Global Maximizers for Spherical Restriction

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Three problems

Geometry

Given $d \geq 2$ and $0 < k < d$, what is the maximal volume of the intersection of the unit cube $[-\frac{1}{2}, \frac{1}{2}]^d$ with a k-dimensional subspace of $\mathbb{R}^d$?

Probability Theory

Given $d, n \geq 2$, what is the probability distribution of an n-step uniform random walk in $\mathbb{R}^d$?

Algebra

Given $d \geq 2$, what is the minimal codimension of a proper subalgebra of the special orthogonal Lie algebra $\mathfrak{so}(d)$?

Fourier Restriction Theory

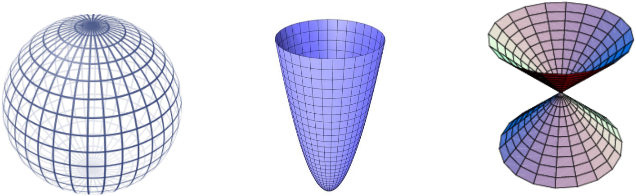
Given $1 \leq p \leq 2$, for which exponents $1 \leq q \leq \infty$ does

$$\int_{\mathbb{S}^{d-1}} |\widehat{g}(\omega)|^q \, \mathrm{d}\sigma(\omega) \lesssim \|g\|^q_{L^p(\mathbb{R}^d)} \quad \text{hold?}$$

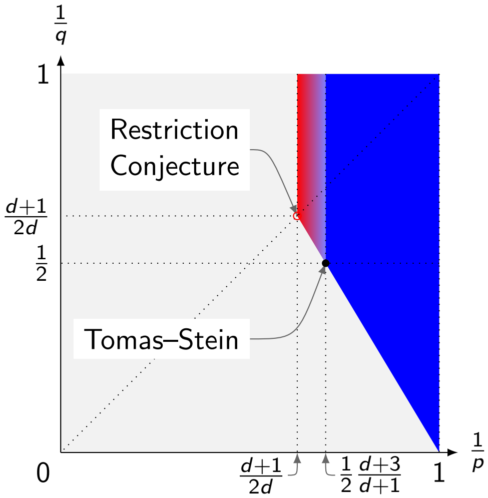
Restriction Conjecture. $1 \leq p < \frac{2d}{d+1}, \quad q \leq \frac{d-1}{d+1} p'$

Tomas–Stein Inequality. $1 \leq p \leq 2\frac{d+1}{d+3}, \quad q = 2$

Curvature plays a role: Any smooth *compact* hypersurface of *nonvanishing* Gaussian curvature will do.



Riesz Diagram



Why should any of this be true?

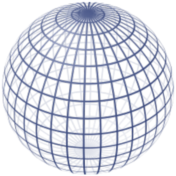
$$\mathcal{R}(g) := \widehat{g}|_{\mathbb{S}^{d-1}} \text{ implies } \mathcal{R}^*(f) = \widehat{f\sigma}, \text{ where}$$

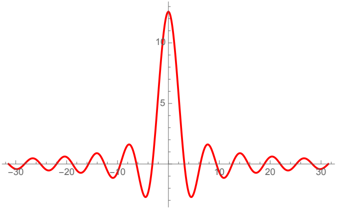
$$\widehat{f\sigma}(x) = \int_{\mathbb{S}^{d-1}} f(\omega) e^{ix\cdot\omega}\, \mathrm{d}\sigma(\omega)$$

$$(\mathcal{R}^* \circ \mathcal{R})(g) = g * \widehat{\sigma}$$

$$|\widehat{\sigma}(\lambda e_d)| = \left| \int_{\mathbb{S}^{d-1}} e^{i\lambda\omega_d}\, \mathrm{d}\sigma(\omega) \right|$$

$$\simeq \left| \int_{\mathbb{R}^{d-1}} e^{i\lambda(1-|\omega'|^2)^{1/2}} \frac{\eta(\omega')\,\mathrm{d}\omega'}{(1-|\omega'|^2)^{1/2}} \right| \lesssim \langle\lambda\rangle^{-\frac{d-1}{2}}$$





Sharp Fourier Restriction Theory

If $d \geq 2$ and $q \geq 2\frac{d+1}{d-1}$, then $\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{L^2(\mathbb{S}^{d-1})}$.

$$\mathbf{T}_{d,q}^q := \sup_{\mathbf{0} \neq f \in L^2} \Phi_{d,q}(f) := \sup_{\mathbf{0} \neq f \in L^2} \frac{\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)}^q}{\|f\|_{L^2(\mathbb{S}^{d-1})}^q}$$

- Maximizers exist if $q > 2\frac{d+1}{d-1}$, or if $(d, q) \in \{(2, 6), (3, 4)\}$
  (Fanelli–Vega–Visciglia'11, Christ–Shao'12, Shao'16, Frank–Lieb–Sabin'16)
- Constants maximize $\Phi_{3,4}$ (Foschi'15)
- Constants maximize $\Phi_{d,4}$ if $d \in \{4, 5, 6, 7\}$ (Carneiro–OS'15)

Theorem (OS–Quilodrán'19)

Let $d \in \{3, 4, 5, 6, 7\}$ and $q \geq 6$ be an even integer. Then constant functions are the unique real-valued maximizers of $\Phi_{d,q}$.

Special case: $(d, q) = (3, 6)$, $\Phi_q := \Phi_{3,q}$, $\mathbf{T}_q := \mathbf{T}_{3,q}$

Step 1: Calculus of Variations

Let f be a maximizer for Φ_6, and normalize it so that $\|f\|_{L^2} = 1$.
Let $\mathcal{E}(f) = \widehat{f\sigma}$ and $\mathcal{E}^*(g) = g^\vee|_{\mathbb{S}^2}$. Then:

$$\begin{aligned}
\|\mathcal{E}\|_{L^2\to L^6}^6 &= \|\mathcal{E}(f)\|_{L^6(\mathbb{R}^3)}^6 = \langle |\mathcal{E}(f)|^4\mathcal{E}(f), \mathcal{E}(f)\rangle \\
&= \langle \mathcal{E}^*(|\mathcal{E}(f)|^4\mathcal{E}(f)), f\rangle_{L^2(\mathbb{S}^2)} \\
&\leq \|\mathcal{E}^*(|\mathcal{E}(f)|^4\mathcal{E}(f))\|_{L^2(\mathbb{S}^2)} \\
&\leq \|\mathcal{E}^*\|_{L^{6/5}\to L^2}\||\mathcal{E}(f)|^4\mathcal{E}(f)\|_{L^{6/5}(\mathbb{R}^3)} \\
&= \|\mathcal{E}^*\|_{L^{6/5}\to L^2}\|\mathcal{E}(f)\|_{L^6(\mathbb{R}^3)}^5 = \|\mathcal{E}\|_{L^2\to L^6}^6
\end{aligned}$$

Euler–Lagrange equation:

$$(|\widehat{f\sigma}|^4\widehat{f\sigma})^\vee|_{\mathbb{S}^2} = \lambda f$$

$$(f\sigma * f_\star\sigma * f\sigma * f_\star\sigma * f\sigma)|_{\mathbb{S}^2} = (2\pi)^{-3}\lambda f$$

where $f_\star(\omega) := \overline{f(-\omega)}$. 40 pages later: f is C^∞-smooth.

Step 2: Symmetrization

Positivity:
(uses evenness of $q = 6$ in a crucial way)

$$\|f\sigma * f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \leq \||f|\sigma * |f|\sigma * |f|\sigma\|_{L^2(\mathbb{R}^3)}$$

Antipodal symmetry:
(can be adapted to general, possibly non-even, $q \geq 2\frac{d+1}{d-1}$)

$$f_\sharp := \sqrt{\frac{|f|^2 + |f_\star|^2}{2}}$$

$$\|f\sigma * f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \leq \|f_\sharp\sigma * f_\sharp\sigma * f_\sharp\sigma\|_{L^2(\mathbb{R}^3)}$$

Therefore:

$$\mathbf{T}_6^6 = \max_{0 \leq f = f_\star \in C^\infty(\mathbb{S}^2) \setminus \{\mathbf{0}\}} \Phi_6(f)$$

Step 3: Operator Theory

Consider the operator

$$T_f : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2),\ T_f(g) := g * K_f$$

with convolution kernel

$$K_f(\xi) := (|\widehat{f\sigma}|^4)^\vee(\xi) = (2\pi)^3(f\sigma * f_\star\sigma * f\sigma * f_\star\sigma)(\xi)$$

Euler–Lagrange boils down to: $T_f(f) = \lambda f$

Kernel K_f:

- $K_f(0) = \|\widehat{f\sigma}\|^4_{L^4(\mathbb{R}^3)}$
- $K_f(\xi) = \overline{K_f(-\xi)}$, for all ξ
- K_f defines a bounded, continuous function on $\mathbb{R}^3$

Operator T_f:

- T_f is self-adjoint and positive definite
- T_f is *Hilbert–Schmidt*: $K_f^\flat(\omega,\nu) := K_f(\omega-\nu) \in L^2(\mathbb{S}^2\times\mathbb{S}^2)$
- T_f is *trace-class*, and $\mathrm{tr}(T_f) = \int_{\mathbb{S}^2} K_f^\flat(\omega,\omega)\, \mathrm{d}\sigma(\omega)$

Step 4: Lie Theory

Fact: $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3), A \mapsto e^A$, is surjective onto SO(3).

$$\Phi_6(f \circ e^{tA}) = \Phi_6(f) = \Phi_6(e^{i\xi\cdot} f), \quad \partial_A f := \frac{\partial}{\partial t}\Big|_{t=0} (f \circ e^{tA})$$

Lemma 1 (New from Old)

Let $f : \mathbb{S}^2 \to \mathbb{R}$ *be non-constant, such that* $f = f_\star \in C^1(\mathbb{S}^2)$ *and* $\|f\|_{L^2} = 1$. *Assume* $T_f(f) = \lambda f$. *Then:*

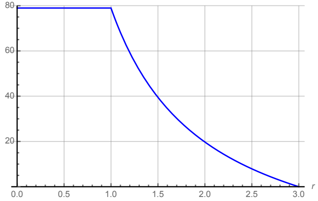
$$T_f(\partial_A f) = \tfrac{\lambda}{5}\partial_A f, \quad \text{for every } A \in \mathfrak{so}(3)$$
$$T_f(\omega_j f) = \tfrac{\lambda}{5}\omega_j f, \quad \text{for every } j \in \{1, 2, 3\}$$

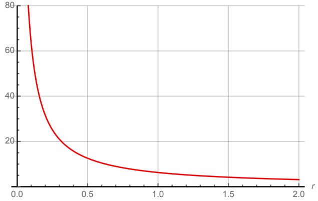
Moreover, there exist $A, B \in \mathfrak{so}(3)$, *such that the set* $\{\partial_A f, \partial_B f, \omega_1 f, \omega_2 f, \omega_3 f\}$ *is linearly independent over* $\mathbb{C}$.

Key: The codimension of a proper, nontrivial subalgebra of $\mathfrak{so}(3)$ is equal to 2. Think $\mathrm{SO}(2) \subsetneq \mathrm{SO}(3)$.

Step 5: Probability Theory

Consider i.i.d. random variables X_1, X_2, X_3, taking values on $\mathbb{S}^2$ with uniform distribution. Then $Y_3 = X_1 + X_2 + X_3$ is the *uniform 3-step random walk* in $\mathbb{R}^3$. If p_3 denotes the probability density of $|Y_3|$, then $(\sigma * \sigma * \sigma)(r) = \sigma(\mathbb{S}^2)^2 p_3(r) r^{-2}$.





Left: Plot of $r \mapsto (\sigma * \sigma * \sigma)(r)$ for $0 \leq r \leq 3$
Right: Plot of $r \mapsto (\sigma * \sigma)(r)$ for $0 \leq r \leq 2$

$$\Phi_4(\mathbf{1}) = (2\pi)^3 \|\mathbf{1}\|_{L^2(\mathbb{S}^2)}^{-4} \|\sigma * \sigma\|_{L^2(\mathbb{R}^3)}^2 = 16\pi^4$$
$$\Phi_6(\mathbf{1}) = (2\pi)^3 \|\mathbf{1}\|_{L^2(\mathbb{S}^2)}^{-6} \|\sigma * \sigma * \sigma\|_{L^2(\mathbb{R}^3)}^2 = 32\pi^5 = 2\pi\Phi_4(\mathbf{1})$$

Step 6: Putting it all together

Any non-constant critical point $f : \mathbb{S}^2 \to \mathbb{R} \in C^1(\mathbb{S}^2)$ **of** Φ_6 **satisfies** $\Phi_6(f) < \Phi_6(\mathbf{1})$**.** Further assume $f = f_\star$, and $\|f\|_{L^2} = 1$. From $T_f(f) = \lambda f$, one checks that $\lambda = \Phi_6(f)$. Thus:

$$\Phi_6(f) = \lambda = \tfrac{1}{2}(\lambda + 5 \times \tfrac{\lambda}{5}) < \frac{1}{2}\text{tr}(T_f) = \frac{1}{2}\int_{\mathbb{S}^2} K_f^\flat(\omega,\omega)\, \mathrm{d}\sigma(\omega) = 2\pi K_f(0) \quad (1)$$

where the strict inequality is a consequence of Lemma 1, together with the fact that all eigenvalues of T_f are strictly positive. But

$$K_f(0) = \|\widehat{f\sigma}\|^4_{L^4(\mathbb{R}^3)} = \Phi_4(f) \leq \Phi_4(\mathbf{1}) \quad (2)$$

where the latter inequality follows from Foschi's result. From the last slide, we have that

$$\boxed{2\pi\Phi_4(\mathbf{1}) = \Phi_6(\mathbf{1})}$$

and so from (1) and $2\pi\times$ (2), it follows that $\Phi_6(f) < \Phi_6(\mathbf{1})$. $\square$

Extension to all even exponents $q \geq 6$

$$\boxed{\Phi_q(\mathbf{1}) \leq \frac{1}{\sigma(\mathbb{S}^2)} \frac{q+6}{q+1} \Phi_{q+2}(\mathbf{1})}$$

Since $\widehat{\sigma}(r) = 4\pi \operatorname{sinc}(r)$, the latter inequality holds **if and only if**

$$\int_0^\infty \frac{|\sin r|^q}{r^{q-2}} \, \mathrm{d}r \leq \frac{q+6}{q+1} \int_0^\infty \frac{|\sin r|^{q+2}}{r^q} \, \mathrm{d}r$$

$$1 - \frac{r^2}{6} \leq \frac{\sin(r)}{r} \leq e^{-r^2/6}, \text{ for all } r \in [0, \pi]$$

K. Ball (1986): Consider the unit cube $Q_d := [-\frac{1}{2}, \frac{1}{2}]^d \subset \mathbb{R}^d$, and let $H \subset \mathbb{R}^d$ be a linear subspace of codimension 1. The $(d-1)$-dimensional volume of $H \cap Q_d$ is at least 1, and at most $\sqrt{2}$.

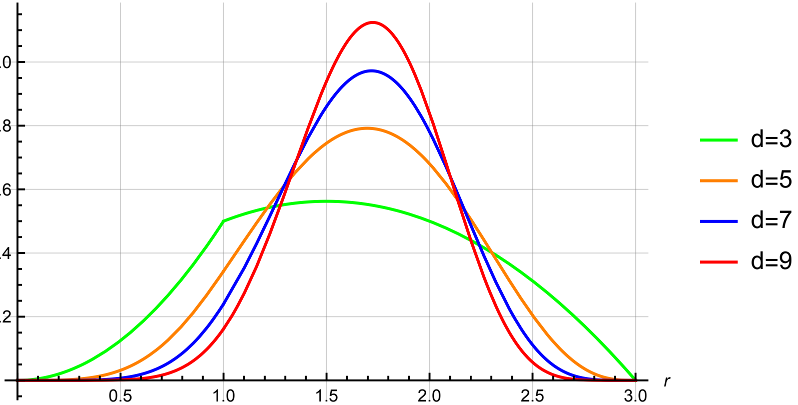
$$\frac{1}{\pi} \int_{-\infty}^\infty \left| \frac{\sin r}{r} \right|^p \mathrm{d}r \leq \sqrt{\frac{2}{\pi}}, \quad p \geq 2$$

Generalization to higher dimensions $d \geq 3$

- The minimal codimension of a proper subalgebra of $\mathfrak{so}(d)$ equals $d-1$ if $d \geq 3, d \neq 4$, but equals 2 if $d = 4$.
- Explicit formulae for the probability distribution of an n-step uniform random walk in $\mathbb{R}^d$ were recently obtained for general n, but only for **odd** values of d. The 2-fold convolution is:

$$(\sigma * \sigma)(r) = \frac{\omega_{d-2}}{2^{d-3}} \frac{1}{r} (4 - r^2)_+^{\frac{d-3}{2}}$$

3-step uniform random walks in $\mathbb{R}^d$



Plot of the function $r \mapsto r^{d-1}(\sigma * \sigma * \sigma)(r)$ for $0 \leq r \leq 3$, when $d \in \{3, 5, 7, 9\}$.

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- Explicit formulae for the probability distribution of an n-step uniform random walk in $\mathbb{R}^d$ were recently obtained for general n, but only for **odd** values of d. The 2-fold convolution is:

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- Since $\widehat{\sigma}(x) = (2\pi)^{\frac{d}{2}} |x|^{-\nu} J_\nu(|x|)$ if $\nu = \frac{d}{2} - 1$,

$$\Phi_{d,q}(\mathbf{1}) \leq \frac{1}{\omega_{d-1}} \frac{q + 2d - \delta_{d,4}}{q+1} \Phi_{d,q+2}(\mathbf{1})$$

if and only if

$$\boxed{\int_0^\infty |J_\nu(r)|^q \, r^{d-1-q\nu} \, \mathrm{d}r \leq \frac{\Gamma(\frac{d}{2})^2}{2^{2-d}} \frac{q+2d-\delta_{d,4}}{q+1} \int_0^\infty |J_\nu(r)|^{q+2} \, r^{d-1-(q+2)\nu} \, \mathrm{d}r}$$

$$E(d,q) := \frac{\Gamma(\frac{d}{2})^2}{2^{2-d}} \frac{\int_0^\infty |J_\nu(r)|^{q+2}\, r^{d-1-(q+2)\nu}\,\mathrm{d}r}{\int_0^\infty |J_\nu(r)|^q\, r^{d-1-q\nu}\,\mathrm{d}r} - \frac{q+1}{q+2d-\delta_{d,4}} \overset{?}{\geq} 0$$

$d \backslash q$	4	6	8	10	12	14	16	18	20
2	*	0.57	5.16	5.32	4.95	4.55	4.19	3.88	3.61
3	0.00	8.33	8.63	8.29	7.85	7.40	6.97	6.57	6.20
4	0.82	4.83	5.67	5.93	5.94	5.83	5.66	5.45	5.24
5	3.98	7.68	9.13	9.77	9.97	9.94	9.77	9.53	9.25
6	2.26	6.16	8.24	9.38	9.97	10.22	10.27	10.18	10.03
7	0.36	4.46	7.03	8.62	9.57	10.10	10.38	10.47	10.45
8	-1.42	2.75	5.66	7.62	8.89	9.70	10.20	10.47	10.60
9	-2.98	1.11	4.25	6.49	8.04	9.10	9.81	10.26	10.54
10	-4.31	-0.39	2.85	5.31	7.09	8.36	9.27	9.89	10.31
11	-5.41	-1.76	1.51	4.11	6.08	7.54	8.62	9.40	9.96

Values of $100 \times E(d,q)$, obtained through numerical evaluation of Bessel integrals and truncated to two decimal places.

Complex-valued maximizers

Theorem (OS–Quilodrán'19)

Let $d \geq 2$ and $q \geq 2\frac{d+1}{d-1}$ be an even integer. Then each complex-valued maximizer of $\Phi_{d,q}$ is of the form

$$ce^{i\xi\cdot\omega}F(\omega)$$

for some $\xi \in \mathbb{R}^d$, some $c \in \mathbb{C} \setminus \{0\}$, and some nonnegative, antipodally symmetric maximizer F of $\Phi_{d,q}$.

Corollary (OS–Quilodrán'19)

Let $d \in \{3, 4, 5, 6, 7\}$ and $q \geq 4$ be an even integer. Then all complex-valued maximizers of $\Phi_{d,q}$ are given by

$$f(\omega) = ce^{i\xi\cdot\omega}$$

for some $\xi \in \mathbb{R}^d$ and $c \in \mathbb{C} \setminus \{0\}$.

Open problems

1. What happens in dimensions $d \geq 8$?
2. Do constants maximize $\Phi_{2,6}$?
3. Are non-zero solutions of the Euler–Lagrange equation for arbitrary exponents $q \geq 2\frac{d+1}{d-1}$ and dimensions $d \geq 2$,

$$(|\widehat{f\sigma}|^{q-2}\widehat{f\sigma})^\vee|_{\mathbb{S}^{d-1}} = \lambda f$$

   necessarily C^∞-smooth even when q is *not* an even integer?
4. Do maximizers of $\Phi_{d,q}$ exist at the endpoint $q = 2\frac{d+1}{d-1}$ if $d \geq 4$? (Frank–Lieb–Sabin'16 have a conditional result along these lines)
5. If so, do constants maximize $\Phi_{d,q}$ if $q = 2\frac{d+1}{d-1}$, for all $d \geq 4$?
6. Conversely, are all real-valued maximizers of $\Phi_{d,q}$ if $q = 2\frac{d+1}{d-1}$ constant?

Thank you very much