# Hawking's Singularity Theorem in General Relativity

**Pedro F. C. Oliveira**

LisMath Seminar

Complexo Interdisciplinar da Universidade de Lisboa
Lisbon, 13th March 2015

Acknowledgements: Prof. José Natário (IST, U. Lisbon)

# Outline

# Outline



TÉCNICO LISBOA

# Outline

1. Fundamentals of Lorentzian geometry
2. Fundamentals of general relativity
   - The Einstein field equations
   - Spacetimes with timelike singularities
3. Causality
4. Hawking's singularity theorem

# Outline

1. Fundamentals of Lorentzian geometry
2. Fundamentals of general relativity
   - The Einstein field equations
   - Spacetimes with timelike singularities
3. Causality
4. Hawking's singularity theorem

# Lorentzian manifolds

- A **Lorentzian manifold** is a pair $(M, g)$, where $M$ is a differentiable manifold and $g$ is a nondegenerate symmetric 2-tensor with signature $(- + \ldots +)$ (called the **metric**).

- $M$ admits a Lorentzian metric iff it is noncompact or $\chi(M) = 0$.

- The simplest example is the **Minkowski spacetime**:

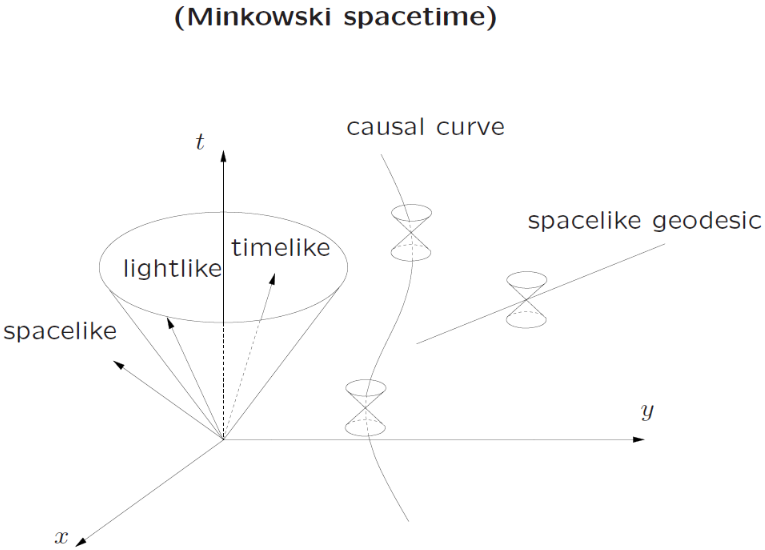
$$M = \mathbb{R}^{n+1}, \quad g = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + \ldots + dx^n \otimes dx^n$$

# Comparison with Riemannian geometry

- Most classic results from Riemannian geometry are valid.
- But not all:
  - Triangle inequality (twin paradox);
  - Hopf-Rinow theorem (geodesic completeness);
  - Vectors $v$ come in three types:
    - **Timelike**: $g(v, v) < 0$
    - **Lightlike**: $g(v, v) = 0$
    - **Spacelike**: $g(v, v) > 0$

  ($v$ is **causal** if it is not spacelike).

# Comparison with Riemannian geometry



# Basic concepts

- A curve $c : I \subset \mathbb{R} \to M$ is called **timelike**, **lightlike**, **spacelike** or **causal** if so is $\dot{c}$.
- If $c$ is a **geodesic**, then its type cannot change.
- $\tau(c)$, the **length** of the timelike curve $c\colon [a, b] \to M$, is the **proper time** between **events** $c(a)$ and $c(b)$.

**Definition**

$(M, g)$ is **singular** if it is not geodesically complete.

# Einstein field equations

- Einstein postulated the equation system $Ric - \frac{S}{2}g = T$ and Hilbert derived it from the variational principle.
- The spherically symmetric vacuum ($T = 0$) case is the **Schwarzschild solution**. In $1+3$ dimensions,

$$g = -\left(1 - \frac{2M}{r}\right) dt \otimes dt + \left(1 - \frac{2M}{r}\right)^{-1} dr \otimes dr + \\ + r^2 \left(d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi\right)$$

The surface $\{r = 2M\}$ is the **black hole** horizon. In the inner region, **every** timelike geodesic reaches the curvature singularity $r = 0$ in **finite time**.

# Einstein field equations

- Consider the Riemannian 3-manifold $(\Sigma, h)$ with constant curvature $k$. $\left(\mathbb{R} \times \Sigma, -dt \otimes dt + a^2(t)h\right)$ is the FLRW universe.

- Friedmann used the Einstein equations to obtain

$$\frac{\dot{a}^2}{2} - \frac{\alpha}{a} = -\frac{k}{2}$$

where the density is $\rho = \frac{6\alpha}{a^3}$.

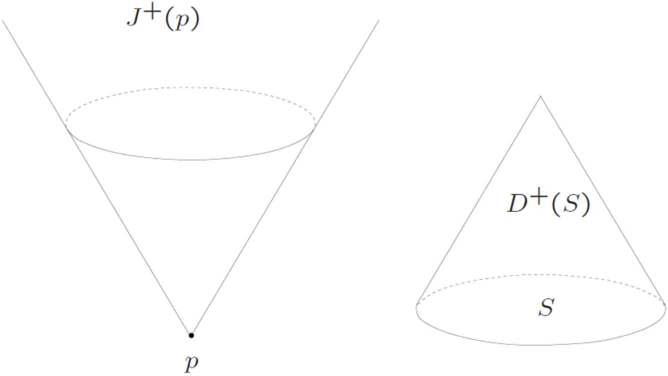
- If $\alpha > 0$, then $a$ either blows up or goes to zero in finite time (**Big Bang**).

# Basic concepts

- Timelike $v$ and $w$ have the same **time orientation** if $\langle v, w\rangle < 0$.
- $(M, g)$ is **time-orientable** if $\exists$ nonvanishing timelike $T \in \mathfrak{X}(M)$. A timelike vector $v \in T_pM$ is **future-pointing** if it has the same time orientation as $T_p$. A timelike curve $c$ is **future-directed** if $\dot{c}$ is future-pointing.
- **Chronological future** of $p \in M$ – set $I^+(p)$ of all points to which $p$ can be connected by a by a future-directed timelike curve. **Causal future** of $p \in M$ – set $J^+(p)$ of all points to which $p$ can be connected by a by a future-directed causal curve.

# Basic concepts

**(Minkowski spacetime)**



# Main results

**Proposition**

***Locally**, the **topological properties** of a time-oriented Lorentzian manifold are similar to those of the Minkowski spacetime. In particular:*

- $q \in I^+(p)$ *and* $r \in I^+(q) \Rightarrow r \in I^+(p)$;
- $I^+(p)$ *is open.*

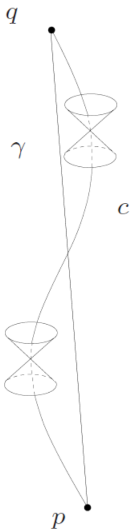
**Proposition (twin paradox)**

*Let* $(M, g)$ *be time-oriented. Every* $p_0 \in M$ *has a geodesically convex open neighbourhood* $V \subset M$ *such that, in* $(V, g_{|V})$*, timelike geodesics have maximal length.*

TÉCNICO LISBOA

# The twin paradox

$$\tau(\gamma) > \tau(c)$$



# Some further concepts

- $(M, g)$ satisfies the **chronology condition** if it does not contain closed timelike curves (cannot be compact).
- $(M, g)$ is **stably causal** if $\exists$ a **global time function**, i.e., a smooth function $t\colon M \to \mathbb{R}$ such that $\mathrm{grad\ t}$ is timelike.
- A future-directed causal curve $c\colon ]a, b[\to M$ is **future-inextendible** if $\lim_{t\to b} c(t)$ does not exist.
- The **future domain of dependence** of $S \subset M$ is

$$D^+(S) = \{p \in M \mid \text{every past-directed inextendible causal curve intersects S}\}$$

**Proposition**

*If $p \in D^+(S)$, then $D^+(S) \cap J^-(p)$ is compact.*

# Essential conditions

**Definition**

(M,g) is said to be **globally hyperbolic** if it is stably causal and all time slices $S_a = t^{-1}(a)$ satisfy $D(S_a) = M$.

- $(M, g)$ will always be assumed to be globally hyperbolic and $S$ will always denote a time slice.
- **Strong energy condition:** $Ric(V, V) \geq 0$ for every timelike $V$.
- Let $n$ be the future-pointing unit normal of $S$ and $c_p$ the geodesic orthogonal to $S$ and tangent to $n$ at $p \in S$.
  The critical values of the **exponential map** ($\exp(t, p) = c_p(t)$) are said to be **conjugate points** to $S$.

# Main results

- If $c_p$ has no conjugate points between $c_p(0) = p$ and $c_p(t_0)$, $\exists$ an open neighbourhood $V$ of $c_p([0, t_0])$ where $\exists$ a family of timelike geodesics tangent to a unit vector field $X = -\mathrm{grad}\ \mathrm{t}$.
- $\theta = \mathrm{div}\ \mathrm{X}$ is called the **expansion**.

**Proposition**

*Let $(M, g)$ satisfy the SEC, $p \in S$ and $\theta = \theta_0 < 0$.*
*Then, if $c_p$ can be extended to a length $\tau_0 = -n/\theta_0$ to the future of $S$, it has a conjugate point.*

# Main results

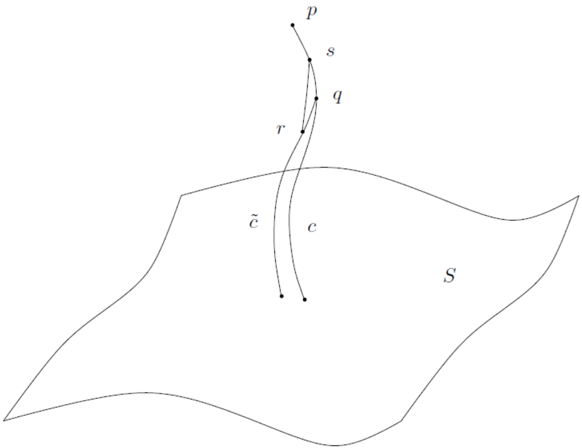
**Proposition**

*Let $c$ be a geodesic through $p \in M$ orthogonal to $S$.*
*Then, if $\exists$ a conjugate point between $S$ and $p$, $c$ does not maximize length.*

**Proposition**

*$\exists$ a timelike curve connecting $S$ to $p$ which has maximal length.*
*This curve is a geodesic.*

# Conjugate points and (non)-maximizing geodesics



# Hawking's singularity theorem

**Theorem**

*If $(M, g)$ satisfies the SEC and $\theta \leq \theta_0 < 0$, then it is singular.*

**Proof**

- Suppose $\exists$ future-directed timelike geodesic $c$ orthogonal to $S$, extendible to proper time $\tau(c) = \tau_0 + \epsilon > \tau_0 = -\frac{n}{\theta_0}$;
- Then, $\exists$ maximal timelike geodesic $\gamma$ connecting $S$ to $c(\tau_0 + \epsilon)$;
- As such, $\tau(\gamma) \geq \tau_0 + \epsilon$;
- However, $\gamma$ has a conjugate point at a distance of at most $\tau_0$ and ceases to be maximizing beyond this point;
- Hence, this yields a contradiction.

□

TÉCNICO LISBOA

# References

- Papers:
  1. S. Hawking, *The Occurrence of Singularities in Cosmology. III. Causality and singularities*, Proc. Roy. Soc. Lon. A 300 (1967), 187201.

- Books:
  1. G. Naber, *Spacetime and Singularities: An Introduction*, Cambridge University Press, 1988.
  2. L. Godinho and J. Natário, *An Introduction to Riemannian Geometry: With Applications to Mechanics and Relativity* (Universitext), Springer, 2014.