# Homotopical dynamics: suspension and duality

## LISMATH SEMINAR

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# Introduction

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In this presentation we will follow a paper [1] by Octavian Cornea in which he studies attractor-repellor pairs.
Here, we will apply homotopy theory in order to study dynamical systems, more particulary, smooth flows.
The thing we want to reach is that the connection maps [2] associated to an attractor-repellor decomposition with respect to both the flow and the inverse flow are Spanier-Whitehead duals.

# Cone and suspension

Let $X$ be a topological space.

**Definition**

*We call $CX$ the cone of $X$ to the space $X \times I/(x,0) \sim (y,0)$.*

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**Definition**

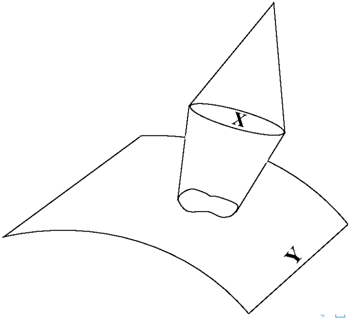
*We call $\Sigma X$ the suspension of $X$ to the space $X \times I/((x,0) \sim (y,0),(x,1) \sim (y,1))$*

# Mapping Cone

Let $f : X \to Y$ be a continuous map between topological spaces.

## Definition

*We call $C_f$ the cone of the map $f$ to the space*
$Y \coprod X \times I/((x,0) \sim (y,0),(x,1) \sim f(x))$



## Cone length

Let $X$ be a path connected space.

**Definition**

*The cone length of $X$, $cl(X)$ is the smallest $n$ such that there are cofibration sequences $Z_{i-1} \to Y_{i-1} \to Y_i$ such that $Y_0$ is contractible and $Y_n$ homotopically equivalent to $X$.*

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The cone length is a topological invariant of a space, but in general is very hard to compute. One way to estimate from above this value is to restrict $Z_i$ to some family of spaces, two invariants that come out of this are $cl_\Sigma(X)$, where we demand $Z_i$ to be an $i-th$ suspension and $cl_S(X)$, where we demand $Z_i$ to be a wedge of spheres.

# LS-Category

**Proposition ([3])**

*Let $M$ be a smooth manifold. For $k$ large enough, we can construct functions on $M \times D^k$ which point inward in the boundary with no more than $Cat(M) + 2$ critical points.*

# LS-Category

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**Definition (Lusternik–Schnirelmann category)**

*Given a manifold $M$, we call $Cat(M)$ to the smallest integer $k$ such that there is an open covering $\bigcup_0^k U_i = M$ such that each open ser in the covering is contractible in $M$.*

# LS-Category

We can get a lower bound of $Cat(M) + 1$ for the inequality above if we can show that the LS-category agrees with the cone length.[4]

**Conjeture**

*For a closed manifold $M$, $Cat(M) = cl(M)$.*

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This conjecture cannot be extended to CW-complexes, and counter examples have been constructed. The inequality $Cat(M) \leq cl(M)$ is already known to hold.
In the cases of $Cat(M) = 1$ and $cl_{\Sigma}(M) = 1$, both are equivalent to the space being, up to homotopical equivalence, a suspension, so the conjecture holds.

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# Spanier-Whitehead duality

## Definition

*Two CW-complexes $X$ and $X'$ are said to be Spanier-Whitehead $m-$duals if there is a map $\mu : X \wedge X' \to S^m$ such that the slant product by $\mu^*(i_m)^*$, $/ : H_q(X') \to H^{m-q}(X)$ gives us an isomorphism.*

The slant product is a product that takes a cohomology class in the product and an homology class in one of the spaces and returns a cohomology class in the other space.
Note: The slant product is usually defined from $H^m(X \times X') \times H_q(X') \to H^{m-q}(X)$, so here we are first pulling back the cohomology class from the wedge to a cohomology class in the product.

## Spanier-Whitehead duality

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We can then define duality between two maps. Assume $(X, X')$ and $(Y, Y')$ are two pairs of $m-$duals, by maps $\mu$ and $\nu$, respectively. We say that two maps $f : X \to Y$ and $g : Y' \to X'$ are duals if for some $k, q$ the maps $\mu(1 \wedge g)$ and $\nu(f \wedge 1)$ give the same map $\eta : \Sigma^k X \wedge \Sigma^q Y' \to S^{m+k+q}$.

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If we allow $k$ and $q$ to be large enough, we can get a duality isomorphism between (stable) classes of maps.

## Connected simple systems

Let $\mathrm{Ho}(\mathcal{T})$ be the homotopy category of topological spaces.
A connected simple system is a subcategory $X$ such that given any two objects $X_1, X_2$ in $X$ the set $mor_X(X_1, X_2)$ has exactly one element. This element is an homotopy class of maps from $X_1$ to $X_2$. We call the maps in this class comparison maps.

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**Proposition**

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**Proposition**

*All comparison maps are homotopy equivalences.*

We can note that given connected simple systems $X$ and $Y$, one object in each, $A \in Ob(X)$ and $B \in Ob(Y)$ and a map $f_{A,B} : A \to B$ we have a morphism of connected simple systems defined by the homotopy class of $f_{A,B}$.

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Connected simple systems will also form a category which we will call $\mathcal{CS}$.

# Cofibration sequences

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When we have a cofibration $f : X \to Y$, we can construct a cofibration sequence $X \xrightarrow{f} Y \xrightarrow{i} C_f \to C_i$. One can prove that $C_i$ is homotopically equivalent to $\Sigma X$. We call the map $\delta : C_f \to \Sigma X$ the connection map.

# Cofibration sequences

We can now to define the funtor $\Sigma : \mathcal{CS} \to \mathcal{CS}$ by using the suspension of spaces and maps that are in the category. Let $X$, $Y$ and $Z$ be objects of $\mathcal{CS}$.

## Definition (Cofibration sequence of connected simple systems)

*A cofibration sequence of connected simple systems is a triple of morphisms between objects in* $X \xrightarrow{i} Y \xrightarrow{p} Z \xrightarrow{\delta} \Sigma X$ *such that, there exists* $X_0 \in Ob(X)$, $Y_0 \in Ob(Y)$, $Z_0 \in Ob(Z)$ *and maps* $i_0$ *and* $p_0$ *such that* $X_0 \xrightarrow{i_0} Y_0 \xrightarrow{p_0} Z_0$ *is a cofibration sequence, with* $[i_0] = i$ *and* $[p_0] = p$. *If the map* $\delta_0 : Z_0 \to \Sigma X_0$ *is the connecting map,* $[\delta_0] = \delta$.

# Duality in connected simple systems

Consider the connected simple system given by the space $S^m$. Let $x, X' \in Ob(\mathcal{CS})$. We can define the smash product $X \wedge X'$ by doing it for all $X_0$ and $X_0'$ spaces in each category.

**Definition (S-W dual connected simple systems)**

*We say that two connected simple systems $X, X'$ are S-W duals if there is a morphism from $X \wedge X'$ to the connected simple system $S^m$ such that all maps from $X_0 \wedge X_0'$ to $S^m$ are duality maps.*

# Duality in connected simple systems

We can do a similar definition to morphisms, given $X, X'$ and $Y, Y'$ pairs of duals, with morphisms $f : X \to Y$ and $g : Y' \to X'$. We say that $f$ and $g$ are duals if for some $k, k'$ we have that every instance of $f$ and $g$ for pairs $X_0, X_0'$, $Y_0, Y_0'$ are duals.

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We say that the cofibration sequences $X \xrightarrow{i} Y \xrightarrow{p} Z \xrightarrow{\delta} \Sigma X$ and $Z' \xrightarrow{p'} Y' \xrightarrow{i'} X' \xrightarrow{\delta'} \Sigma Z'$ are duals if the three pairs of morphisms are duals.

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# Attractor-Repellor pair

Assume we are working on a closed $m-$dimensional manifold $M$ and have a Morse-Smale function $f : M \to \mathbb{R}$. We will call $\gamma$ the gradient flow (and $-\gamma$ the inverse flow). Let $S$ be a compact set invariant by the flow.

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## Definition

*We call $A \subset S$ an attractor if there is an open set(in $S$) $U$ such that $A \subset U \subset S$ and $A$ is the $\omega-$limit set of $U$.*
*We call $A^*$ to the set of points whose orbit does not intersect $A$.*
*We will call $(S, A, A^*)$ an Attractor-Repellor pair.*

# Attractor-Repellor pair

This choice of $A^*$ leads to the following result.

**Proposition**

*Let $(S, A, A^*)$ be an A-R pair. For any $B \subset S$, closed and disjoint from $A$, $\forall_{\epsilon>0}\exists t_0$ such that for all $x \in S$ and $t \geq t_0$, if $\gamma_t(x) \in B$, then $d(x, A^*) < \epsilon$.*

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This means that we can characterize $A^*$ as an $\alpha-$limit set of $S \backslash U$, where this $U$ is the one used to define $A$.

# Conley index theory

Let $N$ be a compact metric space. Let $\gamma : N \times \mathbb{R} \to N$ be a continuous flow and let $S \subset N$ be an isolated invariant set.

**Definition (Index pair[5])**

*A pair $(N_1, N_0)$ of compact sets in $N$ is an index pair for $S$ in $N$ if $N_0 \subset N_1$, $N_1 \backslash N_0$ is a neighbourhood of $S$, $S$ is the maximal invariant set in the closure of $N_1 \backslash N_0$, $N_0$ is positively invariant in $N_1$ and, if for $x \in N_1$ there is a $t \geq 0$ such that $\gamma(x, t) \notin N_1$, then there is a $t_0$ such that the flow is in $N_1$ up to time $t_0$ and in $N_0$ at time $t_0$.*

# Conley index theory

**Theorem**

*Every isolating neighbourhood of $S$ has an index pair.*

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We also have that for two index pairs of $S$, $(N_1, N_0)$ and $(N_1', N_0')$ there are comparison maps $N_1/N_0 \to N_1'/N_0'$. The choice of the map is not canonical, if interested we can use the map from Lemma 4.7 in [5].

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This gives the set of quotients $N_1/N_0$ the structure of a connected simple system, which we will denote by $c_\gamma(S)$. We will call it the Conley index of S with respect to $\gamma$.

# Conley index theory for attractor-repellor decomposition

Let's assume we have an attractor-repellor decomposition $(S; A, A^*)$.

## Definition

*We call $c_\gamma(S; A, A^*)$ to the cofibration sequence of simple systems $c_\gamma(A) \xrightarrow{i} c_\gamma(S) \xrightarrow{p} c_\gamma(A^*) \xrightarrow{\delta} \Sigma c_\gamma(A)$. The maps are defined by the flow and we call $\delta$ the connection map of a given attractor-repellor decomposition.*

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We can represent this cofibration sequence given any triple $N_0 \subset N_1 \subset N_2$ such that $(N_2, N_1)$ is an index pair of $A$, $(N_2, N_0)$ of $S$ and $(N_1, N_0)$ of $A^*$ by $N_1/N_0 \to N_2/N_0 \to N_2/N_1$.

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From now on we will use the notation $I(M)$ to represent the maximal invariant set inside $M$.

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# Lift of a flow

Let $\psi : F \to E \xrightarrow{p} B$ be a locally trivial fibration of manifolds. This means that $p : E \to B$ satisfies the homotopy lifting property. Assume that both $F$ and $B$ are compact.

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Let $\psi : F \to E \xrightarrow{p} B$ be a locally trivial fibration of manifolds. This means that $p : E \to B$ satisfies the homotopy lifting property. Assume that both $F$ and $B$ are compact.
Let $\gamma$ be a flow on $B$, we define $\gamma'$ as follows:

**Definition**

*We say that a flow on $E$ $\gamma'$ is a lift of $\gamma$ if for all $x \in E$ and all $t \in \mathbb{R}$ we have that $p(\gamma'(x,t)) = \gamma(p(x),t)$.*

# Thom index

Given $\psi : F \to E \xrightarrow{p} B$, $\gamma : B \times \mathbb{R} \to B$

## Lemma

*Let $S$ be an isolated invariant set of $\gamma$ and $\gamma'$ be a lift of $\gamma$ to $E$. Then $S' = p^{-1}(S)$ is an isolated invariant set of $\gamma'$ and there is an induced morphism $c_{\gamma'}(S') \to c_{\gamma}(S)$ that depends only on $\psi$ and $\gamma$. This morphism is natural with respect to attractor-repellor pairs.*

# Thom index

## Definition

*We will define $\bar{c}_\gamma^\psi(S)$ as the connected simple system containing as objects the mapping cones of the projections for each index pair of $S$, this means, given an index pair $(N_1, N_2)$ of $S$, we consider the index pair of $S'$ and we will get a projection. The objects will be the mapping cones of the projections. The morphisms are induced by the comparison maps.*

# Cofibration sequences of Thom indexes

Let $(S; A, A^*)$ be an attractor-repellor decomposition.

**Lemma**

*Given the context of the lemma above, there is a cofibration sequence of connected simple systems given by*
$\bar{c}^{\psi}_{\gamma}(A) \to \bar{c}^{\psi}_{\gamma}(S) \to \bar{c}^{\psi}_{\gamma}(A^*) \to \Sigma\bar{c}^{\psi}_{\gamma}(A)$

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Recall now we we have 2 different attractor-reppelor cofibration sequences, one associated only to the flow $c_\gamma(S; A, A^*)$, and the other, associated also to a locally trivial fibration, $\bar{c}^{\psi}_{\gamma}(S; A, A^*)$.

# Particular scenario

Now, we will pass our study to a more particular set.
Consider a Riemannian fiber bundle $\eta : \mathbb{R}^n \to T \to B$. Let $\gamma$ be a flow on $B$ with lift $\gamma'$ to $T$.

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Now, we will pass our study to a more particular set.
Consider a Riemannian fiber bundle $\eta : \mathbb{R}^n \to T \to B$. Let $\gamma$ be a flow on $B$ with lift $\gamma'$ to $T$.
Assume we have a quadratic form $q : \mathbb{R}^n \to \mathbb{R}$ compatible with the metric.
Assume also that there exist local charts $U_i \subset B$ where the restriction of $\eta$ is trivial, so the vector field associated with $\gamma'$ splits in the direct sum of the vector field associated with $\gamma$ and $-\nabla q$.

## Particular scenario

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Consider the associated sphere bundle of $\eta : S(\eta) \xrightarrow{p} B$. Denote by $e(q)$ the fibration $e(q) \to B$ whose total space is the subset of $S(\eta)$ where $q$ is non-positive and whose projection is the restriction of $p$.

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# Thom index isomorphism

**Theorem**

*If $(S; A, A^*)$ is an attractor-repellor decomposition, $c_{\gamma'}(S; A, A^*)$ is isomorphic to $\bar{c}_{\gamma}^{e(q)}(S; A, A^*)$.*

This theorem shows that both the Thom index and the cofibration sequence are fully determined by $\gamma'$.

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**Corollary**

*Let $k$ be the index of the quadratic form $q$. Then we have that $\bar{c}_\gamma^{e(q)}(S;A,A^*) \simeq c_{\gamma'}(S;A,A^*) \simeq \Sigma^k c_\gamma(S;A,A^*)$*

# Duality

Let $B$ be a smooth compact manifold of dimension $n$ embedded on a sphere $S^{n+k}$. Let $\eta$ be the normal bundle of $B$ with a Riemannian metric associated to the total space. Let $S(\eta)$ be the unit sphere bundle. Let $\gamma$ be a flow on $B$.

# Duality

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**Theorem**

*The cofibration sequences $\bar{c}_{\gamma}^{S(\eta)}(S; A, A^*)$ and $c_{-\gamma}(S; A, A^*)$ are Spanier-Whitehead duals by a duality map that depends only on the embedding of $B$ in $S^{n+k}$.*

# Duality

## Proof

This proof can be simplified in four steps. The first one consists in reducing the problem to a flow on $S^{n+k}$. The next step is the construction of some index pairs. Next we prove the duality result for the manifolds in the index pairs and for last we prove for connected simple systems.

# duality

**Corollary**

*We conclude that $\bar{\delta}$ and $\delta^*$ are Spanier-Whitehead duals. In the case that $\eta$ is locally trivial, we also conclude that $\delta$ and $\delta^*$ are duals.*

For the case where $\eta$ is non-trivial, it is possible to find some similar results, where $\delta$ and $\delta^*$ will be Spanier-Whitehead duals modulo some twisting class.

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