Vapnik-Chervonenkis Theory and the Independence Property in Model Theory

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1. VC-dimension: Combinatorics
2. VC-dimension: Model Theory

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2. VC-dimension: Model Theory

Let X be a set, let $\mathcal{F}$ be a family of subsets of X, and let $A \subseteq X$ be a set. Define

$$\mathcal{F} \cap A := \{F \cap A : F \in \mathcal{F}\} .$$

Definition

We say that a family $\mathcal{F}$ *shatters* A if $\mathcal{F} \cap A = \mathcal{P}(A)$.

Remark: If $B \subseteq A$ and $\mathcal{F}$ shatters A, then $\mathcal{F}$ shatters B.

Definition (VC-dimension of a family of sets)

Let $\mathcal{F}$ be a family of sets. The *VC-dimension of* $\mathcal{F}$, $VC(\mathcal{F})$, is defined as:

- The greatest n such that there is a set of cardinality n shattered by $\mathcal{F}$, if such n exists;
- ∞, otherwise.

Remark: If $\mathcal{F}$ shatters a set of cardinality n and does not shatter any set of cardinality $n+1$, then $VC(\mathcal{F}) = n$.

Definition (The shatter function)

Let X be a set, $\mathcal{F} \subseteq \mathcal{P}(X)$. We consider the function $\pi_{\mathcal{F}} : \mathbb{N} \to \mathbb{N}$, defined by:

$$\pi_{\mathcal{F}}(n) = \max_{\substack{A \subseteq X \\ |A| \leq n}} |\mathcal{F} \cap A| \ .$$

Remark: $VC(\mathcal{F}) < n$ if and only if $\pi_{\mathcal{F}}(n) < 2^n$.

Theorem (Sauer–Shelah Lemma)

Let $\mathcal{F}$ be a family of sets with $VC(\mathcal{F}) \leq k$. Then, for $n \geq k$,

$$\pi_{\mathcal{F}}(n) \leq \sum_{i=0}^{k} \binom{n}{i} .$$

In particular, $\pi_{\mathcal{F}}(n) = O(n^k)$.

Definition

Let $\phi(\bar{x}, \bar{y})$ be a formula. We say that a set A is *shattered* by ϕ if there is a family $(\bar{b}_I)_{I \in \mathcal{P}(A)}$ such that for every $\bar{a} \in A$,

$$\mathfrak{C} \models \phi(\bar{a}, \bar{b}_I) \Leftrightarrow \bar{a} \in I \ .$$

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Definition (VC-dimension of a formula)

Let $\phi(\bar{x}, \bar{y})$ be a formula. The *VC-dimension of* ϕ is:

- The greatest natural number n such that there is a set of cardinality n shattered by ϕ, if such n exists;
- ∞, otherwise.

Definition (Independence Property)

A formula $\phi(\bar{x}, \bar{y})$ is said to have the *Independence Property* if $VC(\phi)$ is infinite.
A formula $\phi(\bar{x}, \bar{y})$ is said to be *NIP* (or *dependent*) if ϕ does not have the independence property.

Let $\phi(\bar{x}, \bar{y})$ be a formula. Given $\bar{b} \in \mathfrak{C}$, we define

$$\phi^{\mathfrak{C}}(\bar{x}, \bar{b}) := \{\bar{a} \in \mathfrak{C} : \ \mathfrak{C} \models \phi(\bar{a}, \bar{b})\} \ .$$

Remark

Let $\phi(\bar{x}, \bar{y})$ be a formula, and let $\mathcal{F} = \{\phi^{\mathfrak{C}}(\bar{x}, \bar{b}) : \bar{b} \in \mathfrak{C}\}$. Then

$$VC(\phi) = VC(\mathcal{F}) \ .$$

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Theorem (Shelah)

Let T be a theory, and assume every formula of T of the form $\phi(x, \bar{y})$ is NIP. Then T is NIP.

Definition (o-minimal theory)

A theory T is said to be *o-minimal* if:

- Models of T are linearly ordered;
- For every formula $\phi(x, \bar{b})$ and every $\bar{b} \in \mathfrak{C}$, the set $\phi^{\mathfrak{C}}(x, \bar{b})$ is a finite union of intervals.

Some examples of o-minimal theories:

- DLO;
- RCF;
- $Th(\mathbb{R}, +, -, \cdot, 0, 1, <, e^x)$;
- $\ldots$

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