Higher Lattice Gauge Theory Model

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Topological Phases - 1

Topological phases are classes of matter with long-range entanglement. They may exhibit the following properties:

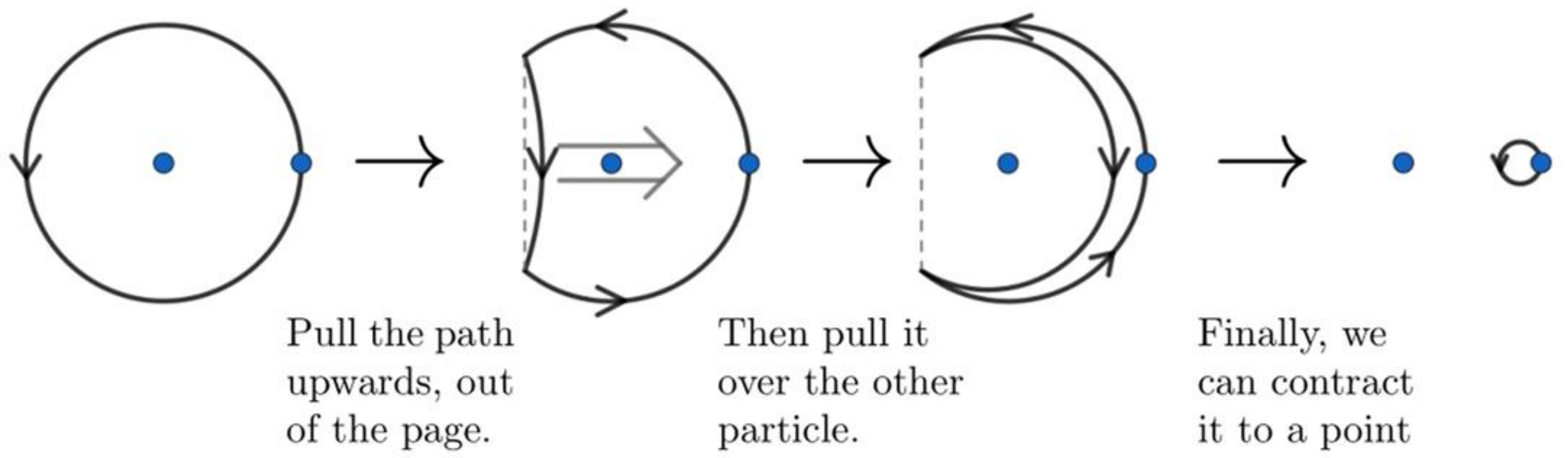
- The ground-states are sensitive to the topology of the manifold we put them on
- Excitations with non-trivial braiding

In the 3D case we have:

- Loop-like excitations
- Bosonic/fermionic point-point braiding

Topological Phases - 2

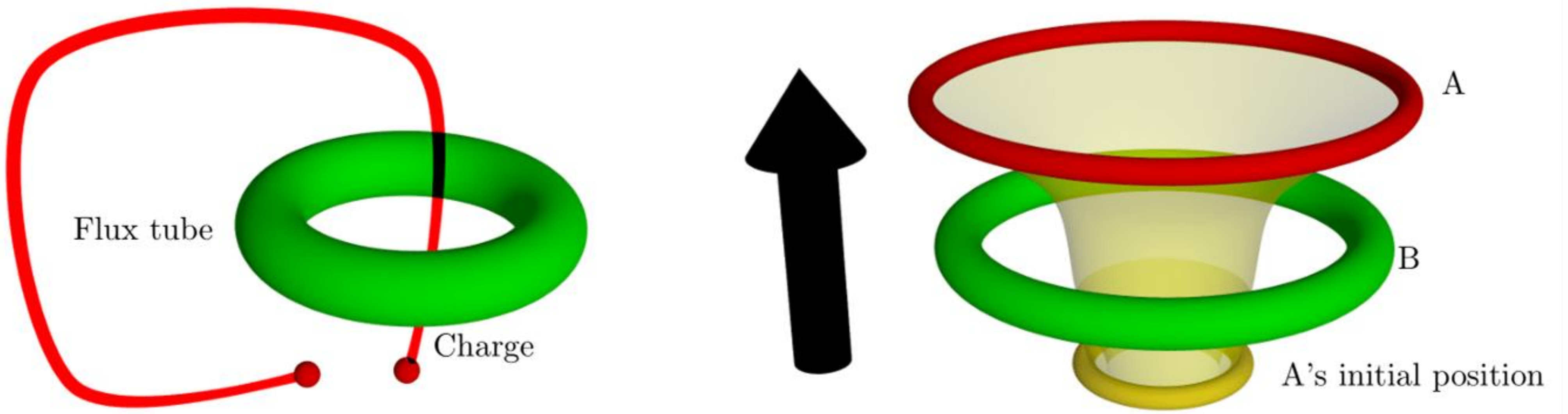
In 3D, braiding between point-like particles is bosonic or fermionic because a full braid is trivial:



Taking a particle all the way around another is equivalent to swapping twice. Therefore the exchange phase squares to 1.

Topological Phases - 3

The non-trivial braiding instead involves loops



Topological Phases - 4

One approach to studying these phases:

- We look at exactly solvable models
- See what kinds of phase are allowed
- We look at the ground states, the excitations and the braiding properties.
- Here we do this for a model based on “higher lattice gauge theory” found by Bullivant et al. [A. Bullivant et al. Phys. Rev. B **95** (2017)]

Introduction

Structure of the talk:

- Consider higher lattice gauge theory (first discussing lattice gauge theory)
- Discuss a model for topological phases in 3D based on it
- Consider how to produce the excitations
- Look at the properties of the excitations
- Discuss the topological sectors

Lattice Gauge Theory

Lattice Gauge Theory - 1

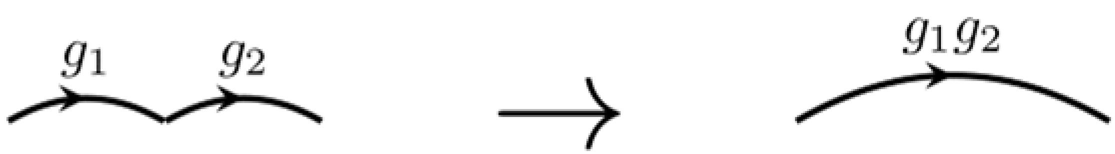
Gauge theory in a continuum:

- Some matter field (e.g electron field)
- A “gauge” field (e.g potentials in E.M)
- A gauge symmetry: field configurations related by gauge transformations are equivalent

Let’s see how these translate to a lattice

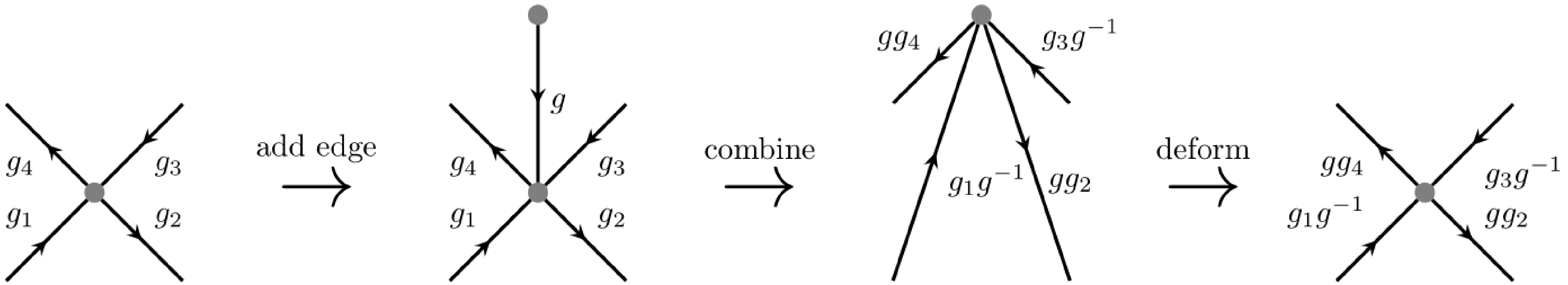
Lattice Gauge Theory -2

- We have a "matter" field on the vertices (which we later take to be trivial)
- A gauge field on edges, valued in some group *G*
- The gauge field describes parallel transport of the matter field across edges
- We can compose edges into paths



Lattice Gauge Theory – 3

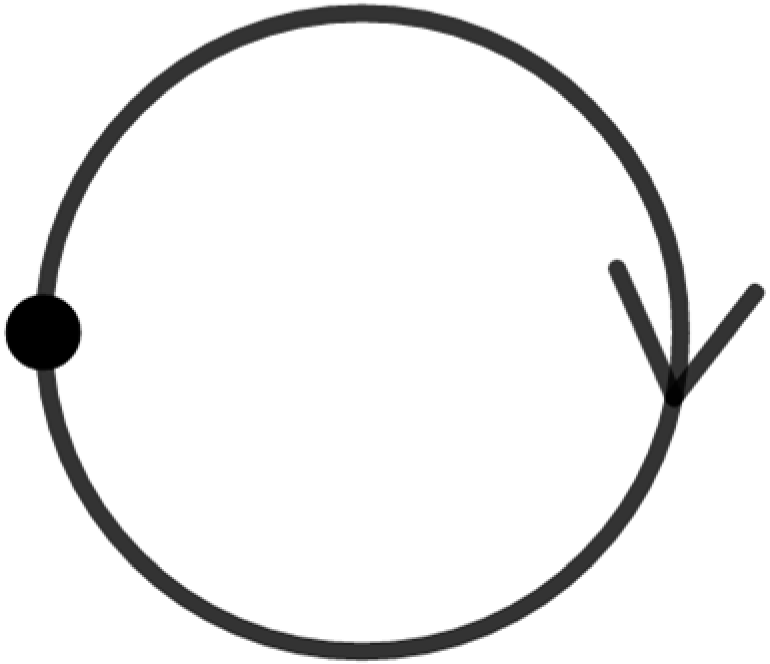
- The gauge symmetry is a set of local operators on each vertex
- The operators act like transporting the vertex (and all attached edges) along an edge
- For a gauge transform labelled by A_v^g:



Lattice Gauge Theory - 4

Physical quantities are gauge-invariant

- Consider a closed loop. Its label is at most conjugated by a gauge transform.
- The conjugacy class is a gauge-invariant quantity



$$g_i \longrightarrow g g_i g^{-1}$$

Lattice Gauge Theory - 5

As an example, consider electromagnetism:

- The group is U(1), i.e a phase
- We have the Aharanov-Bohm effect when a charge is taken around a magnetic flux
- This results in a phase: $e^{i\theta} = e^{iq \oint \vec{A} \cdot d\vec{l}}$
- A non-trivial phase for a closed cycle therefore indicates a non-zero magnetic flux

Lattice Gauge Theory - 6

- For some gauge configurations (sets of labels of the gauge field on each edge), we can make every edge label the identity label 1_G by applying gauge transforms
- Then it is physically equivalent to the case where the parallel transport is trivial
- Therefore the parallel transport of the original configuration must only describe a change of basis
- A non-trivial gauge field describes both a change of basis and a physical “flux”

Quantum Double Model - 1

Now consider making an Hamiltonian model from this theory

- We demote the gauge symmetry to an energetic constraint, with an energy term at each vertex enforcing gauge symmetry
- This is done by averaging over all gauge transforms at the vertex: $A_v = \frac{1}{|G|}\sum_g A_v^g$
- We also punish small loops (plaquettes) with non-trivial group label energetically
- The Hamiltonian is

$$H = -\sum_{\text{vertices, } v} A_v - \sum_{\text{plaquettes, } p} B_p$$

Quantum Double Model - 2

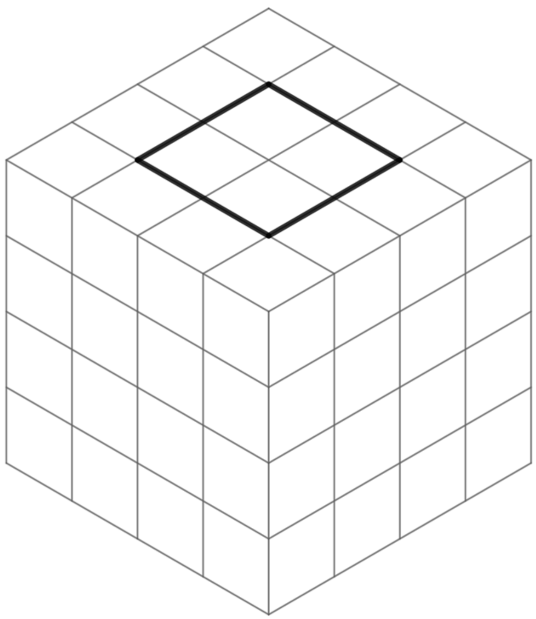
The model obtained by doing this is Kitaev's Quantum Double model [A. Kitaev Annals of Physics **303** (2003)]

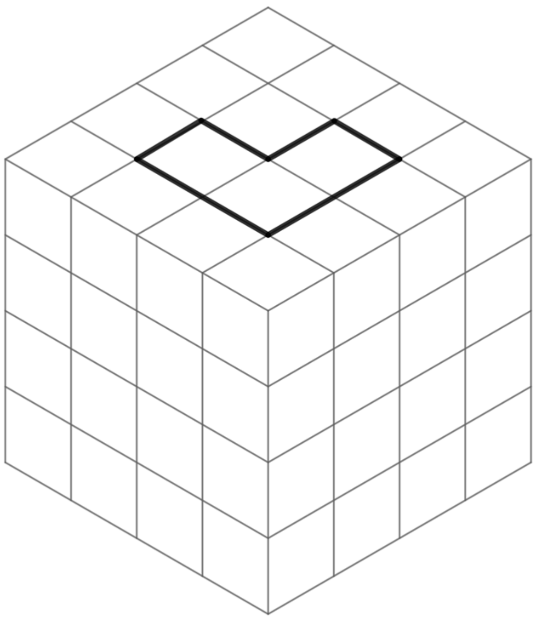
- It is an established model for 2D topological phases
- The excitations are charge-like and flux-like
- The fluxes (magnetic excitations) are associated to plaquettes
- The "charges" (electric excitations) are associated to vertices

Quantum Double Model - 3

We can see how the GSD depends on topology

- On a sphere the plaquette term enforces all closed loops be trivial in the ground state

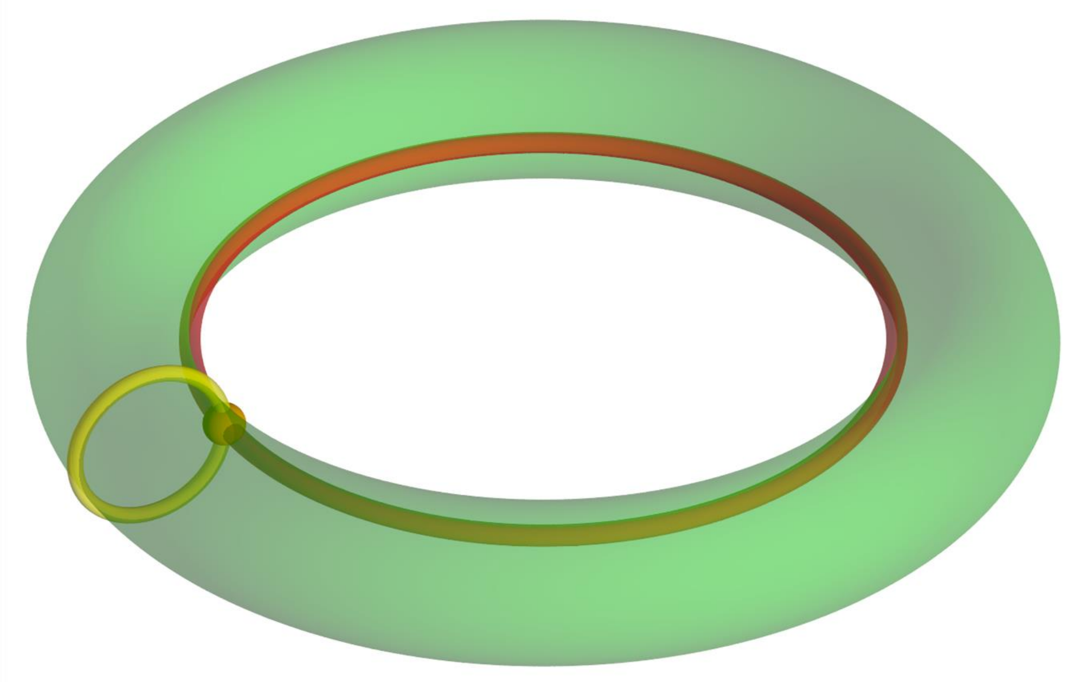




Quantum Double Model - 4

However on a torus there are non-contractible cycles

- The plaquette terms do not affect these
- This means more ways to satisfy the energy terms
- Indeed the flux around the handles labels the ground-states



Higher Lattice Gauge Theory

Higher Lattice Gauge Theory - 1

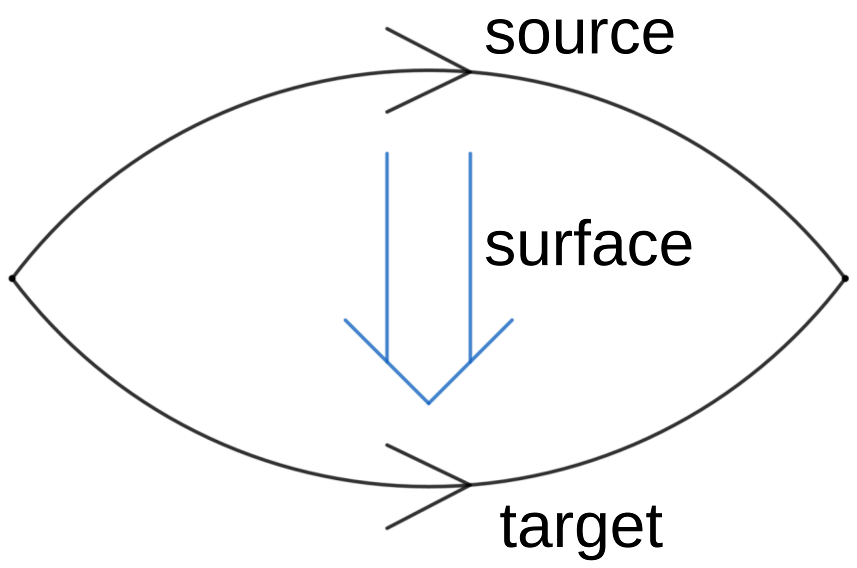
Now we look at Higher Lattice Gauge Theory. We want to generalize Lattice Gauge Theory

In LGT we:

- Consider parallel transport along paths
- Label paths with group elements in G
- We are labelling geometric objects with algebraic ones. So a natural generalization is to label more of the geometry. We already have points and paths. The next objects are therefore surfaces.

Higher Lattice Gauge Theory - 2

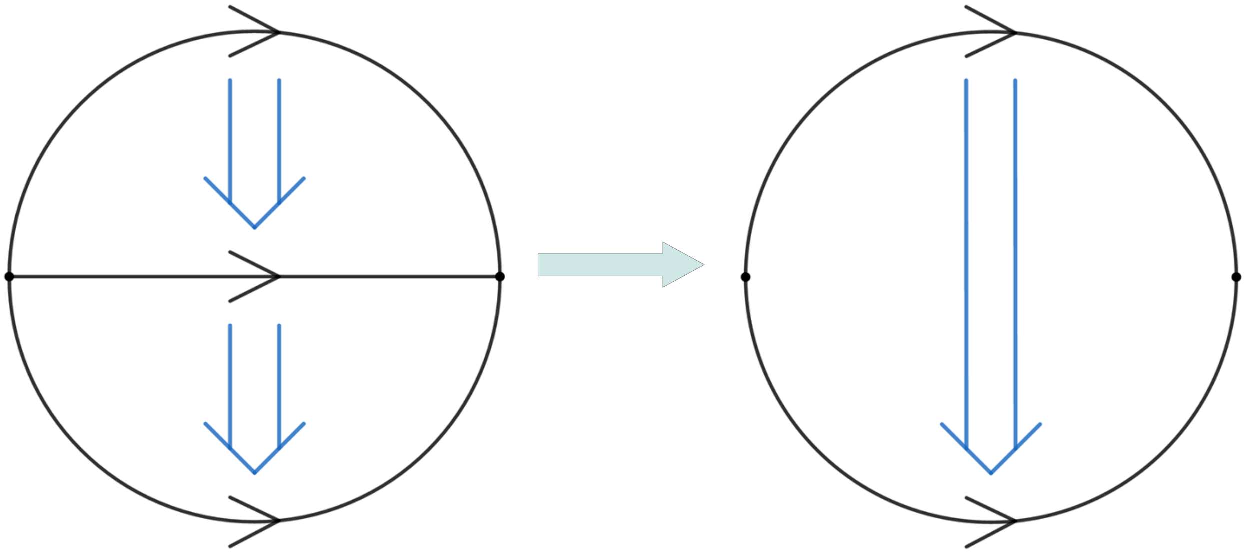
- If paths describe the parallel transport of points, then surfaces describe the parallel transport of paths themselves
- That is, they describe parallel transport of extended objects



Higher Lattice Gauge Theory - 3

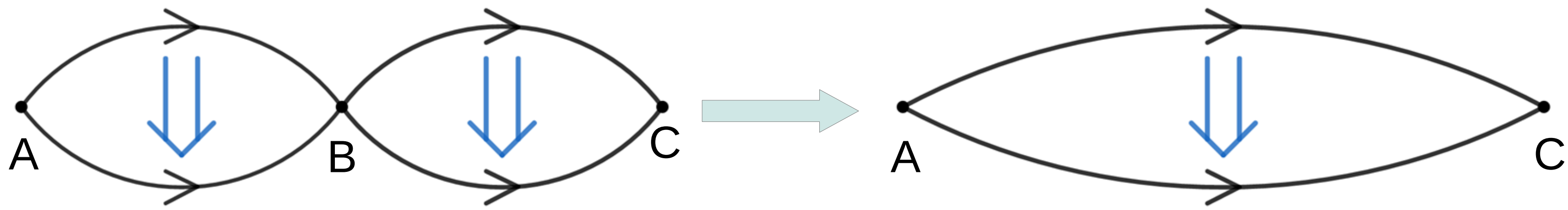
Just as paths may be composed, so may surfaces. They can be composed in two ways

- Vertically:

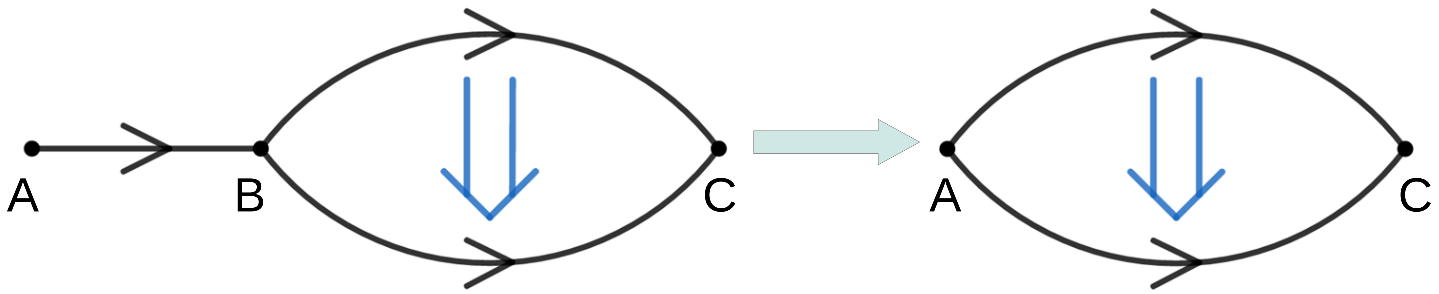


Higher Lattice Gauge Theory - 4

- Horizontally:



- Surfaces can also be composed with edges or paths. This is called whiskering



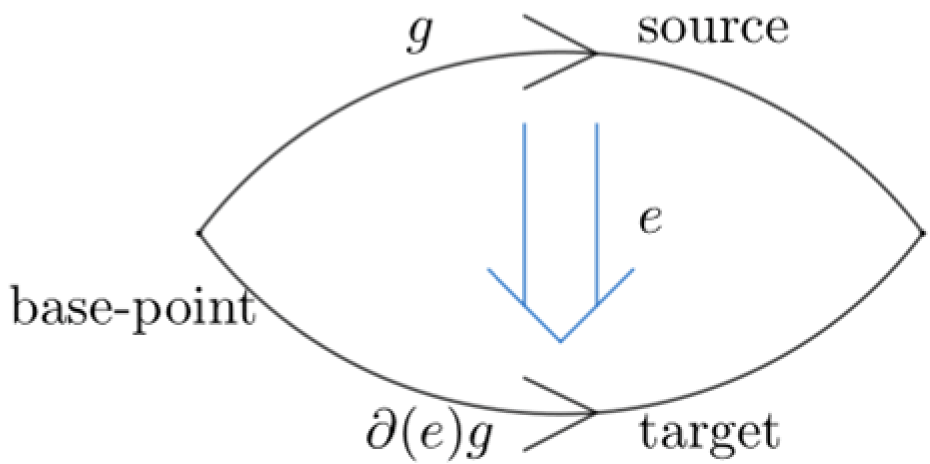
Higher Lattice Gauge Theory - 5

Then we need to determine the correct algebraic structure to describe this.

- The edges are still labelled by the group G as before
- The surfaces will be labelled by a new group E
- This gives a new gauge-field, the 2-gauge field, that lives on the surfaces
- We also have maps between the groups to describe the effect of the various parallel transports

Higher Lattice Gauge Theory - 6

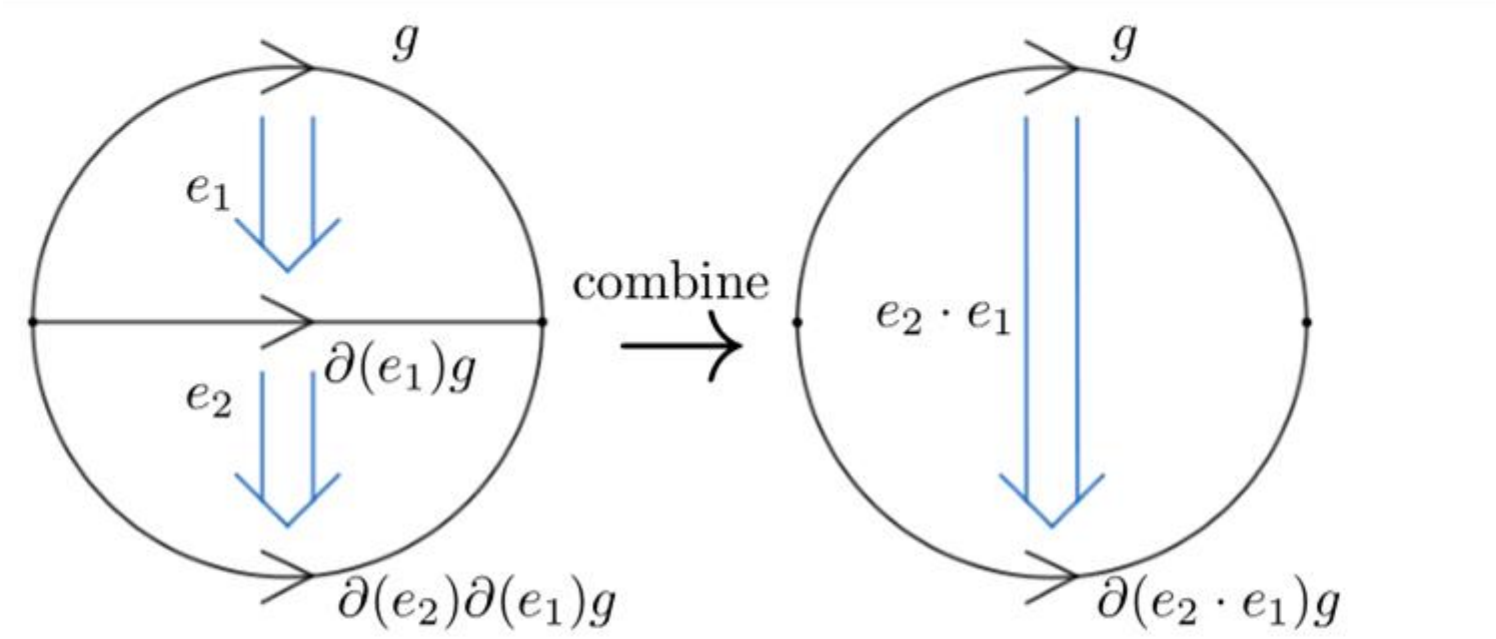
First look back at our parallel transport of paths, adding the labels



- The source path is transported into the target
- The base-point is the vertex at the start of each path
- ∂ is a group homomorphism from E to G

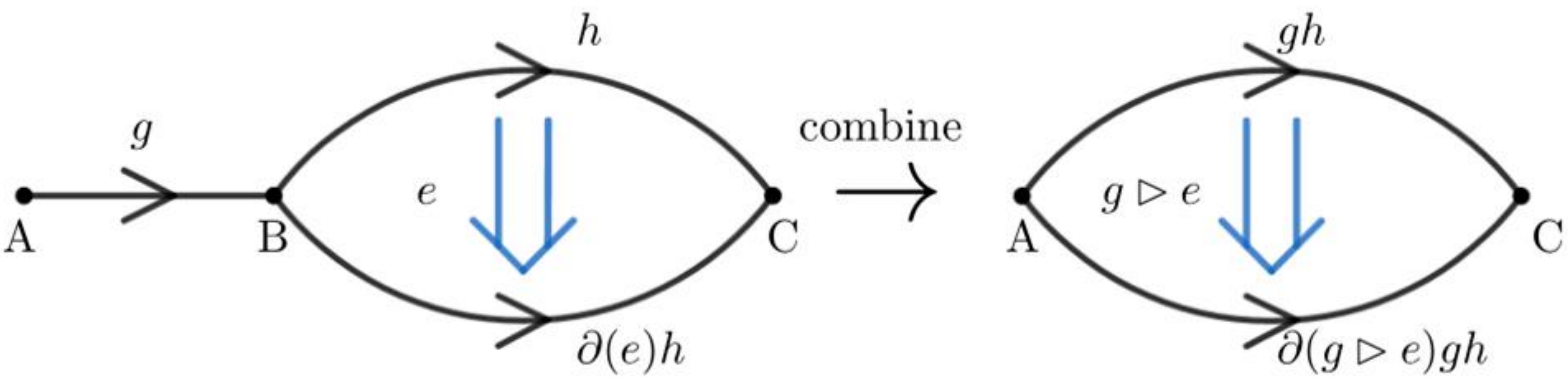
Higher Lattice Gauge Theory - 7

- We can combine surfaces vertically
- We describe this with the group multiplication
- Consistency under vertical composition requires ∂ be an homomorphism



Higher Lattice Gauge Theory - 8

- Whiskering requires a new map, $\triangleright$.
- Whiskering can be thought of as parallel transport of the base-point along an edge
- The group *G* describes transport along edges, so we have some action of *G* on *E*



Higher Lattice Gauge Theory - 9

- The consistency of various diagrams demands that

$$g \triangleright f g \triangleright e = g \triangleright (fe)$$
$$g_1 \triangleright (g_2 \triangleright e) = (g_1 g_2) \triangleright e$$

- This is a group action of *G* on *E*.
- We also have other consistency conditions, called the Peiffer conditions.

$$\partial(g \triangleright e) = g\partial(e)g^{-1} \qquad (P1)$$
$$\partial(e) \triangleright f = efe^{-1} \qquad (P2)$$

- This algebraic structure $(G,\ E,\ \partial,\ \triangleright)$, satisfying these conditions, is called a crossed module

HLGT Gauge Transforms - 1

Now we consider the gauge transforms.

Ordinary gauge transforms:

- change labels around a vertex
- act like parallel transport along edge

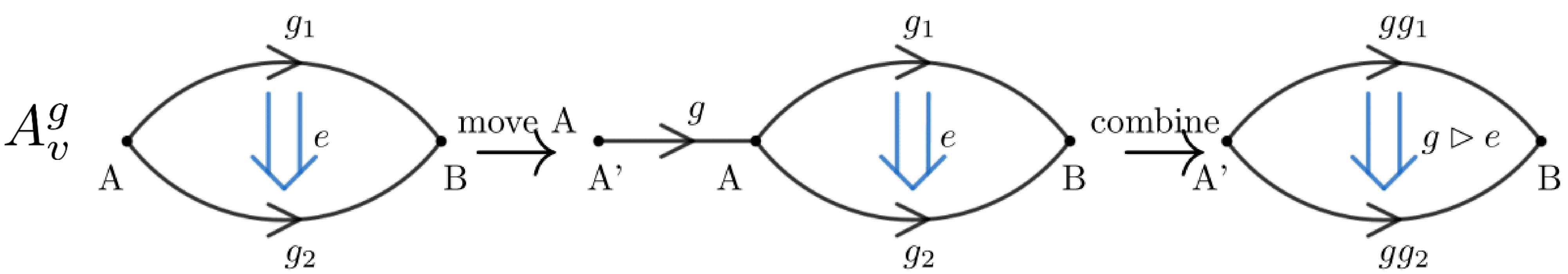
2-gauge transforms:

- change labels around edge
- act like parallel transport along surface

HLGT Gauge Transforms - 2

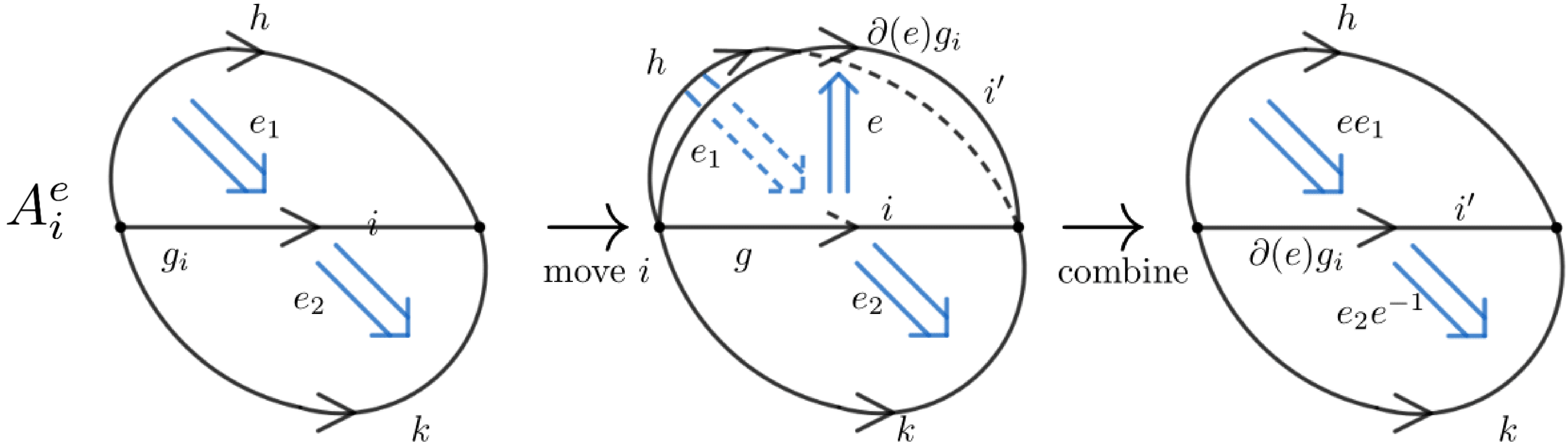
We have a gauge transform on the vertices, labelled A_v^g

- As before (in lattice gauge theory) it acts like parallel transport along an edge
- It acts as before on edges, but also affects surfaces based at that vertex



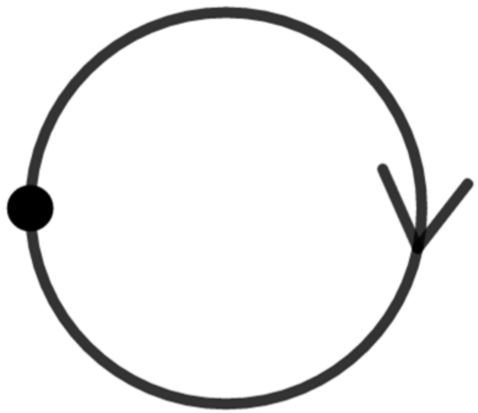
HLGT Gauge Transforms - 3

- In addition to the gauge transforms on the vertices, we have 2-gauge transforms on the edges, labelled A_i^e
- This acts like parallel transport along a surface
- Recall for the vertex transforms we add an edge and combine it, here we add a surface



HLGT Gauge Invariants - 1

We saw in ordinary gauge theory that gauge invariant quantities could be built out of loops.



$$g_i \longrightarrow g g_i g^{-1}$$

What is the equivalent here?

- Loops
- Closed surfaces

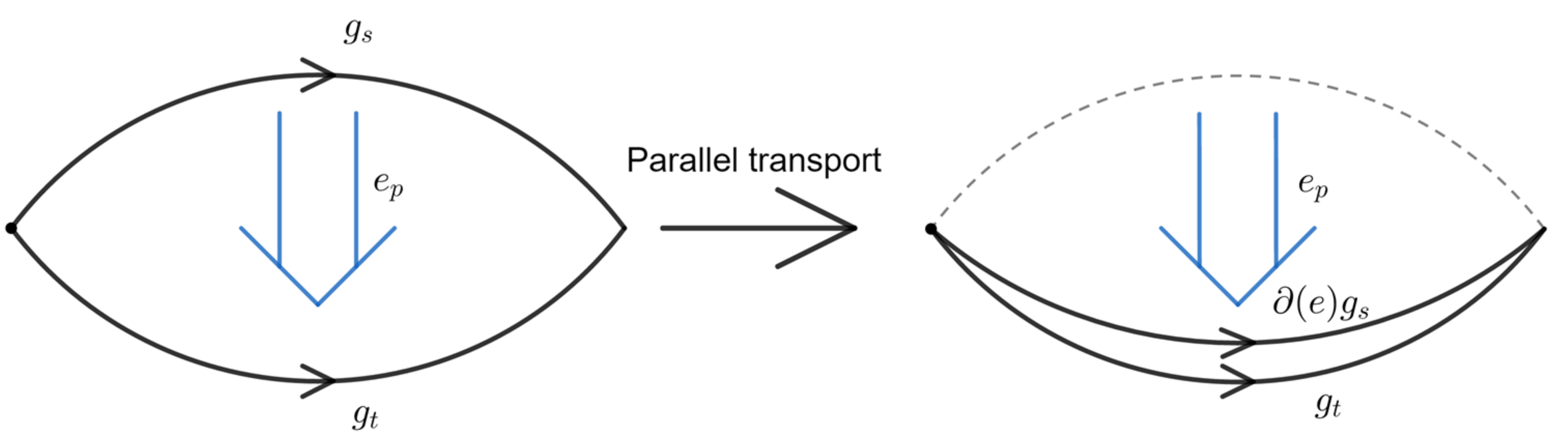
HLGT Gauge Invariants - 2

Closed loop:

- We modify the group element associated to a loop:

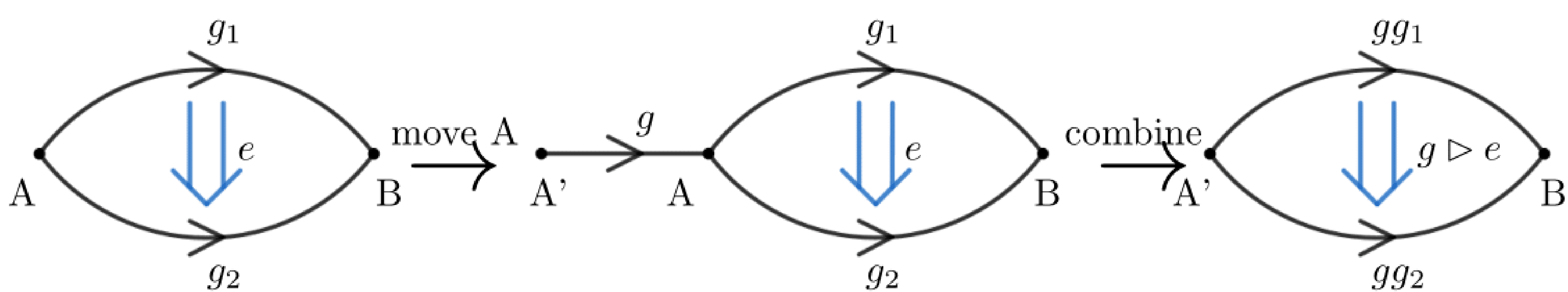
$$g_s g_t^{-1} \to \partial(e) g_s g_t^{-1}$$

- This is due to the parallel transport rules – we have to compare paths at the same position



HLGT Gauge Invariants - 3

- As before, this element is only changed within conjugacy classes by the gauge transforms
- The identity element is in a class of its own, so that a trivial flux is left invariant by the gauge transforms



$$
\begin{aligned}
\partial(e)g_1g_2^{-1} &\to \partial(g \triangleright e)gg_1g_2^{-1}g^{-1} \\
&= g\partial(e)g^{-1}gg_1g_2^{-1}g^{-1} \quad (P1) \\
&= g\partial(e)g_1g_2^{-1}g^{-1}
\end{aligned}
$$

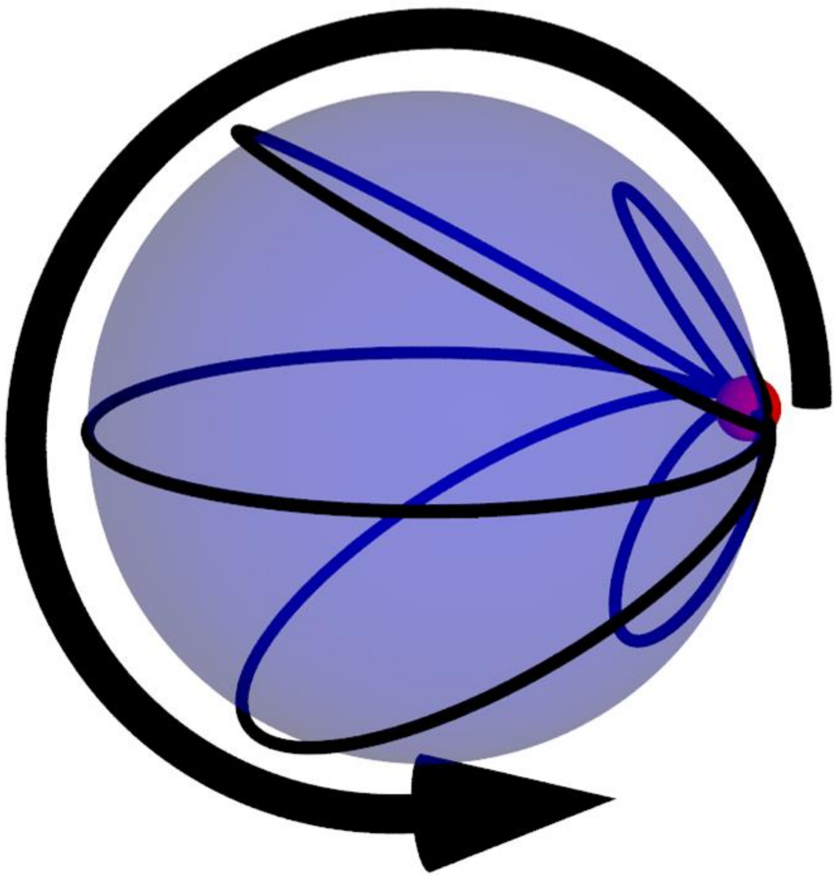
HLGT Gauge Invariants - 4

Closed surface:

- We call the surface label the 2-flux
- The surface label is only changed within a class (not a conjugacy class exactly) by the gauge transforms
- These classes are therefore gauge-invariant quantities
- The identity element is in a class of its own

HLGT Gauge Invariants - 5

The 2-flux on a closed surface corresponds to the process where we nucleate a small loop and pull it over the surface before re-collapsing it



HLGT Hamiltonian Model - 1

Now we consider a Hamiltonian model based on Higher Lattice Gauge Theory

- We work on a “lattice” with directed edges and surfaces with given base-points and orientations
- We put labels from the group G on the edges and from the group E on the plaquettes
- The different possible sets of labels for each edge and plaquette of the lattice are configurations and these form a basis for the Hilbert space

HLGT Hamiltonian Model - 2

Consider the Hamiltonian

- We demote the gauge transforms to energy terms
- We do this by putting energy terms that average over all of these transforms

$$A_v = \frac{1}{|G|} \sum_g A_v^g \qquad A_i = \frac{1}{|E|} \sum_e A_i^e$$

$$H = -\sum_{\text{vertices, } v} A_v - \sum_{\text{edges, } i} A_i + \ldots$$

HLGT Hamiltonian Model - 3

We will also assign energy terms to the closed loops and surfaces we considered earlier, enforcing that they have trivial label

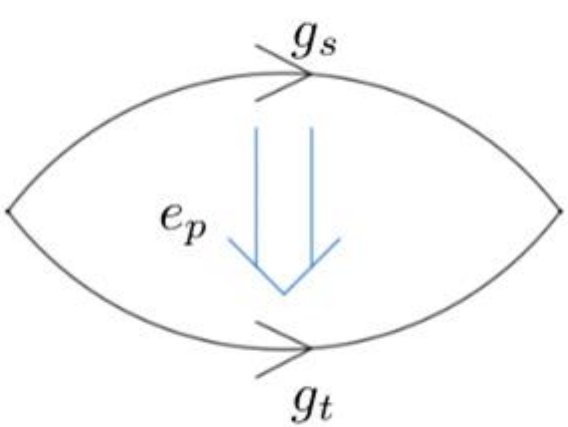
- These energy terms are gauge-invariant, so they commute with the other terms
- Excitations of these terms will correspond to non-trivial fluxes and 2-fluxes

HLGT Hamiltonian Model - 4

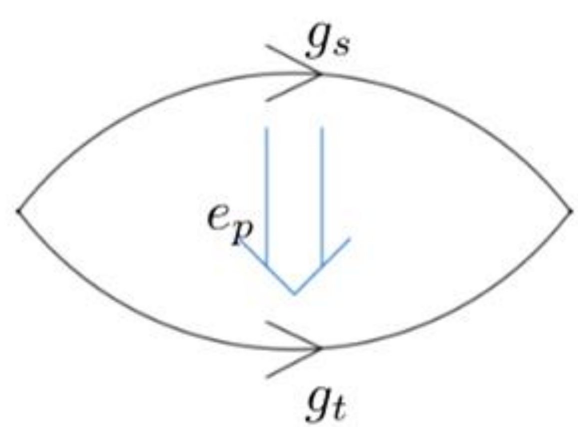
We have an energy term on each plaquette

- We impose the parallel transport rules as an energetic constraint
- This means that we enforce trivial flux for each plaquette

$B_p :$



$= \delta(\partial(e_p)g_s, g_t)$

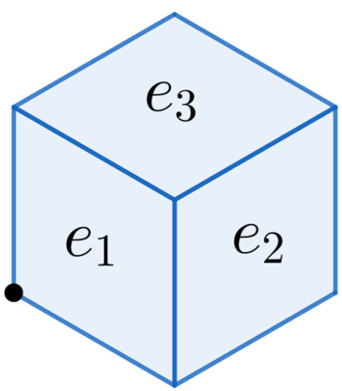


HLGT Hamiltonian Model - 5

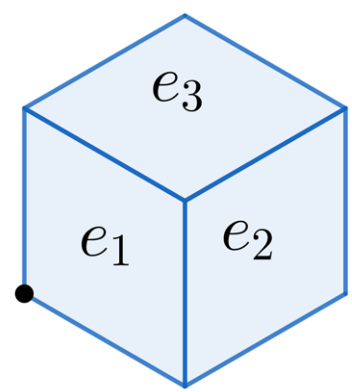
Finally we add an energy term associated to the blobs (3-cells) of the lattice

- These punish closed surfaces (boundaries of blobs) with non-trivial group label

$B_b :$



$= \delta(e(b), 1_E)$



$$e(b) \sim e_1 e_2 e_3 \ldots$$

HLGT Hamiltonian Model - 6

To summarize

$$H = -\sum_{\text{vertices},\ v} A_v - \sum_{\text{edges},\ i} A_i - \sum_{\text{plaquettes},\ p} B_p - \sum_{\text{blobs},\ b} B_b$$

The first two terms enforce gauge symmetry, the latter two restrict the fluxes of closed loops and closed surfaces

The ground state degeneracy depends on the topology of the lattice, as can be seen from the fact that the last two terms are sensitive to the contractible cycles and surfaces

Excitations

Excitations

There are four types of excitations, corresponding to the four energy terms

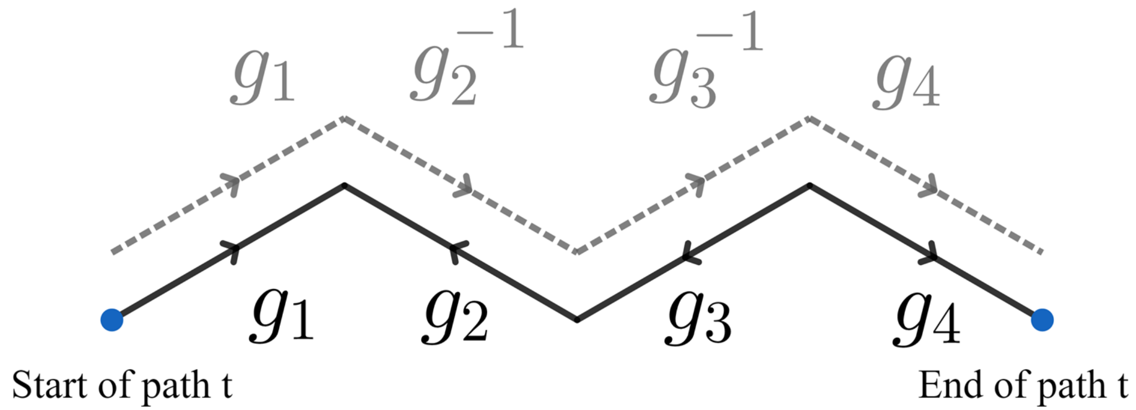
- Two of the types are point-like
- Two are loop-like
- General excitations are combinations of these four types

“Electric” Excitations - 1

- The “electric” excitations are point-like
- They are created by measuring the value of a path element and applying a weight depending on that value:

$$\hat{S}(t) = \sum_g [D^\mu(g)]_{ab}\delta(g, \hat{g}(t))$$

- The excitations are associated to the ends of the path



$$g(t) = g_1 g_2^{-1} g_3^{-1} g_4$$

"Electric" Excitations - 2

$$\hat{S}(t) = \sum_g [D^\mu(g)]_{ab}\delta(g, \hat{g}(t))$$

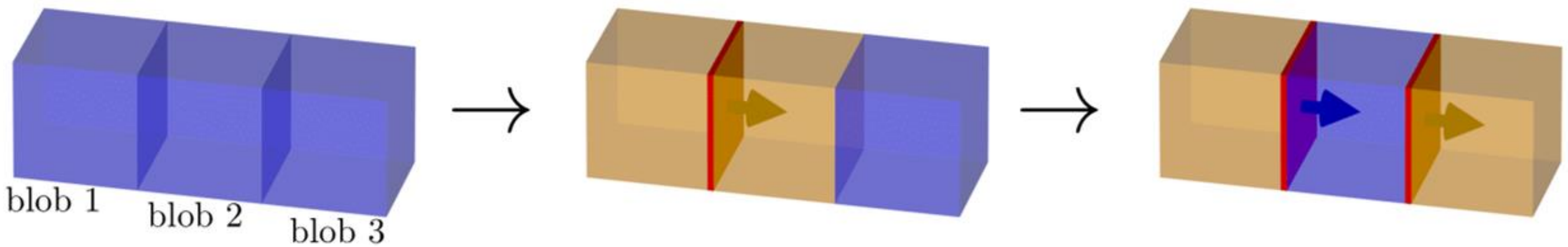
- μ is an irreducible representation of *G*
- $D^\mu(g)$ are the representative matrices
- a and b are the matrix indices
- The irrep labels a conserved charge, we say it describes a *topological sector*
- The matrix indices describe non-conserved local degrees of freedom.

“Blob” Excitations - 1

The other point-like excitations are the “blob” excitations

- They are labelled by group elements of E
- The blob energy terms essentially enforce that a product of surface elements over a blob is trivial
- Then to create excitations, we try changing one plaquette at a time.
- This will create two excitations at the end of an (invisible) string of plaquettes

"Blob" Excitations - 2



- We change the plaquette between blobs 1 and 2
- This leads to excitations of both blobs
- We can correct blob 2 by changing another plaquette
- This excites the third blob, leaving blobs 1 and 3 excited.
- The excitations are produced in pairs
- They are produced by ribbon operators

E-valued Loops - 1

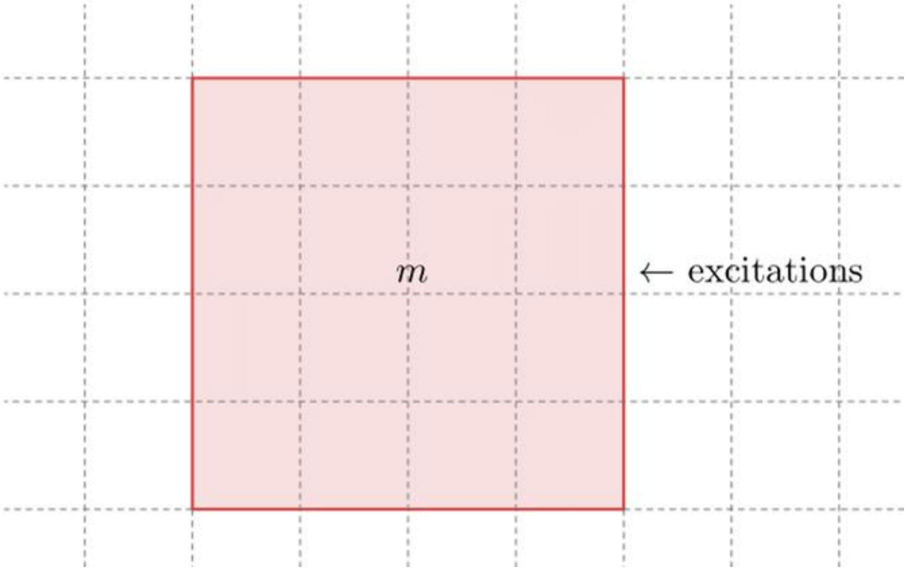
The remaining excitations are loop-like.

First we have the *E*-valued loops

- These are produced by measuring the surface label of a membrane and applying a weight to each possible label
- For a membrane *m*, the membrane operator is

$$\hat{L}^{\mu,a,b} = \sum_{e \in E} [D^{\mu}(e)]_{ab} \delta(e, \hat{e}(m))$$

E-valued Loops - 2



- The edges on the boundary of the membrane are excited
- One of the vertices may also be excited

Flux Tubes -1

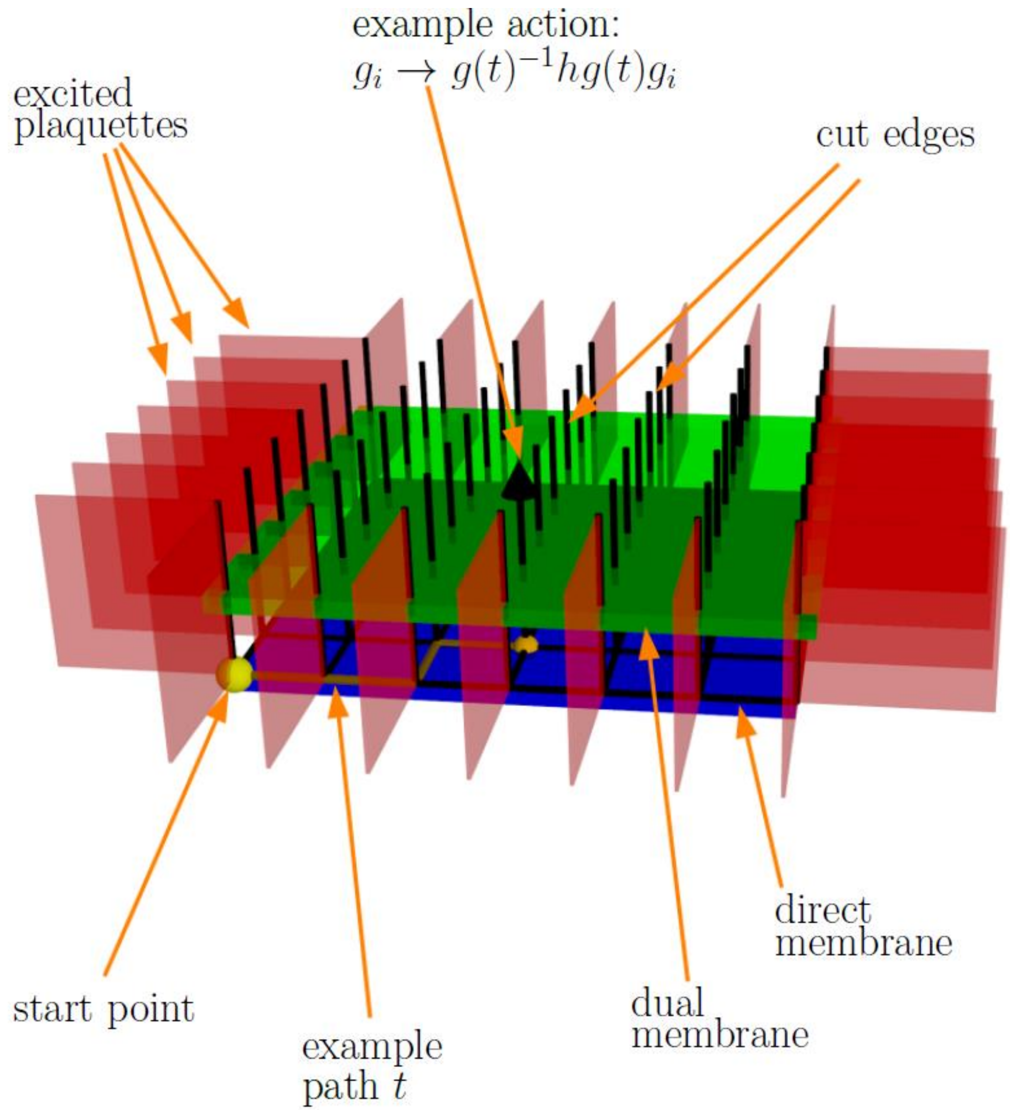
The final excitations (plaquette excitations) are the most complicated

These excitations are simpler in the $\rhd$ trivial case (where $g \rhd e = e$):

- We change the edges cut by a membrane (the "dual membrane")
- We multiply the edge labels by $g(t)^{-1}hg(t)$ where *g(t)* is the path from a privileged start point to the edge being changed

Flux Tubes - 2

- This excites the plaquettes in a loop around the membrane



Condensation and Confinement

Confinement

Normally the excitations can be separated freely, with energy cost only to create them

Here we see confinement of some of the point particles

That is, there is an energy cost to move the excitations

- The electric excitations are confined if their irrep μ has a non-trivial matrix for some element in the image of ∂, that is if $D^{\mu}\big(\partial(e)\big) \neq I$ for some *e* in *E.*
- The blob excitations are confined if their element is not in the kernel of ∂, that is if $\partial(e) \neq 1_G$

Condensation

Another important feature is condensation

A condensation transition is where topologically non-trivial excitations become topologically trivial as the Hamiltonian is changed (condensed excitations)

During this process, other excitations become confined

Condensation - 2

We can consider many HLGT models with the image of ∂ non-trivial as having undergone a condensation/ confinement transition from one with $\partial : E \rightarrow 1_G$

Some of the loop excitations condense, so that they carry trivial topological charge

This means that some of the loop excitations can be produced by operators that act only near the loop, rather than across a whole membrane.

Condensation - 3

The magnetic excitations with label in the image of ∂ are condensed.

The E-valued loops with representations that have trivial restriction to the kernel are condensed. That is an irrep μ is condensed if $D^{\mu}(e_k) = I$ for all e_k in the kernel of ∂.

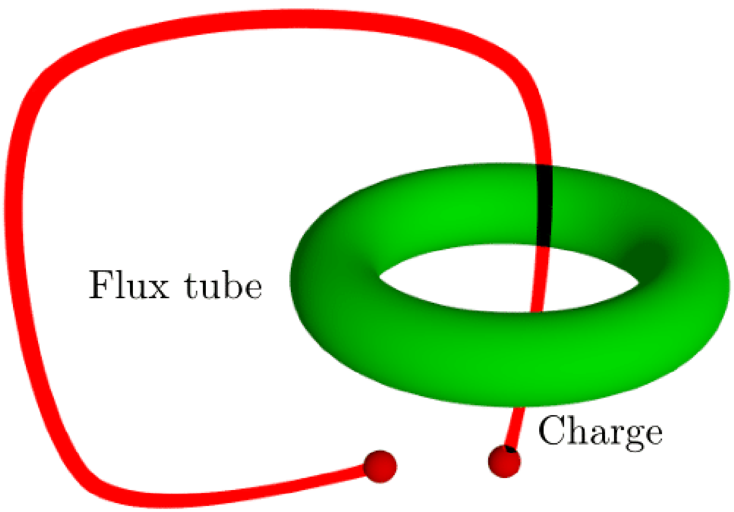
Both condensation and confinement are controlled by ∂.

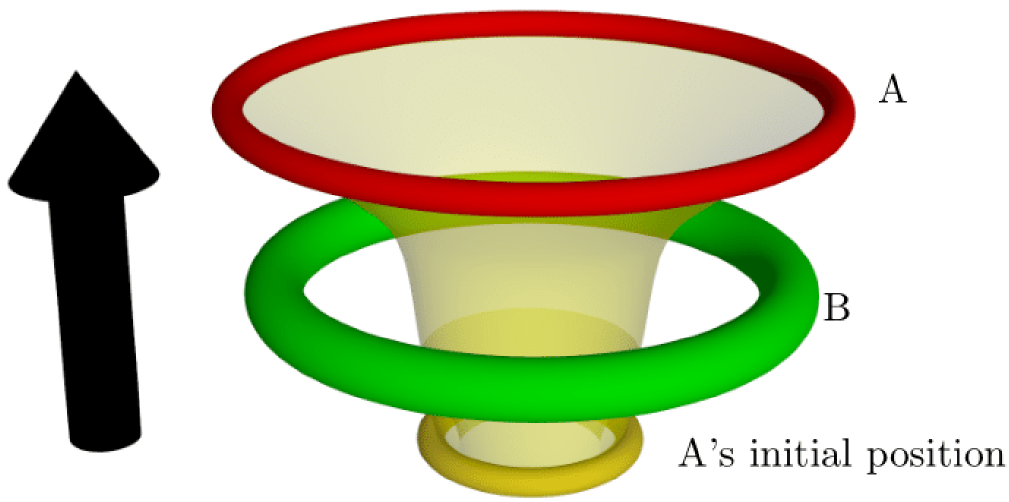
Braiding

Braiding- 1

Next we consider the braiding

Recall that the non-trivial braiding always involves loops

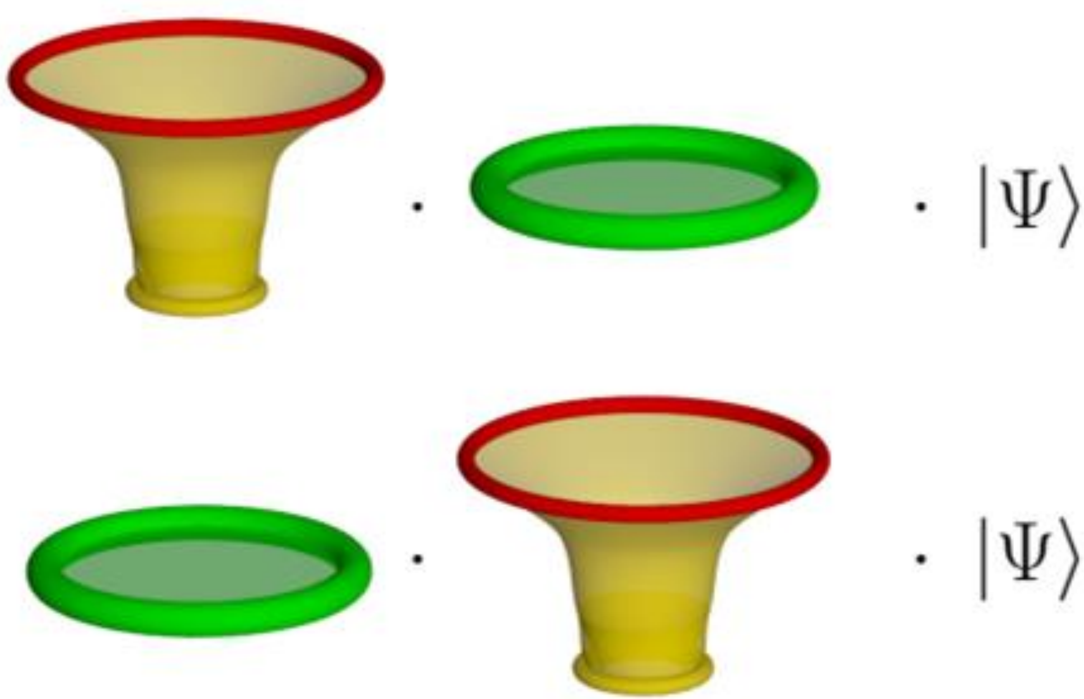




Braiding - 2

We obtain the braiding relations by using the membrane operators

We can relate braiding to a commutation of the operators



Braiding - 3

In the $\triangleright$ trivial case (where all $g \triangleright e=e$) , the braiding is fairly simple:

- There is non-trivial braiding between electric and magnetic excitations
- There is similar braiding between the blobs and *E*-valued loops
- There is also non-trivial loop-loop braiding between magnetic excitations when *G* is non-Abelian
- We see braiding is only between the *G* labelled excitations and between the *E* labelled ones.

Braiding - 4

- For an electric excitation labelled by an irrep μ of G braiding with a magnetic excitation labelled by h, we have a transformation $D^{\mu}(h)$ which is a phase $\mu(h)$ if the irrep is 1D.
- Similarly for a blob excitation labelled by e braiding with a loop excitation labelled by irrep. α of E, we gain a transformation of $D^{\alpha}(e)$.
- Two magnetic loops braid by conjugation of one of the loops.

Braiding - 5

In the $\triangleright$ non-trivial case (but still a special case where *E* is Abelian and ∂ maps onto the centre of *G*), the magnetic excitations braid non-trivially with everything else.

- In this case, they are labelled by two group elements, one from *G* and one from *E*
- Then the excitations are labelled by pairs *(g,e)*.
- We find that two loop excitations *(g,e)* and *(h,f)* now braid to become $(hgh^{-1}, h \triangleright e)$ and $(h, efh \triangleright e^{-1})$.

Topological Sectors

Topological Sectors - 1

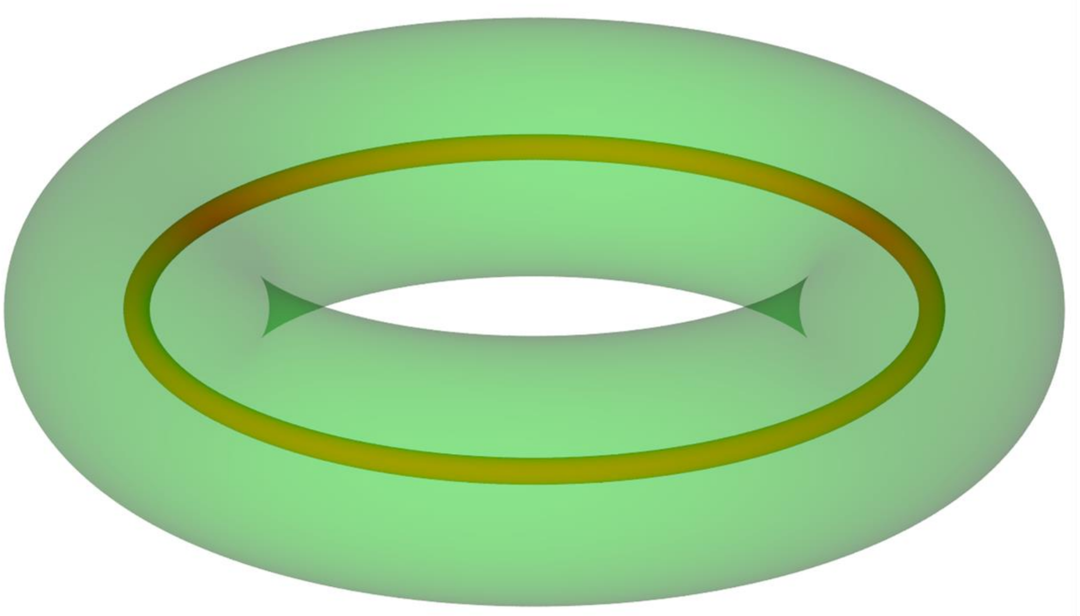
Next we consider topological charge

- Topological charge is a conserved quantum number associated with the excitations
- The charge held within a surface can only be changed by moving charge from inside to outside, not by operators within the surface

Topological Sectors - 2

We measure the charge within a surface, using operators on that surface.

For example, we can measure the charge of a loop by putting an operator on a torus around it.



Topological Sectors - 3

We use closed versions of our membrane operators to measure the charge, following Bombin et al. [H. Bombin and M. A. Martin-Delgado Phys. Rev. B **78**, 115421 (2008)]

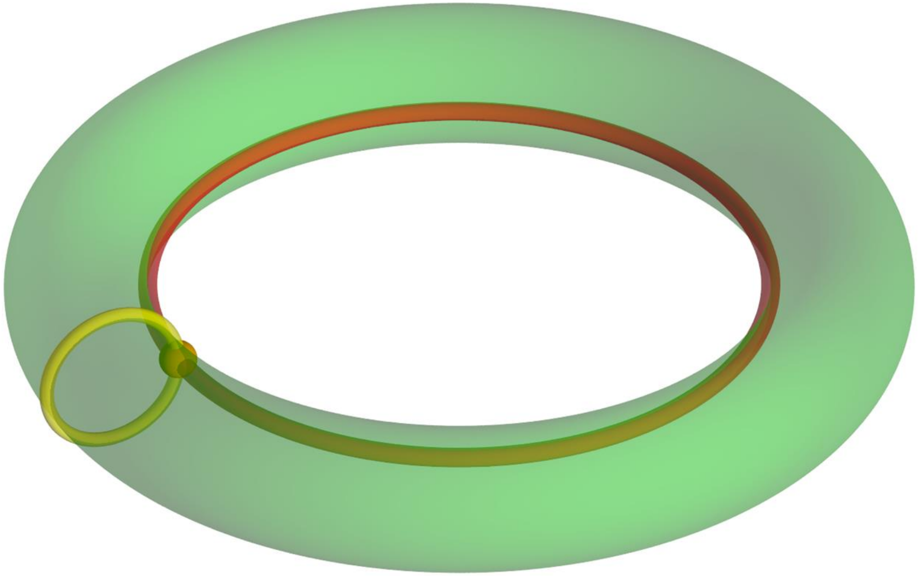
We choose a surface to measure on, then apply every independent operator we can – closed ribbon operators on cycles and membrane operators on the surface itself- subject to restrictions.

The measurement operator must not create excitations, and deforming it without crossing an excitation must not affect the value of charge

Topological Sectors - 4

We can do this on any orientable surface. The charge within a torus is sensitive to loop-like and point-like charge.

- We apply ribbon operators around the two cycles
- We apply membrane operators on the surface



Topological Sectors - 5

The number of allowed measurement operators gives the number of types of topological charge.

- We find that this number is equal to the ground-state degeneracy on the 3-torus
- This is an extension of the 2D result that the number of topological charges is equal to the ground-state degeneracy on a 2-torus (given sufficiently non-trivial braiding)
- This was also found by Bullivant et al. [A. Bullivant and C. Delcamp arXiv: 1909.07937v1 (2019)]

We can explicitly construct the charge measurement operators.

Future Work

Future steps: see where the model fits into the landscape of topological phases as a whole

There have been recent suggestions that a large class of models in 3D (including this one) should relate to a "twisted" version of lattice 1-gauge theory

See under what circumstances the ground-state degeneracy and charge are related for more general models

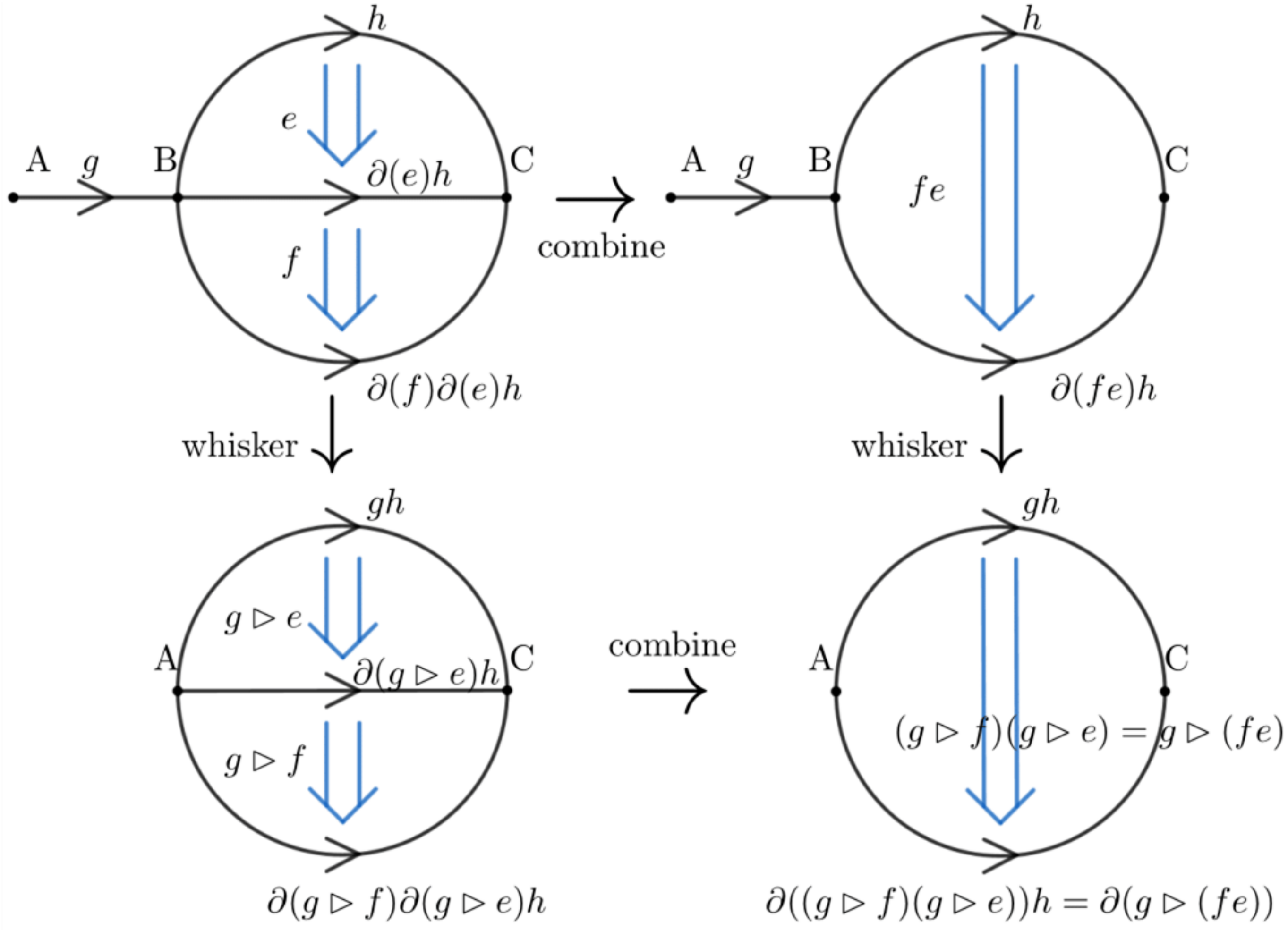
References and Resources

- A. Yu. Kitaev, "Fault-tolerant Quantum Computation by Anyons", Annals of Physics **303** (2003) 2–30
- J.C. Baez and J. Huerta, "An Invitation to Higher Gauge Theory" arXiv:1003.4485 v2 (2015)
- A Bullivant, M. Calcada et al., "Topological phases from higher gauge symmetry in 3+1D", Phys. Rev. B **95**, 155118 (2017)
- A. Bullivant and C. Delcamp, "Excitations in strict 2-group higher gauge models of topological phases" arXiv: 1909.07937v1 (2019)
- H. Bombin and M. A. Martin-Delgado, "A Family of Non-Abelian Kitaev Models on a Lattice: Topological Condensation and confinement" , Phys. Rev. B **78**, 115421 (2008)

Extra Slides

Higher Lattice Gauge Theory

The map $\triangleright$ has properties fixed by these extra diagrams



$$g \triangleright f g \triangleright e = g \triangleright (fe)$$

Higher Lattice Gauge Theory - 7

We also have consistency under composition of edges