On a new class of fractional partial differential equations

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Overview

1. Introduction
2. Fractional Sobolev Spaces
3. Properties of the Fractional Derivative
4. Fractional PDEs

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Fractional Calculus (Motivation)

- Fractional calculus generalize the classical differentiation and integration operator.
- Classically $D^2(f) = (D \circ D)(f) = D(D(f))$
- What is $\sqrt{D} = D^{\frac{1}{2}}$?
- More generally, what is D^a for $a \in \mathbb{R}$? Hopefully, when $a = n \in \mathbb{Z}$, it corresponds to the classical n-th derivative.
- Idea first appeared in a letter from l'Hopital to Leibniz, and introduced in papers by Abel and Liouville independently
- Since then, several different definitions have been proposed and recently many developments are occurring

Fractional Calculus (Applications)

- Fractional advection dispersion equation: for modelling contaminant flow in heterogeneous porous media
- Fractional diffusion equation: for modelling anomalous diffusion processes in complex media (fractional time derivative corresponding to long-time heavy tail decay; fractional spatial derivative corresponding to nonlocal diffusion), including Lévy flight
- Modelling of viscoelastic damping in materials like polymers
- Fractional time acoustical wave equations: for modelling of acoustical waves in complex media such as in biological tissue (attenuation measured in media comes from multiple relaxation phenomena)
- Fractional Schrödinger equation in fractional quantum mechanics: when Brownian-like quantum mechanical paths are replaced by their continuous time-analog, the Lévy like ones

Fractional Laplacian

Recall that the Fourier transform of the Laplacian of $u \in \mathcal{S}$ is given by

$$\widehat{(-\Delta)u}(\xi) = -4\pi^2|\xi|^2\hat{u}(\xi)$$

for every $\xi \in \mathbb{R}^N$.
Naturally, we can define the fractional Laplacian $(-\Delta)^s$, $0 < s < 1$ as

$$\widehat{(-\Delta)^s u}(\xi) = (-4\pi^2|\xi|^2)^s\hat{u}(\xi),$$

which can be rewritten as a singular integral in real space

$$(-\Delta)^s u(x) := c_{N,s}\, p.v. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy.$$

Riesz Potentials

Recall the generalized Riesz potentials of order α for $0 < \alpha < N$ given by the formula

$$I_\alpha * u(x) := \frac{\gamma(N,\alpha)}{|x|^{N-\alpha}} * u(x),$$

where the constant γ is given by

$$\gamma(N,\alpha) := \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\pi^{\frac{N}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}.$$

If $u \in L^p(\mathbb{R}^N)$, $1 < p < \infty$, then

$$I_\alpha * u \in L^{p_\alpha}(\mathbb{R}^N),$$

where

$$\frac{1}{p_\alpha} = \frac{1}{p} - \frac{\alpha}{N}.$$

Riesz Potentials and the Riesz Fractional Derivative

- Riesz potentials satisfy the semi-group property

$$(I_\alpha I_\beta) * u = I_{\alpha+\beta} * u$$

  for $\alpha, \beta > 0$ and $\alpha + \beta < N$.
- The Laplacian maps a potential of order $\alpha + 2$ to a potential of order α,

$$-\Delta(I_{\alpha+2} * u) = I_\alpha * u.$$

Through analytic continuation, the Riesz potential can be extended to negative exponents, and so one arrives at a formula for the fractional Laplacian, or Riesz fractional derivative, given by

$$(-\Delta)^s u(x) = c_{N,s}\, p.v. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy = I_{-2s} * u$$

for $0 < s < 1$.

Riesz Fractional Gradient

Since

$$(-\Delta)^s u(x) = I_{-2s} * u = -\Delta(I_{2-2s} * u),$$

we can naturally define the *s-Riesz fractional gradient* as

$$(D^s u)_j := \frac{\partial^s u}{\partial x_j^s}, \quad j = 1, \ldots, N$$

where

$$\frac{\partial^s u}{\partial x_j^s} := \frac{\partial}{\partial x_j} I_{1-s} * u.$$

Here, $s \in (0,1)$.
Note: For $u \in L^p(\mathbb{R}^N)$ for some $1 < p < \infty$ such that $I_{1-s} * u$ is well-defined, this is defined in the distributional sense, i.e.

$$\langle \frac{\partial^s u}{\partial x_j^s}, v \rangle = (-1)\langle I_{1-s} * u, \frac{\partial v}{\partial x_j} \rangle = -\int_{\mathbb{R}^N} (I_{1-s}) * u \frac{\partial v}{\partial x_j} dx, \quad \forall v \in C_0^\infty(\mathbb{R}^N).$$

Fractional Derivative (integral form)

Writing in integral form, we have the *s-gradient* as

$$D^s u(x) := c_{N,s} \lim_{\epsilon \to 0} \int_{\{|z|>\epsilon\}} \frac{z u(x+z)}{|z|^{N+s+1}} dz$$

for a function u.
We can correspondingly define the *s-divergence* for a vector φ as

$$\mathrm{div}^s \varphi(x) := c_{N,s} \lim_{\epsilon \to 0} \int_{\{|z|>\epsilon\}} \frac{z \cdot \varphi(x+z)}{|z|^{N+s+1}} dz.$$

Note 1: Unlike the classical derivative which depends on only neighbouring points (local property), the fractional derivative involves information on the function further out. It is a nonlocal operator.
Note 2: For $0 < s < 1$, if $u \in C_c^\infty(\mathbb{R}^N)$, then

$$(-\Delta)^s u = -\sum_{j=1}^{N} \frac{\partial^s}{\partial x_j^s} \frac{\partial^s}{\partial x_j^s} u = -\mathrm{div}^s (D^s u).$$

Fractional Derivative (Remarks)

This is a good definition of the fractional derivative, in such a way that the fractional operator does not depend on the chosen basis.
Indeed, this operator is

- translationally invariant,
- rotationally invariant,
- homogeneous of degree $s \in \mathbb{R}$ under isotropic scaling,
- is continuous in the Schwartz space.

Note: These are in fact the properties of the Riesz transform, which we recall, is given by

$$\mathcal{R}_j f(x) := \frac{\Gamma\left(\frac{N+1}{2}\right)}{\pi^{(N+1)/2}} \lim_{\epsilon \to 0} \int_{\{|y|>\epsilon\}} \frac{y_j}{|y|^{N+1}} f(x-y)dy, \quad j = 1, \ldots, N,$$

and it is well-known that the Riesz transform is the only linear operator fulfilling all these properties.

Relation with Riesz Transforms

Indeed, we have the following property: Let $1 \leq p < \infty$ and $s \in (0,1)$. Every $u \in C_c^\infty(\mathbb{R}^N)$ can be expressed as

$$u = I_s * \sum_{j=1}^{N} \mathcal{R}_j \frac{\partial^s u}{\partial x_j^s}$$

where $\mathcal{R}_j$ is the Riesz transform, which can be characterized as a singular integral or 0-order operator with multiplier $\frac{-i\xi_j}{|\xi|}$.

Proof. Since

$$\widehat{\frac{\partial^s u}{\partial x_j^s}} = -(2\pi)^s i\xi_j |\xi|^{-1+s} \hat{u}$$

and $\widehat{\mathcal{R}_j} = \frac{-i\xi_j}{|\xi|}$, the result follows from the identity

$$\widehat{I_s} = (2\pi|\xi|)^{-s}$$

in $\mathcal{S}(\mathbb{R}^N)'$. $\square$

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Fractional Hilbert Spaces

Recall that we can make use of the Fourier transform to define Sobolev spaces as

$$W^{m,2}(\mathbb{R}^N) = H^m(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : \xi \mapsto (1+|\xi|^2)^{m/2}\hat{u}(\xi) \in L^2(\mathbb{R}^N)\}.$$

We can extend this definition to fractional Hilbert spaces.

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \{\xi \mapsto (1+|\xi|^2)^{s/2}\mathcal{F}(u)(\xi)\} \in L^2(\mathbb{R}^N)\} \text{ for } s > 0,$$

and

$$H^s(\mathbb{R}^N) = \{u \in \mathcal{S}'(\mathbb{R}^N) : \{\xi \mapsto (1+|\xi|^2)^{s/2}\mathcal{F}(u)(\xi)\} \in L^2(\mathbb{R}^N)\} \text{ for } s < 0,$$

with norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left\|(1+|\xi|^2)^{s/2}\mathcal{F}(u)\right\|_{L^2(\mathbb{R}^N)}.$$

Fractional Hilbert Spaces (Properties)

It is known that

1. the space $H^s(\mathbb{R}^N)$ with the above defined norm is a Banach space;
2. the space $H^s(\mathbb{R}^N)$ coincides with the classical Sobolev space $W^{m,2}(\mathbb{R}^N)$ if $s = m \in \mathbb{N}$;
3. for $s > 0$, the space $H^{-s}(\mathbb{R}^N)$ coincides with the dual $(H^s(\mathbb{R}^N))'$;
4. the space $\mathcal{S}(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$;
5. for $s > 1/2$ and $N \geq 2$, the functions in $H^s(\mathbb{R}^N)$ have a trace on $\{x_N = 0\}$ that belongs to $H^{s-1/2}(\mathbb{R}^{N-1})$.
6. Conversely, every function in $H^{s-1/2}(\mathbb{R}^{N-1} \times \{0\})$ can be extended in a linear and continuous manner to a function in $H^s(\mathbb{R}^N)$.

Fractional Sobolev Spaces

For general p, we have the following definition of fractional Sobolev spaces,
$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : [u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}} < +\infty \right\},$$
for $0 < s < 1$, with the natural norm

$$\|u\|^p_{W^{s,p}(\Omega)} := \|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\Omega)}.$$

For $p = 2$ with integer s, this definition is the same as the definition of Sobolev spaces with Fourier transform.
The space $W^{s,p}_0(\Omega)$ denotes the closure in $W^{s,p}(\Omega)$ of all smooth functions having a compact support contained in Ω.
Denoting $H^s(\Omega) := W^{s,2}(\Omega)$, we can similarly define $H^s_0(\Omega)$ as the closure in $H^s(\Omega)$ of all smooth functions having a compact support contained in Ω. Moreover, there is an equivalent norm for $H^s_0(\Omega)$, given by

$$\|u\|_{H^s_0(\Omega)} = \|D^s u\|_{[L^2(\Omega)]^n}, \quad 0 < s < 1.$$

Fractional Sobolev Spaces (Embedding Results)

Then we have the following embedding results: Let $0 < s < 1$ and $1 < p < \infty$. Suppose Ω is a Lipschitz open set.

- The space $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$.
- If $0 < s' < s < 1$, then $W^{s,p}(\mathbb{R}^N) \hookrightarrow W^{s',p}(\mathbb{R}^N)$.
- $W^{1,p}(\mathbb{R}^N) \hookrightarrow W^{s,p}(\mathbb{R}^N)$.
- If $sp < N$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \leq Np/(N - sp)$
- If $sp = N$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q < \infty$.
- If $sp > N$, then $W^{s,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, and, more precisely, $W^{s,p}(\Omega) \hookrightarrow C^{0,s-N/p}(\bar{\Omega})$ if Ω is bounded.
- In particular, $H_0^s(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and $L^{2^\#}(\Omega) \hookrightarrow H^{-s}(\Omega) = (H_0^s(\Omega))'$ for $0 < s < 1$, where $2^* = \frac{2N}{N-2s}$ and $2^\# = \frac{2N}{N+2s}$ when $s < \frac{N}{2}$, and if $N = 1$, $2^* = q$ for any finite q and $2^\# = q' = \frac{q}{q-1}$ when $s = \frac{1}{2}$ and $2^* = \infty$ and $2^\# = 1$ when $s > \frac{1}{2}$.

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Properties of the Fractional Derivative (Duality)

Let $s \in (0,1)$. For all Lipschitz compactly supported scalar function f and vector function φ,

$$\int_{\mathbb{R}^N} f \operatorname{div}^s \varphi dx = -\int_{\mathbb{R}^N} \varphi \cdot (D^s f) dx.$$

Proof.

$$\begin{aligned}
\int_{\mathbb{R}^N} f \operatorname{div}^s \varphi dx =& c_{N,s} \int_{\mathbb{R}^N} f(x) \lim_{\epsilon \to 0} \int_{\{|x-y|>\epsilon\}} \frac{(y-x)\cdot\varphi(y)}{|y-x|^{N+s+1}} dydx \\
=& c_{N,s} \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \int_{\{|x-y|>\epsilon\}} f(x) \frac{(y-x)\cdot\varphi(y)}{|y-x|^{N+s+1}} dydx \\
=& -c_{N,s} \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \int_{\{|x-y|>\epsilon\}} \varphi(y) \cdot \frac{(x-y)f(x)}{|x-y|^{N+s+1}} dydx \\
=& -\int_{\mathbb{R}^N} \varphi \cdot (D^s f) dx
\end{aligned}$$

by dominated convergence theorem and Fubini's theorem.

Properties of the Fractional Derivative (Leibniz Rule)

Let $s \in (0,1)$. For all Lipschitz compactly supported scalar functions f, g,

$$D^s(fg) = f(D^s g) + g(D^s f) + D^s_{NL}(f,g),$$

where the nonlinear term

$$D^s_{NL}(f,g)(x) := c_{N,s} \int_{\mathbb{R}^N} \frac{(y-x)(f(y)-f(x))(g(y)-g(x))}{|y-x|^{N+s+1}} dt, \quad \forall x \in \mathbb{R}^N.$$

Moreover, for $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\left\| D^s_{NL}(f,g) \right\|_{L^1(\mathbb{R}^N;\mathbb{R}^N)} \leq c_{N,s} [f]_{W^{\frac{s}{p},p}(\mathbb{R}^N)} [g]_{W^{\frac{s}{q},q}(\mathbb{R}^N)}$$

and similarly

$$\left\| D^s_{NL}(f,g) \right\|_{L^1(\mathbb{R}^N;\mathbb{R}^N)} \leq 2c_{N,s} [f]_{L^\infty(\mathbb{R}^N)} [g]_{W^{s,1}(\mathbb{R}^N)}.$$

A similar Leibniz rule holds for divergence. Note the additional nonlinear term, which makes integration by parts difficult.

Properties of the Fractional Derivative (Leibniz Rule)

Proof. Given f, g compactly supported Lipschitz functions in $\mathbb{R}^N$, we have

$$\begin{aligned}
D^s(fg)(x) =& c_{N,s} \int_{\mathbb{R}^N} \frac{(y-x)(f(y)g(y) - f(x)g(x))}{|y-x|^{N+s+1}} dy \\
=& c_{N,s} \int_{\mathbb{R}^N} \frac{(y-x)(f(y)g(y) - f(y)g(x) + f(y)g(x) - f(x)g(x))}{|y-x|^{N+s+1}} dy \\
=& c_{N,s} \int_{\mathbb{R}^N} \frac{(y-x)f(y)(g(y) - g(x))}{|y-x|^{N+s+1}} dy + g(x) D^s f(x) \\
=& c_{N,s} \int_{\mathbb{R}^N} \frac{(y-x)(f(y) - f(x))(g(y) - g(x))}{|y-x|^{N+s+1}} dy + f(x) D^s g(x) + g(x) D^s f(x).
\end{aligned}$$

We also have that, for any $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned}
\left\| D^s_{NL}(f,g) \right\|_{L^1(\mathbb{R}^N;\mathbb{R}^N)} \leq& C_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(y) - f(x)|}{|x-y|^{\frac{N+s}{p}}} \frac{|g(y) - g(x)|}{|y-x|^{\frac{N+s}{q}}} dydx \\
\leq& c_{N,s} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(y) - f(x)|^p}{|x-y|^{N+s}} dydx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|g(y) - g(x)|^q}{|x-y|^{N+s}} dydx \right)^{\frac{1}{q}}.
\end{aligned}$$

The case $p = \infty$, $q = 1$ is similar. □

Properties of the Fractional Derivative (Extension to Fractional Sobolev Spaces)

With these estimates on the nonlinear term, we can continuously extend our operator

$$D^s_{NL} : \mathrm{Lip}_c(\mathbb{R}^N) \times \mathrm{Lip}_c(\mathbb{R}^N) \to L^1(\mathbb{R}^N; \mathbb{R}^N)$$

to

$$D^s_{NL} : W^{\frac{s}{p},p}(\mathbb{R}^N) \times W^{\frac{s}{q},q}(\mathbb{R}^N) \to L^1(\mathbb{R}^N; \mathbb{R}^N)$$

for any $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Properties of the Fractional Derivative (Estimate on I_s)

Let $H^{s,p}(\mathbb{R}^N)$ be the generalized Sobolev space, defined by

$$H^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : D^s u \in L^p(\mathbb{R}^N)\}$$

for $s > 0$. It holds, for $1 < p < \infty$ and $\epsilon > 0$, that

$$H^{s+\epsilon,p}(\mathbb{R}^N) \hookrightarrow W^{s,p}(\mathbb{R}^N) \hookrightarrow H^{s-\epsilon,p}(\mathbb{R}^N).$$

We now look at some additional properties that correspond to fractional derivatives, that are useful for analysing fractional PDEs.
Estimate on the operator I_s: Let $1 < p < \infty$ and $0 < s < 1$ be such that $sp < N$. Then for all $f \in L^p(\mathbb{R}^N)$ such that $I_s * f$ is well-defined, there exists $C = C(N, p, s) > 0$ such that

$$\|I_s * f\|_{L^{p^*}(\mathbb{R}^N)} \leq C\|f\|_{L^p(\mathbb{R}^N)}$$

where $p^* := \frac{Np}{N-sp}$.
This can be shown just by expanding $I_s * f$.

Properties of the Fractional Derivative (Fractional Sobolev Inequality)

Fractional Sobolev Inequality. Let $1 < p < \infty$ and $0 < s < 1$ be such that $sp < N$. Then there exists a constant $C = C(N, p, s) > 0$ such that

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C\|D^s u\|_{L^p(\mathbb{R}^N)} .$$

This implies the fractional Poincaré inequality for open bounded domains, which is useful for obtaining a priori energy estimates.
Proof. We will show the result for $u \in C_c^\infty(\mathbb{R}^N)$ and extend the result for general $u \in H^{s,p}(\mathbb{R}^N)$ by density. Recall that for $u \in C_c^\infty(\mathbb{R}^N)$, we can rewrite u as $u = I_s * g$ where $g = \sum_{j=1}^N \mathcal{R}_j \frac{\partial^s u}{\partial x_j^s}$.
Note that $g \in L^p(\mathbb{R}^N)$ and $sp < N$. So by the previous estimate on the operator I_s, and the boundedness of the Riesz transform $\mathcal{R}_j : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$, we have
$\|u\|_{L^{p^*}(\mathbb{R}^N)} = \|I_s * g\|_{L^{p^*}(\mathbb{R}^N)} \leq const \times \|g\|_{L^p(\mathbb{R}^N)} \leq const \times const \times \|D^s u\|_{L^p(\mathbb{R}^N)}$ □

Properties of the Fractional Derivative (Fractional Morrey Inequality)

Fractional Morrey Inequality. Let $1 < p < \infty$ and $0 < s < 1$ be such that $sp > N$. Then there exists a constant $M = M(N, p, s) > 0$ such that

$$|u(x) - u(y)| \leq M|x - y|^{s - \frac{N}{p}} \|D^s u\|_{L^p(\mathbb{R}^N)}$$

for all $u \in H^{s,p}(\mathbb{R}^N)$.
This is useful for estimating the variation of the solution.

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Stampacchia Theorem

Stampacchia Theorem. Given V (which, in our case, is $H_0^s(\Omega)$) a Hilbert space, $\mathbb{K} \subset V$ a closed, nonempty, convex set, $L \in V'$ and $a(\cdot,\cdot)$ a bounded coercive bilinear form. Then there exists a unique solution to the variational inequality

$$u \in \mathbb{K} : a(u, v-u) \geq \langle L, v-u \rangle, \quad \forall v \in \mathbb{K}.$$

In the case $\mathbb{K} = V$, one has unique solvability of

$$u \in V : a(u, v) = \langle L, v \rangle, \quad \forall v \in V,$$

which is Lax-Milgram theorem.

Application I: Fractional Dirichlet Problem

Problem. Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Suppose that $f \in L^{2^{\#}}(\Omega)$, and $A : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ is bounded and measurable such that

$$a_*|\xi|^2 \leq A(x)\xi \cdot \xi \leq a^*|\xi|^2$$

for some $a_*, a^* > 0$ and all $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^N$. Then there exists a unique $u \in H_0^s(\Omega)$ such that

$$\int_\Omega A(x) D^s u \cdot D^s v dx = \int_\Omega f v dx$$

for every $v \in H_0^s(\Omega)$.

Dirichlet Problem (Proof) I

Proof. The bilinear mapping $B : H_0^s(\Omega) \times H_0^s(\Omega) \to \mathbb{R}$ defined by the LHS

$$B[u,v] := \int_\Omega A(x) D^s u \cdot D^s v dx$$

is coercive by the strict ellipticity of A and the fractional Poincaré inequality, i.e.

$$B[u,u] \geq a_* \int_\Omega |D^s u|^2 dx \geq c\|u\|_{H_0^s(\Omega)}^2 .$$

Also, the boundedness of A and the Cauchy-Schwarz inequality gives continuity of B, since

$$B[u,v] \leq a^* c' \|u\|_{H_0^s(\Omega)} \|v\|_{H_0^s(\Omega)} .$$

Dirichlet Problem (Proof) II

Similar estimates imply that the map, since $L^{2^{\#}}(\Omega) \subset H^{-s}(\Omega)$,

$$v \mapsto \int_{\Omega} fv$$

is a bounded linear functional on $H_0^s(\Omega)$.
So, one may apply the Lax-Milgram theorem to obtain existence of $u \in H_0^s(\Omega)$ satisfying the equation.
For uniqueness, if u_1, u_2 are two solutions to the Dirichlet problem, then the function $w = u_1 - u_2 \in H_0^s(\Omega)$ and satisfies

$$\int_{\Omega} A(x) D^s w \cdot D^s v dx = 0$$

for every $v \in H_0^s(\Omega)$. Thus, letting $v = w$, we obtain $w \equiv 0$ by the fractional Poincaré inequality. □

Dirichlet Problem (Remark)

Remark. It is possible to extend the Dirichlet problem to arbitrary open domains with generalised Dirichlet data, i.e. find a solution $u \in H^s(\Omega)$ with $u = \varphi \in H^s(\mathbb{R}^N)$ in Ω^c, where φ is given in such a way that $I_{1-s} * \varphi$ is well-defined. Then, $D^s u$ is well-defined everywhere. In this case, we instead use the bilinear form

$$B[\tilde{u}, v],$$

where

$$\tilde{u} = u - \varphi.$$

Application II: Fractional Obstacle Problem

Problem. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary, $f \in L^{2^\#}(\Omega)$, and $A : \Omega \to \mathbb{R}^{N\times N}$ be strictly elliptic, bounded and measurable. Then, for every function $\psi \in H^s(\Omega)$ such that the closed convex sets

$$\mathbb{K}^s_\psi = \{v \in H^s_0(\Omega) : v \geq \psi \text{ a.e. in } \Omega\} \neq 0,$$

there exists a unique $u \in \mathbb{K}^s_\psi$ such that

$$\int_\Omega A(x)D^s u \cdot D^s(v-u)dx \geq \int_\Omega f(v-u) \quad \forall v \in \mathbb{K}^s_\psi.$$

Proof. This is just a direct application of the Stampacchia theorem, since we have already shown that the bilinear form is bounded and coercive previously in the Dirichlet problem. □

Remark 1. The theorem holds, for instance, if $\psi \in H^s_0(\Omega)$.

Remark 2. For the special case of $A = Id$ and $f = 0$,

$$u = P_{\mathbb{K}^s_\psi} 0.$$

Application III: Fractional Variational Inequality with Gradient Constraint I

Problem. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. For positive functions $g \in L^\infty(\Omega)$, consider the nonempty convex sets

$$\mathbb{K}^s_g = \{v \in H^s_0(\Omega) : |D^s v| \leq g \text{ a.e. in } \Omega\}.$$

Suppose that $f_i \in L^1(\Omega)$ and $A : \Omega \to \mathbb{R}^{N\times N}$ is strictly elliptic, bounded and measurable. Then for every

$$g_i \in L^\infty_\nu(\Omega) := \{g \in L^\infty(\Omega) : g(x) \geq \nu > 0 \text{ a.e. } x \in \Omega\},$$

there exists a unique $u_i \in \mathbb{K}^s_{g_i} \cap C^{0,\beta}(\bar{\Omega})$ for all Hölder constants $0 < \beta < 1$, such that

$$\int_\Omega A(x) D^s u_i \cdot D^s(v - u_i)dx \geq \int_\Omega f_i(v - u_i)dx \quad \forall v \in \mathbb{K}^s_{g_i}.$$

Application III: Fractional Variational Inequality with Gradient Constraint II

When $g_1 = g_2$, the solution map $L^1(\Omega) \ni f \mapsto u \in H_0^s(\Omega)$ is $\frac{1}{2}$-Hölder continuous, i.e., for some $C_1 > 0$, we have

$$\|u_1 - u_2\|_{H_0^s(\Omega)} \leq C_1 \|f_1 - f_2\|_{L^1(\Omega)}^{\frac{1}{2}}.$$

Moreover, if in addition, $f_i \in L^{2^\#}(\Omega), i = 1, 2$, and $g_1 = g_2$, then $L^{2^\#}(\Omega) \ni f \mapsto u \in H_0^s(\Omega)$ is Lipschitz continuous, i.e.,

$$\|u_1 - u_2\|_{H_0^s(\Omega)} \leq C_\# \|f_1 - f_2\|_{L^{2^\#}(\Omega)},$$

where $C_\# = \frac{C_*}{a_*} > 0$, where C_* is the constant of the Sobolev embedding $H_0^s(\Omega) \hookrightarrow L^{2^\#}(\Omega)$.

Gradient Constraint Problem (Proof) I

Proof. We first find the solution for $f_i \in L^{2^\#}(\Omega) \subset H^{-s}(\Omega)$. But the assumption on $A(x)$ implies that it defines a continuous bilinear and coercive form over $H_0^s(\Omega)$. Then by the Stampacchia Theorem, we obtain the existence and uniqueness of the solution. The regularity of u_i then follows from the Sobolev embeddings

$$\mathbb{K}_g^s \subset C^{0,\beta}(\bar{\Omega}) \subset L^\infty(\Omega).$$

For the third result, since

$$a_* \|\bar{u}\|^2_{H_0^s(\Omega)} \leq \int_\Omega AD^s\bar{u} \cdot D^s\bar{u} \leq \left\|\bar{f}\right\|_{L^{2^\#}(\Omega)} \|\bar{u}\|_{L^{2^*}(\Omega)} \leq C_* \left\|\bar{f}\right\|_{L^{2^\#}(\Omega)} \|\bar{u}\|_{H_0^s(\Omega)},$$

where $\bar{u} = u_1 - u_2$ and $\bar{f} = f_1 - f_2$, the Lipschitz continuity of the map $f \mapsto u$ follows.

Gradient Constraint Problem (Proof) II

Also, using the Sobolev embedding above, and letting κ be such that

$$\|v\|_{L^\infty(\Omega)} \leq \kappa, \quad \forall v \in \mathbb{K}^s_{g_1},$$

then we have the $\frac{1}{2}$-Hölder continuous estimate with $C_1 = \sqrt{2\kappa/a_*}$ for $f_1, f_2 \in L^{2^\#}(\Omega) \subset L^1(\Omega)$ since

$$a_* \|\bar{u}\|^2_{H^s_0(\Omega)} \leq \left\|\bar{f}\right\|_{L^1(\Omega)} \|\bar{u}\|_{L^\infty(\Omega)} \leq 2\kappa \left\|\bar{f}\right\|_{L^1(\Omega)}.$$

Finally, we can obtain the solvability of the original problem for f_i only in $L^1(\Omega)$ by density, by taking an approximating sequence of $f_i^n \in L^{2^\#}(\Omega)$ such that $f_i^n \to f_i$ in $L^1(\Omega)$ as $n \to \infty$ and using the $\frac{1}{2}$-Hölder continuous estimate for that Cauchy sequence. □

Gradient Constraint Problem (Remarks)

Remark 1. As with the Dirichlet problem, it is possible to extend the variational inequality to arbitrary open domains with generalised Dirichlet data, i.e. $\varphi \in H^s(\mathbb{R}^N)$ such that $I_{1-s} * \varphi$ is well-defined and $|D^s\varphi| \leq g$, and replacing $H^s_0(\Omega)$ in the definition of $\mathbb{K}^s_g$ with the space $\{v \in H^s(\mathbb{R}^N) : v = \varphi \text{ a.e. in } \Omega^c\}$.

Remark 2. If, in addition, A is symmetric, then the variational inequality is equivalent to the optimisation problem

$$u \in \mathbb{K}^s_g : \qquad \mathcal{J}(u) \leq \mathcal{J}(v), \qquad \forall v \in \mathbb{K}^s_g,$$

where $\mathcal{J} : \mathbb{K}^s_g \to \mathbb{R}$ is the convex functional

$$\mathcal{J}(v) = \frac{1}{2}\int_\Omega AD^s v \cdot D^s v - \int_\Omega fv.$$

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