Polar Functions and Measure Transitions

Rodrigo Duarte

Instituto Superior Técnico

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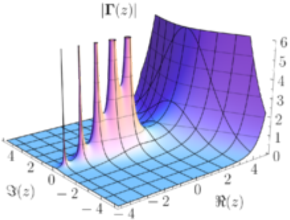

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Definition

$K : \mathbb{R}^2 \to \mathbb{C}$ is said to be a **positive definite kernel** if for any collection $(x_k)_{k=1}^n$, we have

$$\sum_{k,l=1}^{n} K(x_k, x_l)\xi_k \bar{\xi}_l \geq 0, \ \forall \xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n.$$

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A function $f : \mathbb{R} \to \mathbb{C}$ is said to be **positive definite** if for any collection $(x_k)_{k=1}^n$ we have

$$\sum_{k,l=1}^{n} f(x_k - x_l)\xi_k\bar{\xi}_l \geq 0,\ \forall \xi \in \mathbb{C}^n.$$

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- We can always translate from the notion of positive definiteness to the notion of co-positive definiteness.

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- In each vertical strip $-n < \Re(z) < -n+1$, we have the Cauchy-Saalschütz formula

$$\Gamma(z) = \int_0^{+\infty} x^{z-1} \left(e^{-x} - \sum_{m<n} \frac{(-1)^m}{m!} x^m \right) dx.$$

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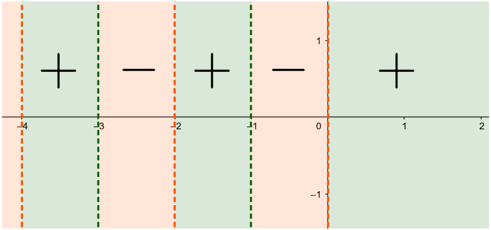

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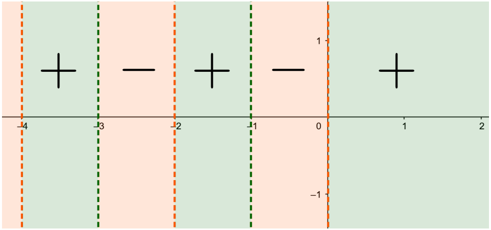

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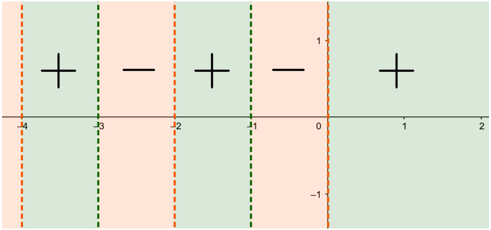

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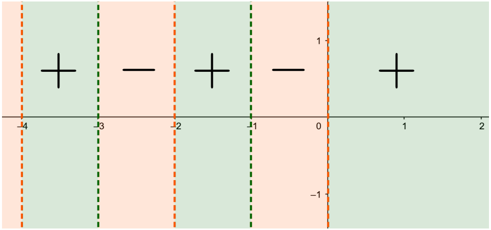

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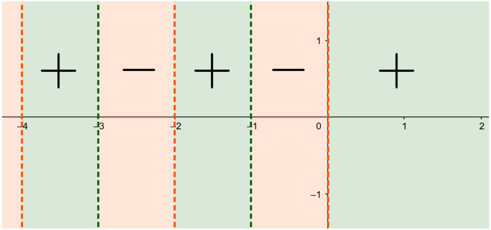

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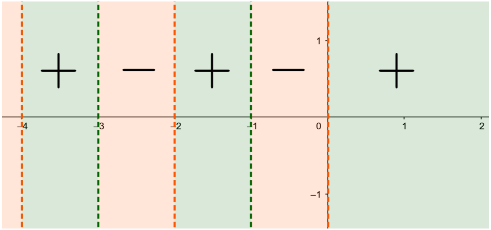

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- Note that the transition from one density to the one on its left follows the simple transition formula:

$$\sigma_{-n}(t) - \sigma_{-(n+1)}(t) = \frac{(-1)^n}{n!} e^{-nt}.$$

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Definition

The Laplace-Fourier transform of a measure μ and the Fourier-Laplace transform are defined by

$$\mathcal{LF}(\mu)(z) = \int_{\mathbb{R}} e^{-zt} d\mu(t), \ \mathcal{FL}(\mu)(z) = \int_{\mathbb{R}} e^{izt} d\mu(t).$$

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Definition

A measure μ over the σ-algebra of measurable sets in $\mathbb{R}$ is exponentially finite with respect to a nonempty interval I, if

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Proposition

Let μ be a nonnegative measure which is exponentially finite with respect to $]a, b[$. Then, we can assume $]a, b[$ is maximal and the function $\mathcal{LF}(\mu)$ is holomorphic and co-positive definite in $T_{a,b}$, with singularities in a and b, if finite.

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Theorem

Let $f : T_{a,b} \to \mathbb{C}$ be some holomorphic function. Then f is co-positive definite if and only if it is the Laplace-Fourier transform of an exponentially finite measure with respect to $]a, b[$. Furthermore, μ is uniquely determined by f.

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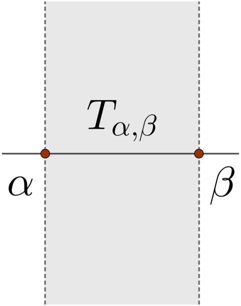
- If $]\alpha, \beta[$ is the maximal interval with respect to which μ is exponentially finite, then f has an analytic continuation $\mathcal{LF}$ to $T_{\alpha,\beta}$.



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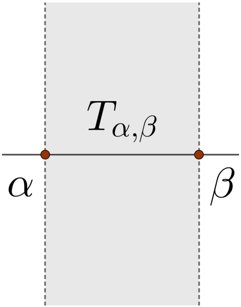

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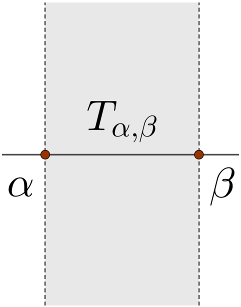
- This result has a direct analog for positive definite functions and Fourier-Laplace transforms.



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Short Detour Through Measure Theory

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Definition

A signed measure μ is σ-finite if there is a countable family $\{A_n\} \subseteq \mathcal{M}$ of measurable sets with finite measure such that

$$X = \bigcup_n A_n.$$

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Definition

Suppose we have two signed measures μ and ν in $(X, \mathcal{M})$. We say that ν is **singular** with respect to μ if there are measurable sets A, B such that $A \cap B = \varnothing$, $A \cup B = X$, $\mu(A) = 0$ and $\nu(B) = 0$. This is written $\mu \perp \nu$.

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Let ν be a signed measure and μ a nonnegative measure in $(X, \mathcal{M})$. We say that ν is **absolutely continuous** with respect to μ if

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This is written $\nu \ll \mu$.

Short Detour Through Measure Theory

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- One defines $L^1(\mu) = L^1(\mu^+) \cap L^1(\mu^-)$ and for $f \in L^1(\mu)$,

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Short Detour Through Measure Theory

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- The relation

  $$\nu(A) = \int_A f d\mu$$

  is also written as $d\nu = f d\mu$. In this case we call f the **density** associated with ν.

Exponentially Finite Signed Measures

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A measure μ on the σ-algebra of measurable sets in $\mathbb{R}$ is an **exponentially finite signed measure** (EFSM) with respect to a nonempty interval I if it is a signed measure and

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Exponentially finite signed measures are σ-finite.

- In particular, there is a Lebesgue decomposition

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  of any EFSM μ, with respect to the Lebesgue measure on $\mathbb{R}$.
- Furthermore, there is a density $\sigma(t)$ such that $d\mu^a(t) = \sigma(t)dt$.

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- We can analogously define left polarized signed measures;
- A signed measure is **bilaterally polarized** if it is both left and right polarized.

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- In particular, f has singularities at $z = \alpha$ and $z = \beta$, if these are finite.

Transition Polynomial

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where $d\mu_{\cdot k}^p(t) = \sigma_{\cdot k}^p(t)dt$ and

$$\begin{cases} \sigma_{lk}^p(t) = -\frac{t^{k-1}}{(k-1)!}(1 - H(t))e^{ct} \\ \sigma_{rk}^p(t) = \frac{t^{k-1}}{(k-1)!}H(t)e^{ct}. \end{cases}$$

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Definition

We define the **transition polynomial** to be

$$\mathcal{P}(t) = \sum_{k=1}^{m} \frac{b_k}{(k-1)!} t^{k-1} = e^{-ct}(\sigma^p_r(t) - \sigma^p_l(t)).$$

Lemma

Let $(\mu_l, \mu_r)_{\alpha,c,\beta}$ be a polarized measure pair with associated function f. Suppose that $\mu^h_{rx_0} := \mu_{rx_0} - \mu^p_{rx_0}$ is exponentially finite with respect to $]\alpha - x_0, \beta - x_0[$, for some $x_0 \in]c, \beta[$. Then,

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Remark:

- There is an analogous result using μ_l and going from $T_{\alpha,c}$ to $T_{c,\beta}$.

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(ii) We have

$$\begin{cases} \mu^s_l = \mu^s_r \\ \sigma_r(t) - \sigma_l(t) = \mathcal{P}(t)e^{ct} \text{ a.e. in } \mathbb{R} \end{cases}$$

where μ^s_l, μ^s_r are exponentially finite with respect to $]\alpha, \beta[$, and $d\mu^a_\cdot(t) = \sigma_\cdot(t)dt$;

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Remarks:

- One could replace r by l in item (i) and (iii);

Main Theorem

Theorem

Let $(\mu_l, \mu_r)_{\alpha,c,\beta}$ be a polarized measure pair. The following are equivalent:

(i) $\mu^h_{rx_0} := \mu_{rx_0} - \mu^p_{rx_0}$ is exponentially finite with respect to $]\alpha - x_0, \beta - x_0[$, for some $x_0 \in]c, \beta[$;

(ii) We have

$$\begin{cases} \mu^s_l = \mu^s_r \\ \sigma_r(t) - \sigma_l(t) = \mathcal{P}(t)e^{ct} \text{ a.e. in } \mathbb{R} \end{cases}$$

where μ^s_l, μ^s_r are exponentially finite with respect to $]\alpha, \beta[$, and $d\mu^a_\cdot(t) = \sigma_\cdot(t)dt$;

(iii) $\mu^h_r =: \mu_r - \mu^p_r$ is a bilaterally polarized EFSM with respect to $]\alpha, \beta[$.

Remarks:

- One could replace r by l in item (i) and (iii);
- The implication (iii) $\Longrightarrow$ (i) is immediate.

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so $\mu_{lx} = \mu^p_{lx} + \mu^h_{rx}$, for $x \in]\alpha, c[$.

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- Considering the Lebesgue decomposition we obtain $\mu_{lx}^s = \mu_{rx}^{hs} = \mu_{rx}^s$. So,

$$e^{-xt} d\mu_l^s = d\mu_{lx}^s = d\mu_{rx}^s = e^{-xt} d\mu_r^s.$$

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- Considering instead the absolutely continuous part we obtain $\mu_{lx}^a = \mu_{lx}^{pa} + \mu_{rx}^{ha} = \mu_{lx}^p + \mu_{rx}^a - \mu_{rx}^p$, so
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- Thus,

$$\begin{aligned} e^{-xt}(\sigma_r - \sigma_l)dt = d(\mu_{rx}^a - \mu_{lx}^a) = d(\mu_{rx}^p - \mu_{lx}^p) = e^{-xt}(\sigma_r^p - \sigma_l^p)dt \\ = e^{-xt}\mathcal{P}(t)e^{ct}dt. \end{aligned}$$

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  - Decay properties of b_k?

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