was proposed by [Crippa, De Lellis, 08]. To show the uniqueness of the flow, they consider a functional of the type

$$\Phi_\delta(t) := \int \log\left(1 + \frac{|\mathbf{X}_1(t,x) - \mathbf{X}_2(t,x)|}{\delta}\right) dx \qquad t \in [0, T];$$

- the assumption $\operatorname{div} \mathbf{b}_t \in L^\infty(\mathbb{R}^d)$ can be weakened to

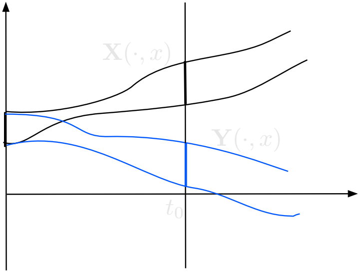
$$\operatorname{div} \mathbf{b}_t \in BMO(\mathbb{R}^d)$$

[Mucha, 2010], [C., Crippa, Spirito 2016].

Idea of the proof - existence and uniqueness

The real difficulty is with uniqueness.

Well posedness of the CE $\Rightarrow$ uniqueness of the flow. (in the class of bounded, compactly supported solutions). Assume by contradiction that there exists a set $A \subseteq \mathbb{R}^d$ such that two flows $\mathbf{X}(\cdot, x)$ and $\boldsymbol{Y}(\cdot, x)$ start at every $x \in A$. Taking a subset, we can assume that the two flows are disjoint at a later time t_0.



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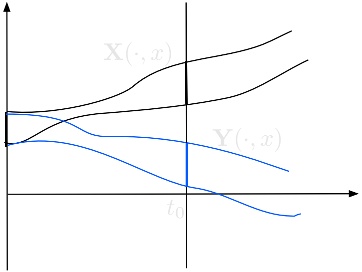

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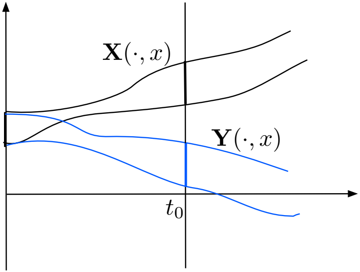

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Proof of the well posedness of the CE.

Proposition ([DiPerna-Lions, '89], [Ambrosio '04])

for any divergence free $\mathbf{b} \in L^1_t W^{1,1}{}_{x,loc}$, $u_0 \in L^\infty_c$ there exists a unique solution $u \in L^\infty_t L^\infty{}_{x,c}$ to

$$\partial_t u_t + \operatorname{div}(\mathbf{b}_t u_t) = 0$$

The statement holds more in general when the integrability of $\nabla \mathbf{b}$ is coupled with the integrability of u (DiPerna-Lions range).

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Proposition ([DiPerna-Lions, '89], [Ambrosio '04])

Let $r, p \in [1, \infty]$ satisfy

$$\frac{1}{p} + \frac{1}{r} \leq 1.$$

Then, for any divergence free $\mathbf{b} \in L^1_t L^1_t W^{1,r}_x$, $u_0 \in L^p_c$ there exists a unique solution $u \in L^\infty_t L^p{}_x$ to

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The statement holds more in general when the integrability of $\nabla \mathbf{b}$ is coupled with the integrability of u (DiPerna-Lions range).

Idea of the proof - well posedness

By linearity, we show that any bounded, compactly supported solution $u(t,x)$ of the CE

$$\partial_t u_t + \operatorname{div}(\mathbf{b}_t u_t) = 0$$

$$u_0(0,\cdot) = 0 \qquad \Longrightarrow \quad u(t,\cdot) \equiv 0 \quad \text{for any } t > 0.$$

Formally, multiply the equation by u and integrate

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} \frac{u(t,x)^2}{2}\,dx &= \int_{\mathbb{R}^d} u\partial_t u\,dx = -\int_{\mathbb{R}^d} u \operatorname{div}(u\mathbf{b})\,dx \\ &= \int_{\mathbb{R}^d} u\nabla u\cdot\mathbf{b}\,dx = \int_{\mathbb{R}^d} \frac{u^2}{2}\operatorname{div}\mathbf{b}\,dx \\ &\le C\int_{\mathbb{R}^d}\frac{u^2}{2}\,dx\end{aligned}$$

This computation doesn't make sense because u is not regular.

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We repeat the computation after convolving the equation with a smooth kernel ρ_ε. The function $u^\varepsilon = u * \rho_\varepsilon$ solves

$$\partial_t u_t^\varepsilon + \operatorname{div}(\mathbf{b}_t u_t^\varepsilon) = \operatorname{div}(\mathbf{b}_t u_t^\varepsilon) - \operatorname{div}(\mathbf{b}_t u_t)^\varepsilon = \mathcal{C}^\varepsilon(u_t, \mathbf{b}_t).$$

To handle the commutator, we employ the lemma

Lemma

If $u \in L_c^\infty(\mathbb{R}^d)$ and $|\nabla \mathbf{b}| \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, then

$$\lim_{\varepsilon \to 0} \mathcal{C}^\varepsilon(u, \mathbf{b}) = 0 \qquad \text{in } L^1(\mathbb{R}^d).$$

Uniqueness $\Rightarrow$ Existence. Consider a smooth approximation $\mathbf{b}^\varepsilon$ of the vector field $\mathbf{b}$ by convolution and consider the approximating flows $\mathbf{X}^\varepsilon$. They converge (in a suitable weak sense) to a limit collection of curves, which is the limit flow by uniqueness.

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Two questions

Question: a.e. uniqueness of integral curves

Does any divergence free $\mathbf{b} \in L^1_t W^{1,p}_x$ admit a unique integral curve (namely, $\gamma \in W^{1,1}(0,T)$ solution of the ODE $\dot\gamma(t) = u(t,\gamma)$) for a.e. initial datum $x \in \mathbb{R}^d$?

Open since the pioneering works of DiPerna-Lions and Ambrosio.

(Related) question: well posedness of the CE

Let $\mathbf{b} \in L^1_t W^{1,p}_x$ divergence free. Is the CE $\partial_t u + \operatorname{div}(\mathbf{b}u) = 0$ well-posed in the class of positive solutions $u \in L^\infty_t L^r_x$ under the minimal summability requirement $\frac{1}{r} + \frac{1}{p^*} < 1$, namely $\frac{1}{p} + \frac{1}{r} < 1 + \frac{1}{d}$?

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The case $p > d$: a.e. uniqueness

For any $\mathbf{b} \in Lip$ then

$$|\mathbf{b}(x) - \mathbf{b}(y)| \leq C|x - y| \quad \forall x, y,$$

Let $\mathbf{X}(\cdot, x)$ be the RLF, γ_x an integral curve from $x \in \mathbb{R}^d$.

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By Gronwall inequality, if $\mathbf{b} \in Lip$ we have everywhere uniqueness. If $\mathbf{b} \in W^{1,p}$, $p > d$, we have a.e. uniqueness [Caravenna, Crippa - Jabin].

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The case $p < d$: uniqueness of RLF

Key observation: for a.e. x it holds

$$\int_0^T g(\mathbf{X}(t,x))\,dt < \infty.$$

Indeed, integrating in x and by incompressibility

$$\int_0^T \|g(\mathbf{X}(t,\cdot))\|_{L^p}\,dt \le C\int_0^T \|g\|_{L^p}\,dt \le CT\|\nabla \mathbf{b}(t,\cdot)\|_{L^p} < \infty.$$

Does the Lusin-Lipschitz inequality imply uniqueness for $p < d$? [Crippa, De Lellis] used it to infer uniqueness of the RLF.

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Main result

If $p < d$ then the a.e. uniqueness for trajectories does not hold.

Theorem ([B.-Colombo-DeLellis, '20])

For every $d \geq 2$, $p < d$ and $s < \infty$ there exist a divergence free velocity field $\mathbf{b} \in C_t(W_x^{1,p} \cap L_x^s)$ and a set $A \subset \mathbb{T}^d$ such that

- $\mathscr{L}^d(A) > 0$;
- *for any $x \in A$ there are at least two integral curves of u starting at x.*

The case $p = d$

Ingredients of proof:

- Ambrosio's superposition principle to connect the a.e. uniqueness of trajectories to uniqueness results for positive solutions to (CE).
- Non-uniqueness theorem for positive solutions to (CE) based on convex integration type techniques borrowed from [Modena-Székelyhidi '18].

What about the critical case $p = d$?

If $\nabla \mathbf{b} \in L^1_t L^{d,1}_x$, the a.e. uniqueness for integral curves holds. Recall that

$$\|f\|_{L^{r,q}} := \left(\int_0^\infty \left(\lambda \mathscr{L}^d(\{|f| \geq \lambda\})^{1/r} \right)^q \frac{dx\lambda}{\lambda} \right)^{1/q}$$

and $L^q \subset L^{d,1} \subset L^d$ for any $q > d$.

EPFL

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- η_x is concentrated on integral curves of $\mathbf{b}$ starting at x;
- we have the representation formula $\mu = (e_t)_\#(\mu_0 \otimes \eta_x)$,

$$\int \phi \, d\mu_t = \int \left(\int \phi(\gamma(t)) \, d\eta_x(\gamma) \right) d\mu_0(x).$$

Superposition solutions are averages of integral curves of u.

Theorem ([Ambrosio '04])

Let $\mathbf{b} : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$, $\mu \in L^\infty_t(\mathscr{M}_+)$ solution of CE with

$$\int_0^T \int |\mathbf{b}(t,x)| \, dx\mu_t(x) \, dxt < \infty.$$

Then it is a superposition solution.

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Maria Colombo

Flows and continuity equation

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Cauchy-Lipschitz thm

Lack of uniqueness

The nonsmooth theory: RLF

A.e. uniqueness of integral curves

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Ambrosio's superposition principle

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A.e. uniqueness of integral curves implies uniqueness of positive solutions to (CE).

Proposition

Let $\mathbf{b} \in L^1_t W^{1,1}_x$ divergence free whose integral curves are unique a.e. and $\mathbf{X}$ its RLF.
Then positive solutions $\mu \in L^\infty_t L^1_x$ to (CE) are unique and have the representation

$$\mu_t = (X_t)_\# \mu_0 \quad \text{for any } t \in [0, T].$$

Indeed, by the superposition principle and a.e. uniqueness of integral curves, $\eta_x \in \mathscr{P}(C([0,T], \mathbb{T}^d))$ must satisfy $\eta_x := \delta_{X(\cdot,t)}$. Hence

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Sharp summability, $p > n$

(Related) question: well posedness of the CE

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Corollary (Caravenna-Crippa 2018)

Let $u \in W^{1,p}$, $p > n$. Positive solutions of CE are well-posed under the minimal summability requirement $u \in L^\infty L^1$.

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Is there a (p-dependent) family of inequalities which interpolates between

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Let $\mathbf{b} \in W^{1,p}$, $1 < p < d$, $\alpha \in [0, \frac{p}{d})$. Then there exists $g \in L^p$ such that

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Remark

The range of α is optimal.

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Corollary

Let $u \in W^{1,p}$, $p < d$. Positive *solutions $u \in L^\infty_t L^r_x$ of the CE are well posed in the range of exponent*

$$\frac{1}{r} + \frac{1}{p} < 1 + \frac{1}{d-1}\frac{p-1}{p}$$

This range but it is strictly contained in the range for which the equations make sense $\frac{1}{p} + \frac{1}{r} < 1 + \frac{1}{d}$. What happens in between?

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This range strictly contains the DiPerna-Lions range $\frac{1}{p} + \frac{1}{r} \leq 1$ but it is strictly contained in the range for which the equations make sense $\frac{1}{p} + \frac{1}{r} < 1 + \frac{1}{d}$. What happens in between? Partial result by

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Nonuniqueness by convex integration

If we produce an example of nonuniqueness of positive solutions of the continuity equations in some range of exponents we have disproved the a.e. uniqueness of integral curves.

Theorem ([B.-Colombo-DeLellis, '20])

Let $d \geq 2$, $p \in (1, \infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$, $r \in [1, \infty]$ be such that

$$\frac{1}{p} + \frac{1}{r} > 1 + \frac{1}{d}.$$

Then there exist

- *a divergence-free vector field $\mathbf{b} \in C_t(W^{1,r}_x \cap L^{p'}_x)$,*
- *a positive, nonconstant $u \in C_t L^p_x$ with $u(0, \cdot) = 1$,*

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Remarks

- The main theorem follows: any velocity field obtained in the previous theorem does not have the a.e. uniqueness for integral curves. Indeed
  - Since $\operatorname{div} \mathbf{b} = 0$, the function $\bar{u} \equiv 1$ solves CE.
  - The u constructed in this theorem is a second distinct solution!
  - As seen before, a.e. uniqueness of integral curves implies uniqueness of positive solutions to (CE).
- The construction is based on convex integration scheme, as in the groundbreaking works [DeLellis-Székelyhidi, '09-'13], [Isett '16] for the Euler equation and [Buckmaster-Vicol '17] for Navier-Stokes.
- The first ill-posedness result for (CE) with Sobolev velocity field has been proven in [Modena-Székelyhidi, '18], [Modena-Sattig, '19].
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The end

Thank you for your attention!