An Introduction to Gerbes

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Table of Contents

Motivation

Complex line and principal $\mathbb{C}^\times$-bundles

Given a smooth manifold M the structures of complex line bundles over M and principal $\mathbb{C}^\times$-bundles over M are equivalent in the following sense:

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Conversely, given a principal $\mathbb{C}^\times$-bundle $P \to M$ we denote by $P \times^{\mathbb{C}^\times} \mathbb{C}$ the complex line bundle defined as follows: $P \times^{\mathbb{C}^\times} \mathbb{C} = P \times \mathbb{C}/\mathbb{C}^\times$ where the action is given by $\lambda \cdot (y, w) = (\lambda^{-1} \cdot y, \lambda w)$.

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The assignments $L \mapsto L^+$ and $P \mapsto P \times^{\mathbb{C}^\times} \mathbb{C}$ give an equivalence of categories between the category of complex line bundles over M and the category of principal $\mathbb{C}^\times$-bundles over M.

Complex line and principal $\mathbb{C}^\times$-bundles

The notion of the tensor product of complex line bundles $L \otimes L'$ defines an abelian group structure on the set of isomorphism classes of line bundles over M. This group is also called the Picard group and denoted by $\mathrm{Pic}^\infty(M)$.

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The equivalent notion for principal bundles is called the contracted product and we also denote it by $P \otimes P'$. It is given by $P \otimes P' = (P \times_M P')/\mathbb{C}^\times$ where $\lambda \cdot (y_1, y_2) = (\lambda^{-1} \cdot y_1, \lambda \cdot y_2)$.

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If we denote the group of isomorphism classes of principal bundles by $\mathrm{Prin}_{\mathbb{C}^\times}(M)$ then it is isomorphic to the Picard group via the equivalence of categories.

Classification of line bundles

Given a line bundle $L \to M$ consider L^+ and choose a trivializing covering of M. Then we can find sections $s_i \in \Gamma(U_i, L^+|_{U_i})$. Hence on double intersections U_{ij} we define $g_{ij} = \frac{s_i}{s_j} : U_{ij} \to \mathbb{C}^\times$. Then (g_{ij}) defines a 1-cocycle with coefficients in the sheaf $\underline{\mathbb{C}}^\times_M$.

Theorem

There is an isomorphism of groups:

$$\mathrm{Pic}^\infty(M) \xrightarrow{\sim} H^1(M, \underline{\mathbb{C}}^\times_M)$$
$$[L] \to [(g_{ij})]$$

Then the exponential short exact sequence implies that

$$\mathrm{Pic}^\infty(M) \cong H^2(M; \mathbb{Z})$$

Bundle Gerbes

The Fundamental Complex

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Let M be a manifold and $\mathcal{U} = \{U_\alpha\}$ an open covering of M. Then we define

$$Y_{\mathcal{U}} = \coprod_{\alpha} U_\alpha$$

This manifold comes equipped with a projection map $\pi : Y_{\mathcal{U}} \to M$ which is a surjective submersion.

The Fundamental Complex

Notice that given any surjective submersion $\pi : Y \to M$ there is an open covering $\mathcal{U}$ of M such that

$$\begin{array}{ccc} Y_{\mathcal{U}} & \longrightarrow & Y \\ & \searrow & \swarrow_{\pi} \\ & M & \end{array}$$

since π allows local sections.

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Therefore we will use the language of surjective submersions $\pi : Y \to M$.

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$Y_{\mathcal{U}}^{[2]}$ is given by the disjoint union of 2-fold intersections of the open covering.

The p-fold fiber product $Y^{[p]}$ comes equipped with a family of projections

$$\pi_i : Y^{[p]} \to Y^{[p-1]}$$
$$(y_1, ..., y_p) \mapsto (y_1, ..., \hat{y}_i, ..., y_p)$$

The Fundamental Complex

Definition/Proposition

Given a surjective submersion $\pi : Y \to M$ consider the space of q-forms on $Y^{[p-1]}$ and define

$$\delta : \Omega^q(Y^{[p-1]}) \to \Omega^q(Y^{[p]})$$

given by

$$\delta(\alpha) = \sum_{i=1}^{p} (-1)^{i-1} \pi_i^*(\alpha)$$

as the alternating sum of the pullback of forms via the projection maps. The resulting complex

$$0 \to \Omega^q(M) \xrightarrow{\pi^*} \Omega^q(Y) \xrightarrow{\delta} \Omega^q(Y^{[2]}) \to \cdots$$

is called the fundamental complex and is exact for all $q = 0, 1, 2, \ldots$.

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- Given a principal $\mathbb{C}^\times$-bundle $P \to Y^{[p-1]}$ we can act with δ in the following way

$$\delta(P) = \pi_1^*(P) \otimes \pi_2^*(P^*) \otimes \pi_3^*(P) \otimes \cdots$$

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- $\delta(\delta(P))$ is canonically the trivial bundle on $Y^{[p+1]}$.

Bundle Gerbes

Definition

A bundle gerbe on a manifold M is a triple (P, Y, M) of manifolds, where $\pi : Y \to M$ is a surjective submersion and $P \to Y^{[2]}$ is a principal $\mathbb{C}^\times$-bundle which carries a product. That is, an isomorphism of principal $\mathbb{C}^\times$-bundles over $Y^{[3]}$

$$m : \pi_3^* P \otimes \pi_1^* P \xrightarrow{\sim} \pi_2^* P$$

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Remark

Given $(y_1, y_2, y_3) \in Y^{[3]}$ the bundle gerbe multiplication gives an isomorphism on the fibers

$$m_{(y_1,y_2,y_3)} : P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \xrightarrow{\sim} P_{(y_1,y_3)}$$

Bundle Gerbes

Remark

Considering the isomorphisms of the fibers, the associativity condition reads

$$\begin{array}{ccc} P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \otimes P_{(y_3,y_4)} & \xrightarrow{m_{(y_1,y_2,y_3)} \otimes \mathrm{id}} & P_{(y_1,y_3)} \otimes P_{(y_3,y_4)} \\ \Big\downarrow {\scriptstyle \mathrm{id} \otimes m_{(y_2,y_3,y_4)}} & & \Big\downarrow {\scriptstyle m_{(y_1,y_3,y_4)}} \\ P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} & \xrightarrow{m_{(y_1,y_2,y_4)}} & P_{(y_1,y_4)} \end{array}$$

for any $(y_1, y_2, y_3, y_4) \in Y^{[4]}$.

Remark

Let $\pi : Y \to M$ be a surjective submersion.

- Given a principal $\mathbb{C}^\times$-bundle Q on Y, the bundle $\delta(Q)$ on $Y^{[2]}$ naturally carries a bundle gerbe product. Indeed, for $(y_1, y_2, y_3) \in Y^{[3]}$ consider the fiber

$$\begin{aligned}(\pi_3^*\delta(Q) \otimes \pi_1^*\delta(Q))_{(y_1,y_2,y_3)} &= \delta(Q)_{(y_1,y_2)} \otimes \delta(Q)_{((y_2,y_3))} \\ &= Q_{y_2} \otimes Q_{y_1}^* \otimes Q_{y_3} \otimes Q_{y_2}^* \\ &\cong Q_{y_3} \otimes Q_{y_1}^* \\ &= \pi_2^*\delta(Q)_{(y_1,y_2,y_3)}\end{aligned}$$

- Given a bundle gerbe (P, Y, M) the bundle gerbe product on P forces the bundle $\delta(P)$ on $Y^{[3]}$ to be trivial. Hence equivalently, one can define a bundle gerbe product as a global section $s \in \Gamma(Y^{[3]}, \delta(P))$ such that $\delta(s) = 1$ is equal to the canonical non-vanishing section of the trivial bundle $\delta(\delta(P))$.

Definition

Let (P, Y, M) and (P', Y', M) be two bundle gerbes over M. Then consider the fiber product $Y \times_M Y' \to M$ which is again a surjective submersion. Let $p_1 : Y \times_M Y' \to Y$ and $p_2 : Y \times_M Y' \to Y'$ projections. Then $p_1^{[2]*} P \otimes p_2^{[2]*} P'$ defines a principal $\mathbb{C}^\times$-bundle on $(Y \times_M Y')^{[2]}$. Denote this bundle by $P \otimes_M P'$ and notice that given $((y_1, y_1'), (y_2, y_2')) \in (Y \times_M Y')^{[2]}$ this bundle has fibers

$$(P \otimes_M P')_{((y_1, y_1'), (y_2, y_2'))} = P_{(y_1, y_2)} \otimes P'_{(y_1', y_2')}$$

therefore this bundle carries an induced bundle gerbe product. This defines $(P \otimes_M P', Y \times_M Y', M)$ as the product bundle gerbe of (P, Y, M) and (P', Y', M).

The trivial Bundle Gerbe

Definition

A bundle gerbe (P, Y, M) is said to be trivial if there exists an isomorphism of bundle gerbes

$$(\phi, \mathrm{id}, \mathrm{id}) : (\delta(Q), Y, M) \to (P, Y, M)$$

where Q is a principal $\mathbb{C}^\times$-bundle over Y.

Dixmier-Douady Class

Want to associate a cohomology class $DD(P, Y) \in H^3(M; \mathbb{Z})$ to any bundle gerbe (P, Y, M).

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- Choose $\mathcal{U} = \{U_\alpha\}$ a good open covering of M such that we obtain sections $s_\alpha : U_\alpha \to Y$. On 2-fold intersections we obtain sections $(s_\alpha, s_\beta) : U_{\alpha\beta} \to Y^{[2]}$.

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- Denote by $P_{\alpha,\beta} = (s_\alpha, s_\beta)^* P$ the pullback principal bundle over $U_{\alpha\beta}$.

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- Since $\mathcal{U} = \{U_\alpha\}$ is a good open covering we have that $U_{\alpha\beta}$ is contractible and therefore there are sections $\sigma_{\alpha\beta} : U_{\alpha\beta} \to P_{\alpha,\beta}$.

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- Since $\mathcal{U} = \{U_\alpha\}$ is a good open covering we have that $U_{\alpha\beta}$ is contractible and therefore there are sections $\sigma_{\alpha\beta} : U_{\alpha\beta} \to P_{\alpha,\beta}$.
- Define a map $g_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to \mathbb{C}^\times$ by

$$m(\sigma_{\alpha\beta}(x) \otimes \sigma_{\beta\gamma}(x)) = g_{\alpha\beta\gamma}(x)\sigma_{\alpha\gamma}(x)$$

(This is possible since the bundle gerbe product induces isomorphisms on the fibers)

Dixmier-Douady Class

- The associativity condition of the bundle gerbe product implies that $g_{\alpha\beta\gamma}$ satisfies the cocycle condition, i.e.

$$[g_{\alpha\beta\gamma}] \in \check{H}^2(\mathcal{U}; \underline{\mathbb{C}}_M^\times) \to H^2(M; \underline{\mathbb{C}}_M^\times)$$

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- The short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\exp(2\pi i -)} \mathbb{C}^\times \to 1$$

induces a long exact sequence in sheaf cohomology. Then using that the sheaf $\underline{\mathbb{C}}_M$ is soft (partition of unity) shows eventually that

$$H^2(M; \underline{\mathbb{C}}_M^\times) \xrightarrow{\sim} H^3(M; \underline{\mathbb{Z}}_M) = H^3(M; \mathbb{Z})$$
$$[g_{\alpha\beta\gamma}] \mapsto DD(P, Y)$$

Properties of the DD-Class

The Dixmier-Douady class satisfies analogue properties such as the first Chern class of a complex line bundle. More precisely we have:

Proposition

Let (P, Y, M) and (Q, X, M) be bundle gerbes on M. Then

1. $DD(P \otimes_M Q, Y \times_M X) = DD(P, Y) + DD(Q, X)$
2. The dual bundle gerbe (P^*, Y, M) satisfies
$$DD(P^*, Y) = -DD(P, Y)$$
3. Given a smooth map $f : N \to M$ then the pullback gerbe (f^*P, f^*Y, N) satisfies
$$DD(f^*P, f^*Y) = f^*DD(P, Y)$$
4. $DD(P; Y) = 0$ if and only if (P, Y, M) is a trivial bundle gerbe.

Stable isomorphism of Bundle Gerbes

The last property shows that isomorphisms of bundle gerbes are too strong to behave well under taking Dixmier-Douady classes. Therefore we introduce the appropriate notion of stable equivalence.

Definition

Two bundle gerbes (P, Y, M) and (Q, X, M) are said to be stably isomorphic if there are principal $\mathbb{C}^\times$-bundles T_1, T_2 over $Y \times_M X$ and isomorphisms

$$p_1^{[2]*}P \otimes \delta(T_1) \cong \delta(T_2) \otimes p_2^{[2]*}Q$$

of principal bundles over $(Y \times_M X)^{[2]}$ commuting with the bundle gerbe product.

Classification of Bundle Gerbes

Theorem

Two bundle gerbes (P, Y, M) and (Q, X, M) have the same Dixmier-Douady class if and only if they are stably isomorphic. In particular we have that the group of stable isomorphism classes of bundle gerbes on M is isomorphic to $H^3(M; \mathbb{Z})$.

The DD-class of a Bundle gerbe and the 3-curvature

Basically the construction of a DD-class has been a generalization of the construction of a cocycle characterizing a complex line bundle. Endowing complex line bundles with a connection, one can show that the cohomology class of the 2-curvature is in relation with the class of the characterizing cocycle. We want to introduce bundle gerbe connections and give an analogue construction of a 3-curvature on bundle gerbes.

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Definition

Let (P, Y, M) be a bundle gerbe. Denote by L the associated complex line bundle over $Y^{[2]}$. Then notice that the bundle gerbe product on P induces a product on L. A bundle gerbe connection is a connection ∇_L on L such that

$$\pi_1^* \nabla_L + \pi_3^* \nabla_L = m_L^{-1} \circ \pi_2^* \nabla_L \circ m_L$$

Bundle Gerbe connection

Remark

- Recall that a connection ∇_L on L is an assignment

$$\nabla_L : \Gamma(L) \to \Omega^1(Y^{[2]}) \otimes \Gamma(L)$$

Then given two line bundles L, L' on $Y^{[2]}$ and two connections ∇_L and $\nabla_{L'}$ there is a connection on $L \otimes L'$ given by

$$(\nabla_L + \nabla_{L'})(s \otimes s') = s \otimes \nabla_{L'}(s') + \nabla_L(s) \otimes s'$$

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- the connection $m_L^{-1} \circ \pi_2^* \nabla_L \circ m_L$ is given by

$$\begin{array}{ccc} \Gamma(\pi_1^* L \otimes \pi_3^* L) & \longrightarrow & \Omega^1(Y^{[3]}) \otimes \Gamma(\pi_1^* L \otimes \pi_3^* L) \\ \big\downarrow m_L & & \big\uparrow \mathrm{id} \otimes m_L^{-1} \\ \Gamma(\pi_2^* L) & \xrightarrow{\pi_2^* \nabla_L} & \Omega^1(Y^{[3]}) \otimes \Gamma(\pi_2^* L) \end{array}$$

Curving and 3-curvature of a Bundle Gerbe

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Given ∇ a bundle gerbe connection on (P, Y, M) let $K = K_\nabla$ denote the corresponding curvature 2-form on $Y^{[2]}$. It can be shown that since ∇ is a bundle gerbe connection we have that $\delta(K) = 0$. By exactness of the fundamental complex we have

$$K = \delta(f)$$

for some $f \in \Omega^2(Y)$.

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Such a choice of f is unique up to pullback of 2-forms on M (See first arrow in fundamental complex). A choice of f is called a curving of the bundle gerbe connection.

Curving and 3-curvature of a Bundle Gerbe

Let f be a curving of ∇ on (P, Y, M), then since K is closed it follows

$$\delta(df) = d\delta(f) = 0$$

hence again by exactness of the fundamental complex there is a 3-form ω on M such that $\delta(\omega) = \pi^*\omega = df$.

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The form ω is called the 3-curvature of the pair (∇, f).

Dixmier-Douady class and the 3-curvature

Proposition

Let (P, Y, M) be a bundle gerbe endowed with a arbitrary bundle gerbe connection ∇. Then the class $[\frac{\omega}{2\pi i}] \in H^3(M;\mathbb{R})$ is a real cohomology class which is given by the image of $DD(P, Y)$ under the induced map $H^3(M;\mathbb{Z}) \to H^3(M;\mathbb{R})$. In particular $[\omega]$ is independent of the choice of connection and curving.

Tautological Bundle Gerbe

Assume we are given a manifold M and a closed, complex-valued, integral 3-form $\omega \in \Omega^3(M)$.

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Recall ω is integral, if $[\frac{\omega}{2\pi i}] \in H^3(M; \mathbb{C})$ seen as a cohomology class with complex coefficients via the De Rham isomorphism, lies in the image of $H^3(M; \mathbb{Z}) \to H^3(M; \mathbb{C})$.

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We want to construct a bundle gerbe (P, Y, M) on M and endow it with a bundle gerbe connection such that the 3-curvature equals ω. The construction is split into two parts:

Part 1: The tautological principal bundle

Consider M a 1-connected manifold and ω a closed, complex-valued, integral 2-form on M. Choose a basepoint $m_0 \in M$ and denote by P_0M the Fréchet manifold of smooth paths $\gamma : [0,1] \to M$ starting at m_0.

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The map $\pi : P_0M \to M$ given by $\gamma \mapsto \gamma(1)$ is a surjective submersion and we identify $P_0M^{[2]} \cong \Omega_0 M$ (use reparametrization with sitting instants at the endpoints).

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Define $g : \Omega_0 M \to \mathbb{C}^\times$ as follows: Given $(\mu_1, \mu_2) \in P_0M^{[2]}$ we have by the fact that M is 1-connected that there is a homotopy $H : I^2 \to M$ with fixed endpoints between μ_1 and μ_2.

$$g(\mu_1, \mu_2) := \exp\left(\int_{I^2} H^*\omega\right)$$

The fact that ω is integral shows that this map is well defined.

Part 1: The tautological principal bundle

The map g satisfies a cocycle condition: For $(\mu_1, \mu_2, \mu_3) \in P_0M^{[3]}$ we have

$$g(\mu_1, \mu_2)g(\mu_2, \mu_3) = g(\mu_1, \mu_3)$$

Therefore $P = (P_0M \times \mathbb{C}^\times) / \sim$ defines a principal $\mathbb{C}^\times$-bundle on M, where $(\mu_1, z_1) \sim (\mu_2, z_2)$ if and only if $\mu_1(1) = \mu_2(1)$ and $z_2 = g(\mu_1, \mu_2) \cdot z_1$.

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Considering the evaluation map $ev : P_0M \times I \to M$ given by $(\gamma, t) \mapsto \gamma(t)$ define $A = \int_{[0,1]} ev^*\omega$ 1-form on P_0M. Then the 1-form $\hat{A} = A + z^{-1}dz$ descends to the principal bundle P and defines a connection 1-form.

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Using the fact $dA = \pi^*\omega$ one can show that $d\hat{A} = pr^*\omega$ where $pr : P \to M$, i.e. ω is the curvature of the connection on P.

Part 2: The tautological bundle gerbe

Let M be 2-connected manifold and ω a closed, complex-valued, integral 3-form on M. Then $P_0 M^{[2]}$ is 1-connected and we define a closed, integral 2-form on $P_0 M^{[2]}$ via the evaluation map $ev : S^1 \times \Omega_0 M \to M$, i.e.

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Need to define a bundle gerbe product on P such that (P, P_0M, M) will be our tautological bundle gerbe.

Part 2: The tautological bundle gerbe

Notice $P = (P_0(\Omega_0 M) \times \mathbb{C}^\times)/\sim$ hence given $(\mu_1, \mu_2, \mu_3) \in P_0 M^{[3]}$ we define

$$\begin{aligned} m_{(\mu_1,\mu_2,\mu_3)} : P_{(\mu_2,\mu_3)} \otimes P_{(\mu_1,\mu_2)} &\to P_{(\mu_1,\mu_3)} \\ [H, z] \otimes [V, w] &\mapsto [H \circ V, zw] \end{aligned}$$

where H homotopy with fixed endpoints between (μ_2, μ_3) and V homotopy with fixed endpoints between (μ_1, μ_2).

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Then the 2-form $\nu = \int_{[0,1]} ev^*\omega$ on $P_0 M$ is a curving for the connection $\hat{A}$ since we have $\delta(\nu) = A$. Then the 3-curvature of the pair $(\hat{A}, \nu)$ is ω since $d\nu = \pi^*\omega$.

Gerbes via sheaves of groupoids

Sheaves redefined

Definition

Let X be a topological space, then a sheaf $\mathcal{A}$ on X is an assignment which

- associates to every local homeomorphism $f : Y \to X$ a set $\mathcal{A}(Y \xrightarrow{f} X)$ and to a diagram of local homeomorphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$ a pullback map

$$g^{-1} : \mathcal{A}(Y \xrightarrow{f} X) \to \mathcal{A}(Z \xrightarrow{f \circ g} X)$$

  which is compatible with composition of diagrams $W \xrightarrow{h} Z \xrightarrow{g} Y \xrightarrow{f} X$, i.e. $(g \circ h)^{-1} = g^{-1} \circ h^{-1}$.
- For every open set $V \subset X$ and every surjective local homeomorphism $f : Y \to V$ the sequence

$$\mathcal{A}(V \hookrightarrow X) \xrightarrow{f^{-1}} \mathcal{A}(Y \to X) \rightrightarrows \mathcal{A}(Y \times_M Y)$$

  is an equalizer.

Motivation for descent

Let G be a Lie group and consider $f : Y \to X$ a surjective submersion. Given a principal G-bundle $P \to X$ on X let $Q = f^*P$ be the pullback bundle on Y.

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Let π_1, π_2 denote the projections $Y \times_X Y \to Y$ then we have a natural isomorphism

$$\phi : \pi_1^* Q \to \pi_2^* Q$$

since $f \circ \pi_1 = f \circ \pi_2$. This isomorphism satisfies the following cocycle condition

$$\pi_1^*(\phi) \circ \pi_3^*(\phi) = \pi_2^*(\phi)$$

where $\pi_i : Y^{[3]} \to Y^{[2]}$ the projections omitting the i-th factor.

Motivation for descent

Conversely, let $p : Q \to Y$ be a principal G-bundle on Y together with an isomorphism $\phi : \pi_1^* Q \to \pi_2^* Q$ over $Y^{[2]}$ which satisfies the cocycle condition.

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Now want to see if we can recover from this data a principal bundle $P \to X$ such that $Q \cong f^* P$.

Indeed, we can define $P = Q/\sim$ where the equivalence relation is constructed using the isomorphism ϕ and uses the crucial fact that it satisfies the cocycle condition.

Motivation for descent

The aim of this discussion is to show that if we fix a surjective submersion $f : Y \to X$ there is a one-to-one correspondence (up to isomorphism) between

$$\{\text{Principal } G\text{-bundles on } X\} \leftrightarrow \left\{ \begin{array}{c} \text{Principal } G\text{-bundles on } Y \\ \text{with descent datum w.r.t. } f : Y \to X \end{array} \right\}$$

where descent datum for a principal bundle $Q \to Y$ is an isomorphism $\phi : \pi_1^* Q \to \pi_2^* Q$ satisfying the cocycle identity.

Presheaf / Sheaf of categories

Definition

A presheaf of categories $\mathcal{C}$ over a topological space X consists of the following data

1. for every local homeomorphism $f : Y \to X$ a category $\mathcal{C}(Y \to X)$
2. for every diagram $Z \xrightarrow{g} Y \xrightarrow{f} X$ a functor

$$g^{-1} : \mathcal{C}(Y \to X) \to \mathcal{C}(Z \to X)$$

3. for every diagram $W \xrightarrow{h} Z \xrightarrow{g} Y \xrightarrow{f} X$ of local homeomorphisms an invertible natural transformation

$$\theta_{g,h} : h^{-1} \circ g^{-1} \xrightarrow{\sim} (g \circ h)^{-1}$$

Presheaf / Sheaf of categories

Definition

A presheaf of categories $\mathcal{C}$ is said to be a sheaf of categories if the following two descent properties are satisfied:

- (D1): For every two objects $A, B \in \mathcal{C}(Y \xrightarrow{f} X)$ the assignment

$$(Z \xrightarrow{g} Y) \mapsto \mathrm{Hom}_{\mathcal{C}(Z \to X)}(g^{-1}(A), g^{-1}(B))$$

  defines a sheaf on Y denoted by $\underline{\mathrm{Hom}}(A, B)$.

- (D2): Given a local homeomorphism $f : Y \to X$ and a surjective local homeomorphism $g : Z \to Y$, the functor g^{-1} induces an equivalence of categories

$$\mathcal{C}(Y \xrightarrow{f} X) \to \mathrm{Desc}(\mathcal{C}, g)$$

  where $\mathrm{Desc}(\mathcal{C}, g)$ is the category with objects given by pairs (A, ϕ) where $A \in \mathcal{C}(Z \to X)$ and $\phi : \pi_1^{-1}(A) \to \pi_2^{-1}(A)$ an isomorphism in $\mathcal{C}(Z \times_Y Z \to X)$ satisfying the cocycle condition.

Example of a sheaf of groupoids

Let X be a manifold. Then to every submersion $Y \to X$ we assign the category $\mathrm{Bund}_G(Y)$ with objects given by principal G-bundles on Y and morphisms given by isomorphisms of principal bundles (i.e. a groupoid).

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The descent condition (D2) is satisfied as we have seen in the motivating example.

Example of a sheaf of groupoids

To show descent condition (D1) let P, Q be principal bundles over Y for some submersion $Y \xrightarrow{f} X$. Then consider the assignment:

$$V \subset Y \mapsto \mathrm{Hom}_{\mathrm{Bund}_G(V)}(P|_V, Q|_V)$$

This clearly defines a presheaf on Y. Hence what we need to check is that given any open covering $\{U_\alpha\}$ of V and a family of isomorphisms $\phi_\alpha : P|_{U_\alpha} \to Q|_{U_\alpha}$ with the property that

$$\phi_\alpha|_{U_{\alpha\beta}} = \phi_\beta|_{U_{\alpha\beta}}$$

then there is an isomorphism $\phi : P|_V \to Q|_V$ that restricts to ϕ_α.

This is true for general principal bundles over V, hence (D1) is satisfied.

Gerbe as sheaf of groupoids

Definition

Let M be a manifold and A a Lie group. A gerbe with band $\underline{A}_M$ over M is a sheaf of groupoids $\mathcal{C}$ on M satisfying the following properties:

- (G1): Given any object $Q \in \mathcal{C}(Y \xrightarrow{f} M)$ the automorphism sheaf on Y denoted by $\underline{\mathrm{Aut}}(Q)$ is locally isomorphic to $\underline{A}_Y$ to the sheaf of A-valued functions on Y.
- (G2): Given objects $Q_1, Q_2 \in \mathcal{C}(Y \xrightarrow{f} M)$ there is a surjective submersion $g : Z \to Y$ such that $g^{-1}(Q_1) \cong g^{-1}(Q_2)$ isomorphic.
- (G3): There is a surjective submersion $f : Y \to M$ such that the category $\mathcal{C}(Y \to M)$ is non-empty.

Example of a Gerbe

Let M be a manifold and A an abelian Lie group.

Definition

An $\underline{A}_M$-torsor over M is a sheaf $\mathcal{T}$ together with a morphism of sheaves $\underline{A}_M \times \mathcal{T} \to \mathcal{T}$ inducing actions for every open subset $U \subset M$

$$\underline{A}_M(U) \times \mathcal{T}(U) \to \mathcal{T}(U)$$

which are simply transitive and such that there is an open covering $\{U_\alpha\}$ such that $\mathcal{T}(U_\alpha)$ are non-empty.

Remark

Notice that torsos are just a sheaf theoretic version of a principal A-bundle. Indeed, there is an equivalence of categories

$$\begin{aligned} \mathrm{Bund}_A(M) &\longrightarrow \mathrm{Tors}_{\underline{A}_M}(M) \\ P &\longmapsto U \mapsto \Gamma(U, P) \end{aligned}$$

Example of a Gerbe

Consider the following assignment:

$$Y \xrightarrow{f} M \mapsto \mathrm{Tors}_{\underline{A}_Y}(Y)$$

Then this defines a presheaf of groupoids. In fact the descent conditions (D1) and (D2) are both satisfied, i.e. this defines a sheaf of groupoids.

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Given a torsor $\mathcal{T} \in \mathrm{Tors}_{\underline{A}_Y}(Y)$ we have that

$$\underline{\mathrm{Aut}}(\mathcal{T}) \xrightarrow{\sim} \underline{A}_Y$$

since as a torsor the action of $\underline{A}_Y$ is simply transitive. This motivates property (G1).

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Given $\mathcal{T}_1, \mathcal{T}_2 \in \mathrm{Tors}_{\underline{A}_Y}(Y)$ we use the fact that any torsor is locally isomorphic to the trivial torsor (like principal bundles are locally trivial). Hence we can find an open covering $\{U_\alpha\}$ of Y trivializing both $\mathcal{T}_1$ and $\mathcal{T}_2$, i.e. $g^{-1}\mathcal{T}_1 \cong g^{-1}\mathcal{T}_2$ where $g : \coprod U_\alpha \to Y$. This motivates property (G2).

Example of a Gerbe

The property (G3) is also satisfied, since obviously $\mathrm{Tors}_{\underline{A}_M}(M)$ non-empty. Hence $\mathrm{Tors}_{\underline{A}_M}(-)$ is a gerbe of band $\underline{A}_M$ also called the trivial gerbe.

Classification of gerbes

Similarly as for bundle gerbes there exist notions of product gerbes and equivalences of gerbes.

Theorem

(i) Let M be a manifold and A an abelian Lie group. Then there is a group isomorphism between the group of gerbes with band $\underline{A}_M$ up to equivalence and the group $H^2(M; \underline{A}_M)$.

(ii) In the case $A = \mathbb{C}^\times$ a gerbe with band $\underline{\mathbb{C}}^\times_M$ is called a Dixmier-Douady gerbe and there is a group isomorphism between the group of DD-gerbes up to equivalence and the group $H^3(M; \mathbb{Z})$.

The DD-gerbe associated to a bundle gerbe

Let M be a manifold and (P, Y, M) a bundle gerbe. We want to associate to (P, Y, M) a sheaf of groupoids $\mathcal{G}$ which is a DD-gerbe with $DD(P, Y) = DD(\mathcal{G}) \in H^3(M; \mathbb{Z})$.

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Let $U \subset M$ be an open subset, then we define $\mathcal{G}(U)$ to be the groupoid with objects given by pairs (Q, η) where $Q \to Y_U$ are principal $\mathbb{C}^\times$-bundles over Y_U and η an isomorphism of principal bundles

$$\begin{array}{ccc} \delta(Q) & \xrightarrow{\quad\eta\quad} & P|_{Y_U^{[2]}} \\ & \searrow \quad \swarrow & \\ & Y_U^{[2]} & \end{array}$$

i.e. (Q, η) are local trivializations of P. Morphisms in $\mathcal{G}(U)$ are then isomorphisms of principal bundles which commute with the isomorphisms η.

The DD-gerbe associated to a bundle gerbe

One can check that $\mathcal{G}$ satisfies the descent properties (D1) and (D2) and that conditions (G1),(G2) and (G3) are satisfied, i.e. $\mathcal{G}$ defines a DD-gerbe. Moreover one has that

$$DD(P, Y) = DD(\mathcal{G})$$

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