Quasi-analytical solution of an investment problem with decreasing investment cost due to technological innovations

Cláudia Nunes[1*]
joint work with: Carlos Oliveira and Rita Pimentel

[*]CEMAT, Técnico Lisboa, Universidade de Lisboa

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Motivation

Optimal stopping problems: decide when to take some action, in order to maximize the expected value of the profit or to minimize the expected value loss.

American call options

- Call option
  - Contract between buyer and seller
  - Fixed maturity (T), underlying asset, and strike price (K)
  - Option is exercised if price stock is larger than K.
- American

The option may be exercised until maturity time T

When should the buyer exercise his option?
Not too late nor too early

Exit problem

- A firm is active in the market;
- The market is declining, and therefore the firm may decide to exit, paying some sunk cost.

Investment problem

- A firm is not yet active in the market;
- The market is in expansion, and therefore the firm may decide to invest, paying some sunk cost.
- Once in the market, due to technological innovations, the firm may invest in producing a new and more efficient product. When should the firm invest?

When should the firm exit the market?

When should the firm invest in the market?

Not too late nor too early

The QUESTION

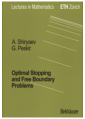
When should we take the decision, in order to maximize the return of our investment?

The ANSWER

Use optimal stopping approach to find such time

Classical references

- Peskir, Goran, and Albert Shiryaev. Optimal stopping and free-boundary problems. Birkhäuser Basel, 2006.



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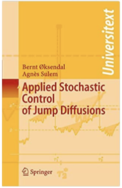

Alternative approaches

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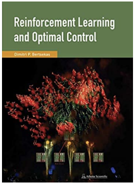
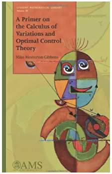
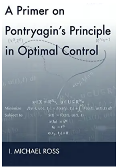
- Mesterton-Gibbons, Mike. A primer on the calculus of variations and optimal control theory. Vol. 50. American Mathematical Soc., 2009.
- Ross, I. Michael. A primer on Pontryagin's principle in optimal control. Collegiate Publ., 2009.

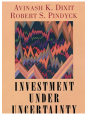






Applications

- Dixit, Avinash K., Robert K. Dixit, and Robert S. Pindyck. Investment under uncertainty. Princeton university press, 1994.



- Thompson, Gerald L. Optimal Control Theory: Applications to Management Science and Economics. Springer, 2006.
- Atkinson, Michael P., Zheng Cao, and Lawrence M. Wein. "Optimal stopping analysis of a radiation detection system to protect cities from a nuclear terrorist attack." Risk Analysis: An International Journal 28.2 (2008): 353-371.
- Mrázek, Pavel, and Mirko Navara. "Selection of optimal stopping time for nonlinear diffusion filtering." International Journal of Computer Vision 52.2-3 (2003): 189-203.

Value Function

Find V, with:

$$\boxed{V(x) = \sup_{\tau \in S} E_x \left[\int_0^{\tau} e^{-rs} G(X(s)) ds + e^{-r\tau} h(X(\tau)) \right]}$$

(inf, sup, with h or without... all the *same*) where:

- S: set of stopping times adapted to the filtration generated by the stochastic process X
- τ: stopping time
- e^{-rs}: killing factor
- G: running payoff, gain function
- h: terminal payoff

Examples

- Terminal reward

$$V(x) = \sup_{\tau \in \mathcal{S}} E_x \left[e^{-r\tau} G(X(\tau)) \right]$$

(e.g. American Option)

- Integral reward

$$V(x) = \sup_{\tau \in \mathcal{S}} E_x \left[\int_0^{\tau} e^{-rs} G(X(s)) ds - e^{-r\tau} I \right]$$

(e.g. Exit Option)

$$V(x) = \sup_{\tau \in \mathcal{S}} E_x \left[\int_{\tau}^{\infty} e^{-rs} G(X(s)) ds - e^{-r\tau} I \right]$$

(e.g. Investment Option)

Characterization of the Value Function

Without loss of generality, we use the following formulation:

$$\boxed{V(x) = \sup_{\tau \in S} E[G(X_\tau)]}$$

So the optimal stopping problem consists in finding the quantity V and the stopping time τ^* at which the supremum is attained (if τ^* exists...)
Assumption: $E\left[\sup_{t \geq 0} |G(X_t)| \, X_0 = x\right] < \infty, \forall x$.
Introduce two sets:

$$\text{continuation set } C = \{x : V(x) > G(x)\}$$
$$\text{stopping set } D = \{x : V(x) = G(x)\}$$

$$\boxed{\tau^* = \inf\{t \geq 0 : X_t \in D\}}$$

Ansatz

If the process X is **Markovian**, then we

- guess a candidate solution
- verify that it is indeed a solution

$$\begin{cases} \mathcal{L}_X V \geq 0 \\ V(x) > G(x), x \in C, V(x) = G(x), x \in D \end{cases}$$

Diffusion processes

If X is a diffusion process, then the above conditions are equivalent to:

$$\begin{cases} \mathcal{L}_X V(x) = 0 & x \in C \\ V(x) = G(x) & x \in D \\ \left.\frac{\partial V(x)}{\partial x}\right|_{\partial C} = \left.\frac{\partial G(x)}{\partial x}\right|_{\partial C} & \end{cases}$$

Questions:

- $\mathcal{L}_X$ is a differential operator, which means:

  We need to solve a differential equation

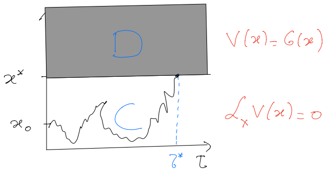
- We do not know C and D and even their shape.

  We need to propose C and D

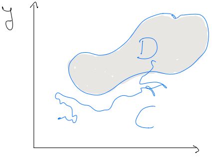
A free-boundary problem:

$$\min(\mathcal{L}_X V(x), V(x) - G(x)) = 0$$

- if X is an homogeneous 1-dimensional process, usually it is possible to find the solution (V and ∂C) explicitly, and then we verify that indeed it is a solution.



- But what if X has more dimensions? Can we solve the PDE explicitly? And in case we can, it is always the case that we can guess a shape for C and D? Can we always verify that what we have is the solution of the optimization problem?



Motivation

Solve an investment problem, with two sources of uncertainty:

- Price
- Investment cost

where the investment cost has sample paths with discontinuities due to exogeneous jumps driven by a Poisson process.
Special feaures:

- 2-dimensional problem
- in one of the dimensions we do not have a diffusion (we have a jump process)

Problem set-up

Model assumptions:

- Price of the product

$$dP_t = \mu P_t dt + \sigma P_t dW_t, \quad \text{with } P_0 = p > 0$$

- Investment cost
  - Case 1:

  $$I_t = I(1+\phi)^{N_t}, \quad \text{with} \quad I_0 = (I+1)\phi^n,$$

  (investment cost increases whenever we have a jump)
  - Case 2:

  $$I_t = I\phi^{N_t}, \quad \text{with} \quad I_0 = I\phi^n,$$

  (investment cost decreases whenever we have a jump)

with $N_t \sim Poi(\lambda t)$ and $\phi < 1$.

Investment Problem

$$V(p,n) = \sup_{\tau \geq 0} E_{p,n}\left[e^{-r\tau}\left(\frac{P_\tau}{r-\mu} - I_\tau\right)\right] \equiv E_{p,n}\left[e^{-r\tau} g_{N_\tau}(P_\tau)\right]$$

$g_n(p)$: expected perpetual return when the investment takes place with price p and the investment cost downsized n times.

(Some) Related literature

- Technology adoption: Huisman (2001), Hagspiel et al (2015)
- Two sources of uncertainty: Dixit and Pindyck (1994), Adkins and Paxson (2011)
- Jump-diffusion processes: Murto (2007), Nunes and Pimentel (2017)

HJB equations

In order to solve the Investment Problem, we need to:

- Solve the HJB equation:

$$\min(\mathcal{L}_{P,I}V(p,n), V(p,n) - g_n(p)) = 0$$

  where $\mathcal{L}_{P,I}$ is the infinitesimal generator of the two-dimensional process (P, I)

- Find the Investment Region (Stop region).

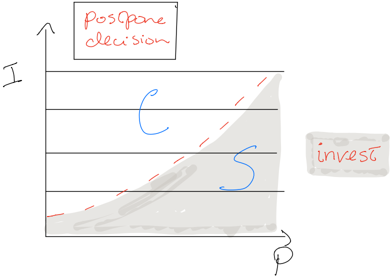
Differential-difference equation

In the continuation region, the following equation must hold

$$(\lambda + r)V(p,n) - \mu p V'(p,n) - \frac{\sigma^2}{2} p^2 V''(p,n) = \lambda V(p,n+1)$$

How to solve this equation?

Investment region



Investment should occur when the price is high and the investment cost is low.

But where is the boundary between continuation and stopping?

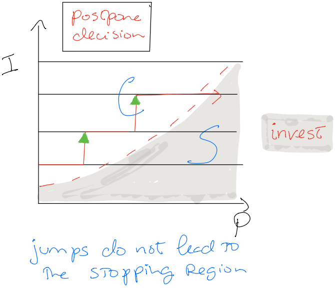
Explicit solutions of the differential equation

Case 1

$$I_t = I(1+\phi)^{N_t}, \quad \text{with} \quad I_0 = (I+1)\phi^n,$$

(investment cost increases whenever we have a jump)

- We stay in the waiting/continuation region whenever a jumps occurs.
- Thus the investment region can only be hitted due to an increase in the price.



Case 1

$$I_t = I(1+\phi)^{N_t}, \quad \text{with} \quad I_0 = (I+1)\phi^n,$$

- If the running function g is homogeneous:

$$g_n(p) = n^{\alpha} h(p), \quad \alpha \in \mathbb{N}$$

then

one may transform the problem in a one-dimensional one, with state variable

$$Q = I^{\alpha} P$$

In this case, in order to solve the problem, we proceed as follows:

- Apply Itô's formula to compute $dQ(x)$ and find $\mathcal{L}_Q$
- Solve the 1-dimensional differential equation $\mathcal{L}_Q V(q) = 0$ (analitically)
- Guess the waiting and stopping region
- Prove the HJB equations
- Go back to the original processes, P and I.

Solution of the Optimal Stopping Time Problem

$$\tau^\star = \inf\{t > 0 : I_t^\alpha P_t \geq A\}$$

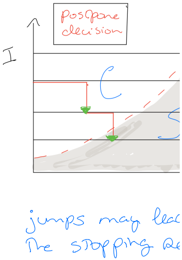
(Nunes and Pimentel. *Analytical solution for an investment problem under uncertainties with shocks.* European Journal of Operational Research 259.3 (2017): 1054-1063.)

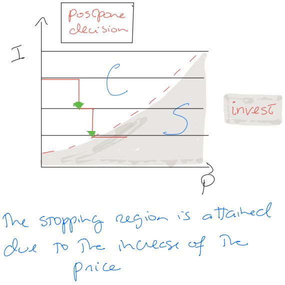
Case 2

$$I_t = I\phi^{N_t}, \quad \text{with} \quad I_0 = I\phi^n,$$

(investment cost decreases whenever we have a jump)

- A jump may lead to the stopping region (crossing the boundary)
- But the stopping region may also be attained by an increase in the price (hitting the boundary)





In the continuation region, the following equation must hold

$$(\lambda + r)V(p,n) - \mu p V'(p,n) - \frac{\sigma^2}{2} p^2 V''(p,n) = \lambda V(p,n+1)$$

Difficulty: the solution in the state (p, n) depends on the solution of the same equation in the state $(p, n+1)$.

- $(p, n+1)$ is still in the continuation region: $(\lambda + r)V(p,n) - \mu p V'(p,n) - \frac{\sigma^2}{2} p^2 V''(p,n) = \lambda V(p,n+1)$
- $(p, n+1)$ is in the stopping region, and therefore it is optimal to invest. $(\lambda + r)V(p,n) - \mu p V'(p,n) - \frac{\sigma^2}{2} p^2 V''(p,n) = \lambda V(p,n+1)$

So the value of the firm at state (p, n) depends recursively on all the levels above $(n+1, n+2, \ldots)$.

Quasi-Analytical Solution

- Analytical solution of the problem is not known;
- Numerical solutions: few proposals

Here we propose an approach that converges to the analytical solution, based on

Truncation of the problem

The investment time is bounded by a random time

$$V^{\overline{n}}(p,n) = \sup_{0 \le \tau \le \tau_{\overline{n}}} E_{p,n}\left[e^{-r\tau} g_{N_\tau}(P_\tau)\right]$$

$\tau_{\overline{n}}$ is the time of the $\overline{n}$th arrival, where $\overline{n}$ is the maximum number of allowed jumps. We assume that

If no decision is taken until $\tau_{\overline{n}}$, then this time will be the optimal investment time.

How do we proceed?

- Fix $\overline{n}$ *large*;
- Solve $V^{\overline{n}}(p,n)$

When $\overline{n} \to +\infty$:

- $\tau_{\overline{n}} \nearrow +\infty$;
- the solution of the truncated problem also converges to the solution of the original problem

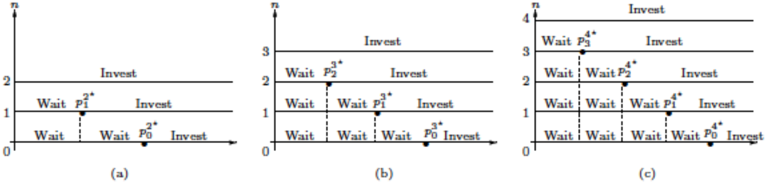


Figure 1: Waiting and investment regions when (a) $\overline{n} = 2$, (b) $\overline{n} = 3$ and (c) $\overline{n} = 4$.

$p_n^{\overline{n}\star}$ is the trigger price for the investment when the maximum number of jumps of innovations is $\overline{n}$ and n jumps have already occurred.

Equations and boundary conditions for the truncated problem

- In the continuation region, for $n = 0, 1, 2, \cdots, \overline{n} - 1$:

$$(r+\lambda) v_n^{\overline{n}}(p) - \mu p (v_n^{\overline{n}})'(p) - \frac{\sigma^2}{2} p^2 (v_n^{\overline{n}})''(p) - \lambda v_{n+1}^{\overline{n}}(p) = 0$$

- In the stopping region: $v_n^{\overline{n}}(p) = g_n(p)$;
- $v_{\overline{n}}^{\overline{n}}(p) = g_{\overline{n}}(p)$
- $v_n^{\overline{n}}(p)$ is of class C^1 for all p and $n \leq \overline{n}$

The case $\overline{n} = 2$

This case can be handled:

- Explicitly, not too messy
- The general case follows along similar lines

How does it work?

Backwards!

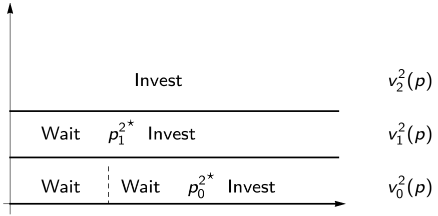


Figure: Waiting and investment regions when $\overline{n} = 2$

1. $v_2^2(p) = \frac{p}{r-\mu} - I\phi^2$: value of the investment in perpetuity
2. $$v_1^2(p) = \begin{cases} A_{1,0,0}^2 p^{d_1} + \lambda \left[\frac{p}{(r-\mu)(r+\lambda-\mu)} - \frac{I\phi^2}{r+\lambda} \right], & 0 < p < p_1^{2^\star} \\ \frac{p}{r-\mu} - I\phi, & p \geq p_1^{2^\star} \end{cases} :$$
   value function of a standard investment problem

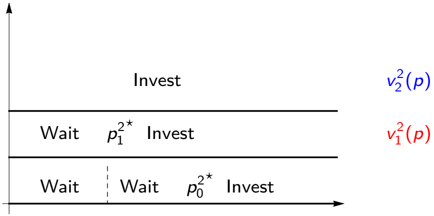


Figure: Waiting and investment regions when $\overline{n} = 2$

1. $v_0^2(p)$:
   1. for $0 < p < p_1^{2^\star}$, either we wait for 2 innovations $\left(\left(\frac{\lambda}{r+\lambda}\right)^2 \left[\frac{p(r+\lambda)^2}{(r-\mu)(r+\lambda-\mu)^2} - I\phi^2\right]\right)$ or 1 innovation followed by an increase of the price $(A_{0,0,1}^2 \ln p p^{d_1})$, or no innovations $(A_{0,0,0}^2 p^{d_1})$;
   2. for $p_1^{2^\star} \leq p \leq p_0^{2^\star}$, either we need one innovation $\left(\left(\frac{\lambda}{r+\lambda}\right) \left[\frac{p(r+\lambda)}{(r-\mu)(r+\lambda-\mu)} - I\phi\right]\right)$ or we have just variation in the price (that may go up - positive root, or go down, negative root).

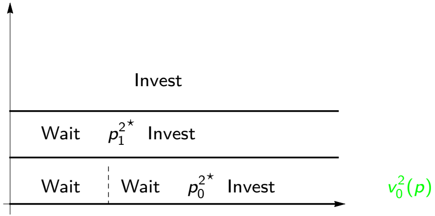


Figure: Waiting and investment regions when $\overline{n} = 2$

All the constants and thresholds can be derived analytically, except $p_0^{2^\star}$

General $\overline{n}$

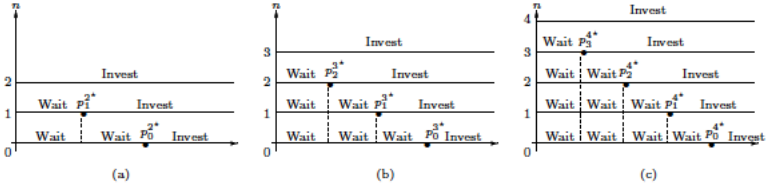


Figure 1: Waiting and investment regions when (a) $\overline{n}=2$, (b) $\overline{n}=3$ and (c) $\overline{n}=4$.

$$v_n^{\overline{n}}(p) = \begin{cases} v_{n,k}^{\overline{n}}(p) & p_{\overline{n}-k}^{\overline{n}^\star} \le p < p_{\overline{n}-1-k}^{\overline{n}^\star}, \ k = 0, 1, 2, \ldots, \overline{n}-1-n \\ \frac{p}{r-\mu} - I\phi^n & p > p_n^{\overline{n}^\star} \end{cases}$$

$$v^{\bar{n}}_{n,k}(p) = \overbrace{\sum_{j=0}^{\bar{n}-1-(n+k)} A^{\bar{n}}_{n,k,j} \ln p^j \; p^{d_1} + B^{\bar{n}}_{n,k,j} \ln p^j \; p^{d_2}}^{\text{different ways that the investment may occur due to an increase of the price}}$$

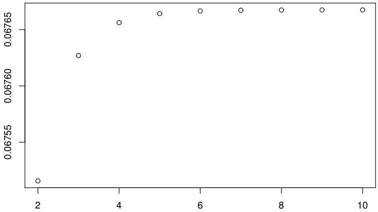
$$+ \underbrace{\left(\frac{\lambda}{r+\lambda}\right)^{\bar{n}-(n+k)} \left[\frac{p(r+\lambda)^{\bar{n}-(n+k)}}{(r-\mu)(r+\lambda-\mu)^{\bar{n}-(n+k)}} - I\phi^{\bar{n}-(n+k)}\right]}_{\text{perpetual value of investment due to } \bar{n}-(n+k) \text{ jumps}}$$

Convergence result

The solution of the truncated problem converges to the solution of the original one , i.e.,

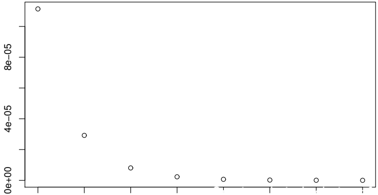
$$\lim_{\overline{n}\to\infty} V^{\overline{n}}(p,n) = V(p,n) \qquad \forall (p,n) \in]0,+\infty[\times\mathbb{N}.$$

r=0.05	$\sigma = 0.1$	$\mu = 0.03$	$\lambda = 0.1$	$\phi = 0.9$	$I = 1$



$p_0^{\bar{n}*}$

$p_0^{\bar{n}*} - p_0^{\bar{n}-1*}$



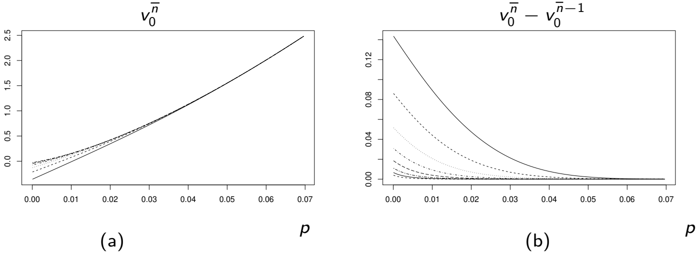
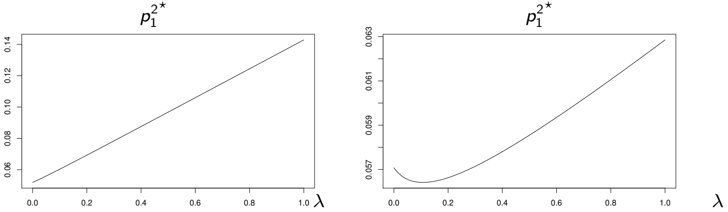


Figure: (a) Plot of the functions $v_0^{\overline{n}}$, for $\overline{n} = 2, \cdots, 7$ where the functions appear in an increasing way in $\overline{n}$. (b). Plot of the functions $v_0^{\overline{n}} - v_0^{\overline{n}-1}$, for $\overline{n} = 3, \cdots, 10$, where the functions appear in a decreasing way in $\overline{n}$.

Analytical results for $p_1^{2^\star}$ $(\overline{n} = 2)$

Proposition

- *Increases with σ and decreases with μ (agrees with the standard result);*
- *Increases with λ for "small values" of ϕ but has a non-monotonic behavior with λ for "large values" of ϕ*
- *Increases with ϕ if $r \geq \lambda$; in case $r < \lambda$, it increases with ϕ if $\phi < \frac{1}{2}\frac{r+\lambda}{\lambda}$ and decreases afterwards.*



(a)

(b)

Figure: Investment threshold $p_1^{2^\star}$ as a function of λ, for (a) $\phi = 0.9$ and (b) $\phi = 0.99$.

In the above example, for a larger number of jumps, the trigger price is smaller than for a smaller number of jumps.
Intuition: after some downwarding shocks in the investment cost, the investor is willing to invest even if the price is not so large

Can we find situations where the opposite hold? And if so, the way to solve is fundamental different?

How do we fix $\overline{n}$?

In some problems, we may have a hint regarding this truncation level

Would it be possible to construct the truncation process in such a way that once the upper bound $\bar{n}$ is reached, investment does not occur?

Yes, but the calculations would be quite different and probably more evolved. But, on the other hand, we would recover some economical interpretation.

What if the jump size is a random variable, taking values in $[0, k]$?

This would mean that once a jump occurs, it can increase the investment cost or it can decrease it (mix of the two cases). Interesting but very challenging!

Thank you for your attention!