Some Results on the Foundations of Elementary Plane Euclidean and Non-Euclidean Geometries

Pedro Miguel dos Santos Pinto

Lisbon Mathematics PhD Seminar, March 20, 2015

LISBOA
UNIVERSIDADE DE LISBOA

Outline

1. Some History...

Outline

1. Some History...
2. Elementary Geometries
   - Hilbert planes
   - Pythagorean planes
   - Euclidean planes
   - Tarski's elementary Geometry

Outline

Objective

In this presentation we will see some important metamathematical results about some elementary axiom systems of plane geometry.

Objective

In this presentation we will see some important metamathematical results about some elementary axiom systems of plane geometry.

By "*elementary*" one means, in the words of Alfred Tarski:

Objective

In this presentation we will see some important metamathematical results about some elementary axiom systems of plane geometry.

By "*elementary*" one means, in the words of Alfred Tarski:

"(...) that part of Euclidean geometry which can be formulated and established without the help of any set-theoretical devices." [4]

Some History

Some History

Euclid's *Elements* was the first axiomatic presentation of mathematics, based on his five postulates and "common notions".

Some History

Euclid's *Elements* was the first axiomatic presentation of mathematics, based on his five postulates and "common notions". In 1899, David Hilbert presented a rigorous revision of Euclid's axiomatics, filling in the many gaps in his definitions and proofs.

Some History

Euclid's *Elements* was the first axiomatic presentation of mathematics, based on his five postulates and "common notions". In 1899, David Hilbert presented a rigorous revision of Euclid's axiomatics, filling in the many gaps in his definitions and proofs.

Researchers who succeeded Hilbert accomplished the disengagement of elementary geometry from the system of real numbers:

Some History

Euclid's *Elements* was the first axiomatic presentation of mathematics, based on his five postulates and "common notions". In 1899, David Hilbert presented a rigorous revision of Euclid's axiomatics, filling in the many gaps in his definitions and proofs.

Researchers who succeeded Hilbert accomplished the disengagement of elementary geometry from the system of real numbers:

"*Elementary Euclidean geometry is a much more ancient and simple subject than the axiomatic theory of real numbers (...)*" [1]

Some History

Euclid's *Elements* was the first axiomatic presentation of mathematics, based on his five postulates and "common notions". In 1899, David Hilbert presented a rigorous revision of Euclid's axiomatics, filling in the many gaps in his definitions and proofs.

Researchers who succeeded Hilbert accomplished the disengagement of elementary geometry from the system of real numbers:

"*Elementary Euclidean geometry is a much more ancient and simple subject than the axiomatic theory of real numbers (...)*" [1]

As Robin Hartshorn [3] said, [without real numbers] "*the true essence of geometry can develop most naturally and economically.*"

Hilbert planes

Hilbert planes

In David Hilbert's *Grundlagen der Geometrie* ("Foundations of Geometry"), published in 1899, Hilbert provided axioms reparing the many gaps in Euclid's proofs.

Hilbert planes

In David Hilbert's *Grundlagen der Geometrie* ("Foundations of Geometry"), published in 1899, Hilbert provided axioms reparing the many gaps in Euclid's proofs.

Hilbert worked with a two-sorted first-order language (first-order variables are used to denote "points" and "lines"), a relation of betweenness, a relation of congruence and a relation of incidence (to mean that a point x lies on a line ℓ).

Hilbert planes

In David Hilbert's *Grundlagen der Geometrie* ("Foundations of Geometry"), published in 1899, Hilbert provided axioms reparing the many gaps in Euclid's proofs.

Hilbert worked with a two-sorted first-order language (first-order variables are used to denote "points" and "lines"), a relation of betweenness, a relation of congruence and a relation of incidence (to mean that a point x lies on a line ℓ).

Five groups of axioms:

- I Incidence
- II Betweenness (or Order)
- III Congruence
- IV Parallelism axiom
- V Continuity axiom

Models of the axioms in I, II and III are called **Hilbert planes**.

Models of the axioms in I, II and III are called **Hilbert planes**.

The study of Hilbert planes allows for the study of straightedge and compass constructions in plane geometry (Greenberg [1]).

Pythagorean planes

Pythagorean planes

The axioms for a Hilbert plane eliminate the possibility that there are no parallels at all – given a line ℓ and a point P not on ℓ, the "standard construction" proves that exists a line m passing through P parallel to ℓ.

Pythagorean planes

The axioms for a Hilbert plane eliminate the possibility that there are no parallels at all – given a line ℓ and a point P not on ℓ, the "standard construction" proves that exists a line m passing through P parallel to ℓ.
Now include a new axiom (by John Playfair):

Pythagorean planes

The axioms for a Hilbert plane eliminate the possibility that there are no parallels at all – given a line ℓ and a point P not on ℓ, the "standard construction" proves that exists a line m passing through P parallel to ℓ.

Now include a new axiom (by John Playfair):

HILBERT'S EUCLIDEAN AXIOM OF PARALLELS (IV)

For every line ℓ and every point P not on ℓ, there does not exist more than one line through P parallel to ℓ.

Pythagorean planes

The axioms for a Hilbert plane eliminate the possibility that there are no parallels at all – given a line ℓ and a point P not on ℓ, the "standard construction" proves that exists a line m passing through P parallel to ℓ.

Now include a new axiom (by John Playfair):

HILBERT'S EUCLIDEAN AXIOM OF PARALLELS (IV)

For every line ℓ and every point P not on ℓ, there does not exist more than one line through P parallel to ℓ.

For Hilbert planes, this axiom is equivalent to Euclid's fifth postulate.

Models of I, II, III and IV are called **Pythagorean planes**.

Models of I, II, III and IV are called **Pythagorean planes**.
The Pythagorean Theorem holds for all right triangles in a Pythagorean plane – because the theory of similar triangles holds.

Models of I, II, III and IV are called **Pythagorean planes**.
The Pythagorean Theorem holds for all right triangles in a Pythagorean plane – because the theory of similar triangles holds.

If $\mathcal{P}$ is a Pythagorean plane then it's possible, by fixing a line ℓ and a point "O" on that line, to define an ordered field F.
That field will be called a Pythagorean field since for all $a, b \in F$ $\sqrt{a^2 + b^2} \in F$.

Models of I, II, III and IV are called **Pythagorean planes**.
The Pythagorean Theorem holds for all right triangles in a Pythagorean plane – because the theory of similar triangles holds.

If $\mathcal{P}$ is a Pythagorean plane then it's possible, by fixing a line ℓ and a point "O" on that line, to define an ordered field F.
That field will be called a Pythagorean field since for all $a, b \in F$ $\sqrt{a^2 + b^2} \in F$.

If we now consider the Cartesian plane F^2, we will have $F^2 \simeq \mathcal{P}$.

Euclidean planes

Euclidean planes

Some results of the plane geometry in Euclid's *Elements* may still fail in Pythagorean planes.
Consider the following statements:

Euclidean planes

Some results of the plane geometry in Euclid's *Elements* may still fail in Pythagorean planes.
Consider the following statements:

TRIANGLE THEOREM (EUCLID I.22)

Given three segments such that the sum of any two is greater than the third, a triangle can be constructed having its sides congruent to those segments.

Euclidean planes

Some results of the plane geometry in Euclid's *Elements* may still fail in Pythagorean planes.
Consider the following statements:

TRIANGLE THEOREM (EUCLID I.22)

Given three segments such that the sum of any two is greater than the third, a triangle can be constructed having its sides congruent to those segments.

LINE-CIRCLE AXIOM

If a line passes through a point inside a circle, then it intersects the circle in two distinct point.

CIRCLE-CIRCLE AXIOM

If a circle passes through a point inside and a point outside another circle, then the two circles intersect in two distinct points.

CIRCLE-CIRCLE AXIOM

If a circle passes through a point inside and a point outside another circle, then the two circles intersect in two distinct points.

- The Line-Circle and the Circle-Circle axioms are examples of *elementary* continuity axioms – elementary in the sense that they only refer to lines and/or circles.

CIRCLE-CIRCLE AXIOM

If a circle passes through a point inside and a point outside another circle, then the two circles intersect in two distinct points.

• The Line-Circle and the Circle-Circle axioms are examples of *elementary* continuity axioms – elementary in the sense that they only refer to lines and/or circles.

• In his very first proposition I.1 – the construction of an equilateral triangle on any base –, Euclid makes use of the Circle-Circle continuity:

CIRCLE-CIRCLE AXIOM

If a circle passes through a point inside and a point outside another circle, then the two circles intersect in two distinct points.

- The Line-Circle and the Circle-Circle axioms are examples of *elementary* continuity axioms – elementary in the sense that they only refer to lines and/or circles.

- In his very first proposition I.1 – the construction of an equilateral triangle on any base –, Euclid makes use of the Circle-Circle continuity:

Euclid I.1

Euclid I.1 does not holds in all Hilbert planes.
(It does, however, holds in all Pythagorean planes, by a different construction, using the fact that the Pythagorean field $F \ni \sqrt{3}$.)

In Hilbert planes, Circle-Circle, Line-Circle and the Triangle Theorem are all equivalent.

In Hilbert planes, Circle-Circle, Line-Circle and the Triangle Theorem are all equivalent.

Any model of I, II, III, IV and the Circle-Circle Axiom is called an **Euclidean plane**.

In Hilbert planes, Circle-Circle, Line-Circle and the Triangle Theorem are all equivalent.

Any model of I, II, III, IV and the Circle-Circle Axiom is called an **Euclidean plane**.

Any Euclidean plane is isomorphic to a Cartesian plane F^2, where F is some arbitrary *Euclidean field* – an ordered field in which every positive element has a square root, i.e.,
$\forall a \in F\ (a > 0 \rightarrow \exists b \in F\ b^2 = a)$.

To sum up...

To sum up...

- I, II, III $\Rightarrow$ Hilbert planes

To sum up...

- I, II, III $\Rightarrow$ Hilbert planes
- $+$ IV $\Rightarrow$ Pythagorean planes

To sum up...

- I, II, III ⇒ Hilbert planes
- + IV ⇒ Pythagorean planes
- + Circle-Circle Axiom ⇒ Euclidean planes

To sum up...

- I, II, III $\Rightarrow$ Hilbert planes
- $+$ IV $\Rightarrow$ Pythagorean planes
- $+$ Circle-Circle Axiom $\Rightarrow$ Euclidean planes

and also...

To sum up...

- I, II, III ⇒ Hilbert planes
- + IV ⇒ Pythagorean planes
- + Circle-Circle Axiom ⇒ Euclidean planes

and also...

Every plane geometric proposition in Euclid's *Elements* can be proved from I, II, III, IV and the Circle-Circle Axiom.

Tarski's elementary Geometry

Tarski's elementary Geometry

Tarski's first order axiom system for Euclidean geometry has a single primitive notion of "point" (is one-sorted) and two relations among points:

Tarski's elementary Geometry

Tarski's first order axiom system for Euclidean geometry has a single primitive notion of "point" (is one-sorted) and two relations among points:

- Betweenness – $\beta(x, y, z)$ (y is between x and z)

Tarski's elementary Geometry

Tarski's first order axiom system for Euclidean geometry has a single primitive notion of "point" (is one-sorted) and two relations among points:

- Betweenness – $\beta(x, y, z)$ (y is between x and z)
- Equidistance – $xy \equiv uv$ ($\equiv(x, y, u, v)$, x is as distant from y as u is from v)

Tarski's elementary Geometry

Tarski's first order axiom system for Euclidean geometry has a single primitive notion of "point" (is one-sorted) and two relations among points:

- Betweenness – $\beta(x, y, z)$ (y is between x and z)
- Equidistance – $xy \equiv uv$ ($\equiv(x, y, u, v)$, x is as distant from y as u is from v)

He defined the relation of "collinearity" of three points in terms of β, and so he did not need "line" as a primitive notion:

"x, y and z are collinear" $\Leftrightarrow \beta(x, y, z) \vee \beta(x, z, y) \vee \beta(y, x, z)$

If we consider

DEDEKIND'S CUT AXIOM (V)

$$\forall X, Y \left(\exists z \forall x, y\, (x \in X \wedge y \in Y \rightarrow \beta(z, x, y)) \\ \rightarrow \exists u \forall x, y\, (x \in X \wedge y \in Y \rightarrow \beta(x, u, y))\right)$$

If we consider

DEDEKIND'S CUT AXIOM (V)

$$\forall X, Y\ (\exists z \forall x, y\ (x \in X \wedge y \in Y \rightarrow \beta(z, x, y)) \\ \rightarrow \exists u \forall x, y\ (x \in X \wedge y \in Y \rightarrow \beta(x, u, y)))$$

Then, any model of I, II, III, IV and V is called a **real Euclidean plane**, ($\mathbb{R}^2$).

If we consider

DEDEKIND'S CUT AXIOM (V)

$$\forall X, Y\ (\exists z \forall x, y\ (x \in X \land y \in Y \to \beta(z, x, y)) \\ \to \exists u \forall x, y\ (x \in X \land y \in Y \to \beta(x, u, y)))$$

Then, any model of I, II, III, IV and V is called a **real Euclidean plane**, ($\mathbb{R}^2$).

Tarski gave an "elementary" version of this axiom:

If we consider

DEDEKIND'S CUT AXIOM (V)

$$\forall X, Y \ (\exists z \forall x, y \ (x \in X \land y \in Y \to \beta(z, x, y)) \\ \to \exists u \forall x, y \ (x \in X \land y \in Y \to \beta(x, u, y)))$$

Then, any model of I, II, III, IV and V is called a **real Euclidean plane**, ($\mathbb{R}^2$).
Tarski gave an "elementary" version of this axiom:

ELEMENTARY AXIOM SCHEMA OF CONTINUITY ($(\mathrm{V})_{\mathrm{elem}}$)

$$\exists z \forall x, y \ (\varphi(x) \land \psi(y)) \to \beta(z, x, y)) \\ \to \exists u \forall x, y \ (\varphi(x) \land \psi(y)) \to \beta(x, u, y)$$

for every formulas φ and ψ

We call the models of Tarski's elementary plane geometry **Tarski-elementary Euclidean planes**.

We call the models of Tarski's elementary plane geometry **Tarski-elementary Euclidean planes**.

Any such model is isomorphic to a Cartesian plane F^2, where F is a *real-closed ordered field* – an Euclidean field in which every polynomial of odd degree has a root.

We call the models of Tarski's elementary plane geometry **Tarski-elementary Euclidean planes**.

Any such model is isomorphic to a Cartesian plane F^2, where F is a *real-closed ordered field* – an Euclidean field in which every polynomial of odd degree has a root.

To sum up...

We call the models of Tarski's elementary plane geometry **Tarski-elementary Euclidean planes**.

Any such model is isomorphic to a Cartesian plane F^2, where F is a *real-closed ordered field* – an Euclidean field in which every polynomial of odd degree has a root.

To sum up...

Hilbert planes $\supset$ Pythagorean planes $\supset$ Euclidean planes $\supset$
$\supset$ Tarski-elementary Euclidean planes $\supset$ the real Euclidean plane

Arithmetic

Arithmetic

Recall that the Second-Order Arithmetic theory (in the language with 0, S, $+$, $\cdot$, $=$, $<$ and $\in$), contains as one of its axioms

INDUCTION AXIOM

$$\forall X \left(0 \in X \wedge \forall x \left(x \in X \rightarrow S(x) \in X\right)\right) \rightarrow \forall x \left(x \in X\right)$$

Arithmetic

Recall that the Second-Order Arithmetic theory (in the language with 0, S, $+$, $\cdot$, $=$, $<$ and $\in$), contains as one of its axioms

INDUCTION AXIOM

$$\forall X \left(0 \in X \wedge \forall x \left(x \in X \rightarrow S(x) \in X\right)\right) \rightarrow \forall x \left(x \in X\right)$$

and that if we replace it by its "elementary" axiom schema of induction,

ELEMENTARY AXIOM SCHEMA OF INDUCTION

$$\left[\varphi(0) \wedge \forall x \left(\varphi(x) \rightarrow \varphi(S(x))\right)\right] \rightarrow \forall x\, \varphi(x),$$

for every formula φ.

Arithmetic

Recall that the Second-Order Arithmetic theory (in the language with 0, S, $+$, $\cdot$, $=$, $<$ and $\in$), contains as one of its axioms

INDUCTION AXIOM

$$\forall X \left(0 \in X \wedge \forall x \left(x \in X \rightarrow S(x) \in X\right)\right) \rightarrow \forall x \left(x \in X\right)$$

and that if we replace it by its "elementary" axiom schema of induction,

ELEMENTARY AXIOM SCHEMA OF INDUCTION

$$\left[\varphi(0) \wedge \forall x \left(\varphi(x) \rightarrow \varphi(S(x))\right)\right] \rightarrow \forall x\, \varphi(x),$$

for every formula φ.

we end up with an axiom system for First-Order Peano Arithmetic.

Definition

A theory is decidable if there is an algorithm for determining whether or not an arbitrary statement in the language of the theory is provable.

Definition

A theory is decidable if there is an algorithm for determining whether or not an arbitrary statement in the language of the theory is provable.

It is well known that PA is **incomplete** – there exists a sentence φ such that neither φ nor $\neg\varphi$ is provable in PA – and **essentially undecidable** – i.e., any consistent extension of PA is undecidable. In particular, the set $\{\varphi \mid \varphi \text{ is true in } \mathbb{N}\}$ is undecidable.

Definition

A theory is decidable if there is an algorithm for determining whether or not an arbitrary statement in the language of the theory is provable.

It is well known that PA is **incomplete** – there exists a sentence φ such that neither φ nor $\neg\varphi$ is provable in PA – and **essentially undecidable** – i.e., any consistent extension of PA is undecidable. In particular, the set $\{\varphi \mid \varphi \text{ is true in } \mathbb{N}\}$ is undecidable.

Also...
(K.Gödel) No finitary proof of the consistency of PA is possible.

Geometry

In comparison...

Geometry

In comparison...

[Recall that every Tarski-elementary Euclidean plane is isomorphic to F^2, where F is some real-closed ordered field.]

Geometry

In comparison...

[Recall that every Tarski-elementary Euclidean plane is isomorphic to F^2, where F is some real-closed ordered field.]

As Descartes showed, every geometric statement φ about the plane F^2 translates into an algebraic statement φ^* about F, and so:

Geometry

In comparison...

[Recall that every Tarski-elementary Euclidean plane is isomorphic to F^2, where F is some real-closed ordered field.]

As Descartes showed, every geometric statement φ about the plane F^2 translates into an algebraic statement φ^* about F, and so:

$$TEG \vdash \varphi \quad \leftrightarrow \quad RCOF \vdash \varphi^*,$$

where TEG denotes the Tarski's Elementary Geometry theory and RCOF the theory of real-closed ordered fields.

Theorem (Tarski)

RCOF is a ***complete*** *theory.*

Theorem (Tarski)

RCOF is a ***complete*** *theory.*

Fact 1. RCOF is a **decidable** theory.

Fact 2. $\{\varphi \in \mathcal{L}_{RCOF} \mid \mathbb{R} \models \varphi\}$ is decidable.

Theorem (Tarski)

*RCOF is a **complete** theory.*

Fact 1. RCOF is a **decidable** theory.

Fact 2. $\{\varphi \in \mathcal{L}_{RCOF} \mid \mathbb{R} \vDash \varphi\}$ is decidable.

Also...

Theorem (Tarski)

*RCOF is a **complete** theory.*

Fact 1. RCOF is a **decidable** theory.

Fact 2. $\{\varphi \in \mathcal{L}_{RCOF} \mid \mathbb{R} \vDash \varphi\}$ is decidable.

Also...
Contrary to arithmetic, it is possible to give a finitary proof of consistence for RCOF (e.g. Ferreira [6]).

Theorem (Tarski)

*RCOF is a **complete** theory.*

Fact 1. RCOF is a **decidable** theory.

Fact 2. $\{\varphi \in \mathcal{L}_{RCOF} \mid \mathbb{R} \vDash \varphi\}$ is decidable.

Also...
Contrary to arithmetic, it is possible to give a finitary proof of consistence for RCOF (e.g. Ferreira [6]).

The theory of the Tarski-elementary Euclidean planes is **complete**, **decidable** and has a finitary proof of consistence.

Final Remarks

Surprisingly...

Final Remarks

Surprisingly...
Ziegler proved that any finitely axiomatizable first-order theory of fields, having the real number field $\mathbb{R}$ as a model must be undecidable.

This includes the theory of fields, the theory of ordered fields, the theories of Pythagorean ordered fields and of Euclidean fields.

Final Remarks

Surprisingly...
Ziegler proved that any finitely axiomatizable first-order theory of fields, having the real number field $\mathbb{R}$ as a model must be undecidable.

This includes the theory of fields, the theory of ordered fields, the theories of Pythagorean ordered fields and of Euclidean fields.

Theorem

*The theory of Euclidean fields is **undecidable**.*

As before, we have

As before, we have

$$EPG \vdash \varphi \quad \leftrightarrow \quad EF \vdash \varphi^*,$$

where EPG denotes the Euclidean plane Geometry theory and EF the theory of Euclidean fields.

As before, we have

$$EPG \vdash \varphi \;\; \leftrightarrow \;\; EF \vdash \varphi^*,$$

where EPG denotes the Euclidean plane Geometry theory and EF the theory of Euclidean fields.

It follows that the Euclidean plane Geometry is **undecidable**:

"Elementary Euclidean geometry is genuinely creative, not mechanical" [1]

References

[1] Greenberg, M. J. 2010. "*Old and New Results in the Foundations of Elementary Plane Euclidean and Non-Euclidean Geometries*", The American Mathematical Monthly, vol.117, no.3, March 2010, pp.198-219

[2] Greenberg, M. J. 2007. "*Euclidean and Non-Euclidean Geometries: Development and History*, 4^{th} Revised and Enlarged Edition, New York: Freeman

[3] Hartshorne, R. 2000. "Teaching Geometry According to Euclid", Notices of the Amer.Math.Soc. 47 (4): 460-465

[4] Tarski, A. 1959. "*What is Elementary Geometry?*" in The axiomatic method, with special reference to geometry and physics (L. Henkin et al., editors), North-Holland Publishing Company, Amsterdam: 16-29

[5] Givant, S. and Tarski, A. 1999. "*Tarski's System of Geometry*", Bulletin of Symbolic Logic 5, no. 2, 175-214

[6] Ferreira, F. and Fernandes, A. 2002. " *Groundwork for Weak Analysis*", Journal of Symbolic Logic 67 (2), 557-578