Morse Homology and Floer Homology

LisMath seminar

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Plan of the presentation

1. Motivation from Hamiltonian mechanics and the Arnol′d conjecture;
2. Morse homology and Lagrangian mechanics;
3. Floer homology and proving the Arnol′d conjecture;
4. Viterbo's theorem.

Hamiltonian mechanics

Consider a conservative force F acting on Euclidean space $\mathbb{R}^n$, with a potencial V such that $\nabla V = -F$. The energy of a particle of unit mass in this space is described by the **Hamiltonian function**

$$\begin{aligned} H \colon \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R} \\ (q, p) &\mapsto \frac{1}{2}|p|^2 + V(q). \end{aligned}$$

The motion of such a particle is given by a curve $t \mapsto (q(t), p(t))$ satisfying the equations

$$\begin{cases} q' = p, \\ p' = F. \end{cases}$$

For a general Hamiltonian function $H\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ motion is described by **Hamilton's equations**

$$\begin{cases} q_i' = \dfrac{\partial H}{\partial p_i}, \\[2ex] p_i' = -\dfrac{\partial H}{\partial q_i}. \end{cases}$$

The symplectic viewpoint

Hamilton's equations are an ODE associated to the **Hamiltonian vector field**

$$X_H = \left(\frac{\partial H}{\partial p_1}, \cdots, \frac{\partial H}{\partial p_n}, -\frac{\partial H}{\partial q_1}, \cdots, -\frac{\partial H}{\partial q_n} \right).$$

The space $\mathbb{R}^n \times \mathbb{R}^n$ has a canonical 2-form

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i$$

satisfying

$$\omega_0 (X_H, v) = -(DH)v, \qquad \forall v \in \mathbb{R}^{2n}.$$

Definition

Let M be a smooth manifold. A smooth 2-form ω on M is a **symplectic form** *if*

- *It is* **closed***: $d\omega = 0$;*
- *It is* **non-degenerate***: if $\omega_x(v, w) = 0$ for all $w \in T_x M$ then $v = 0$.*

We say that (M, ω) is a **symplectic manifold***.*

The **Hamiltonian vector field** associated to the **Hamiltonian function** $H\colon M \to \mathbb{R}$ satisfies

$$\omega_x\left(X_H(x), v\right) = -(dH)_x v, \qquad \forall x \in M, v \in T_x M.$$

A few remarks:

- $(\mathbb{R}^{2n}, \omega_0)$ is a symplectic manifold
- Symplectic manifolds have even dimension;
- Locally, symplectic manifolds are **symplectomorphic** to $(\mathbb{R}^{2n}, \omega_0)$;
- Symplectic manifolds are orientable: ω^n is a volume form;
- Orientable surfaces have symplectic forms (any area form);
- The cotangent bundle has a canonical symplectic form generalizing $(\mathbb{R}^{2n}, \omega_0)$.

An Arnol'd conjecture

Given a **time-dependent** Hamiltonian function $H\colon \mathbb{R}\times M \to \mathbb{R}$ we can define the **Hamiltonian flow** $\phi^{\bullet}_{X_H}\colon \mathbb{R}\times M \to M$ by:

$$\begin{cases} \dfrac{\mathrm{d}\phi^t_{X_H}(x)}{\mathrm{d}t} = X_{H_t}\left(\phi^t_H(x)\right), \\ \\ \phi^0_{X_H}(x) = x. \end{cases}$$

We call $\phi^1_{X_H}$ a **Hamiltonian symplectomorphism**. We say that a fixed point $\phi^1_{X_H}(x) = x$ is **non-degenerate** if

$$(d\phi^1_{X_H})_x - \mathrm{id}_{T_xM}\colon\ T_xM \to T_xM$$

is invertible.

Conjecture (Arnol'd)

Let (M, ω) be a compact symplectic manifold and $H\colon [0,1] \times M \to \mathbb{R}$ be a smooth time-dependent Hamiltonian function. Suppose that the fixed points of the **time-1 Hamiltonian flow** $\phi^1_{X_H}$ *are* **non-degenerate**. *Then*

$$\#\mathrm{Fix}\left(\phi^1_{X_H}\right) \geq \sum_{k=0}^{\dim M} b_k(M; \mathbb{F}),$$

where $b_k(M;\mathbb{F})$ are the Betti numbers of M with coefficients in $\mathbb{F}$.

Proven for $\mathbb{F} = \mathbb{Q}$ by Fukaya and Ono.

This Arnol′d conjecture gives a relation between the Hamiltonian dynamics with the topology of the manifold. It is specific to Hamiltonian dynamics: the Lefschetz fixed-point theorem yields

$$\mathrm{Fix}\left(\phi^1_{X_H}\right) \geq |\chi(M)|.$$

Example

On the flat 2-torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ with the induced symplectic form ω_0 we have $\#\mathrm{Fix}(\phi^1_H) \geq 4$.

On the other hand, any translation is a symplectomorphism (homotopic to $\mathrm{id}_{\mathbb{T}^2}$) with no fixed points.

Morse inequalities

Let M be a manifold and $f \in C^\infty(M)$.

Definition

We say that $p \in M$ is a **critical point** of M if $(df)_p = 0$. We say that p is a **non-degenerate** critical point if the Hessian $(d^2 f)_p$ is non-degenerate. We say that f is a **Morse function** if all its critical points are non-degenerate.

Theorem (Morse inequalities)

Let M be a compact manifold and f a Morse function on M. Then

$$\#\mathrm{Crit}(f) \geq \sum_{k=0}^{\dim M} b_k(M; \mathbb{F}).$$

The Hessian and Morse index

When $(df)_p = 0$ we can define the Hessian as a bilinear map

$$\left(d^2 f\right)_p (X_p, Y_p) = X_p \cdot Y \cdot f$$

for vector fields X and Y.
Given local coordinates $(x_1, \cdots, x_n)$ it is represented by the Hessian matrix

$$\left[\frac{\partial^2 f}{\partial x_i x_j}(p)\right]_{i,j=1}^{n}.$$

The signature of this quadratic form does not depend on the choice of local coordinates.

Definition

The **index** of p is the dimension of the largest linear subspace of $T_p M$ on which $(d^2 f)_p$ is negative-definite.

The basic building blocks of Morse theory

Denote $M^a = f^{-1}(]-\infty, a[)$.

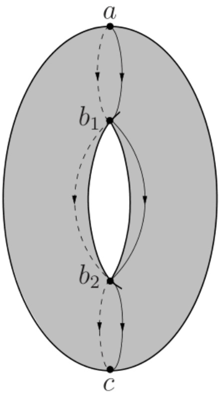
1. Between critical values the topology does not change;
2. When we cross a critical point of index k the homotopy type changes by adding a k-cell.

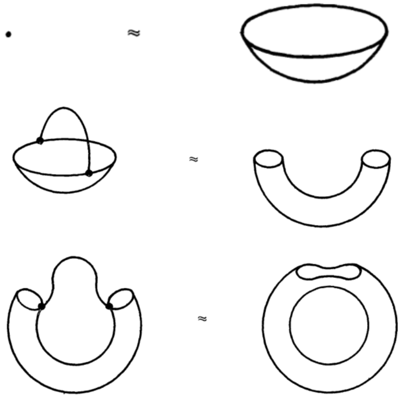
$$\text{Closed manifold} \xleftrightarrow{\text{homotopy}} \text{CW complex}$$

$$\text{Critical point} \longleftrightarrow \text{Cell}$$

$$\text{Index} \longleftrightarrow \text{Dimension}$$

Height function on the 2-torus





Morse theory and CW homology

Denoting by $C(M, f)$ the aforementioned CW complex:

$$H_\bullet^{\text{sing}}(M) \cong H_\bullet^{\text{CW}}(C(M, f)) \cong H_\bullet\left(\mathbb{Z} \cdot \text{Crit } f, \partial\right),$$

where Crit f is graded by index and ∂ is some differential.
As a direct consequence we have the Morse inequalities

$$\#\text{Crit}_k f \geq b_k(M; \mathbb{F}),$$

for any field $\mathbb{F}$.

Example

A Morse function on $\mathbb{RP}^2$ has at least three critical points.

Morse homology

Can we define an appropriate differential ∂ in a self-contained way? The basic idea is to count trajectories of the negative-gradient $-\nabla f$ between critical points — for this purpose we need a Riemannian metric g on M.

Given $p_\pm \in \operatorname{Crit} f$ define $\mathcal{M}(p_-, p_+)$ as the set of curves $c \colon \mathbb{R} \to M$ such that

$$\begin{cases} \dot{c}(s) + \nabla f(c(s)) = 0, \\ \lim_{s \to \pm\infty} c(s) = p_\pm. \end{cases}$$

$\mathbb{R}$ action: $(\sigma \cdot c)(s) := c(s + \sigma)$

$$\mathcal{L}(p_-, p_+) = \mathcal{M}(p_-, p_+)/\mathbb{R}$$

For $p \in \mathrm{Crit}_k f$ define

$$\partial p = \sum_{q \in \mathrm{Crit}_{k-1} f} |\mathcal{L}(p, q)| q, \qquad \mod 2$$

1. How can we guarantee that $\mathcal{L}(p, q)$ is a finite set?
2. How can we show that $\partial^2 = 0$?

We can identify

$$\mathcal{M}(p,q) = W^u(p) \cap W^s(q)$$

where

$$W^u(p) = \left\{ x \in M : \lim_{t \to -\infty} \phi^t_{-\nabla f}(x) = p \right\}$$
$$W^s(q) = \left\{ x \in M : \lim_{t \to +\infty} \phi^t_{-\nabla f}(x) = q \right\}$$

are the unstable/stable manifolds of $-\nabla f$.

By the stable manifold theorem, these are indeed manifolds (diffeomorphic to balls), of dimension

$$\dim W^u(p) = \mathrm{ind}_f(p),$$
$$\dim W^s(q) = n - \mathrm{ind}_f(q).$$

Morse-Smale condition:

$$W^u(p) \pitchfork W^s(q) \Rightarrow \dim \mathcal{M}(p,q) = \mathrm{ind}_f(p) - \mathrm{ind}_f(q)$$

If $p \neq q$:

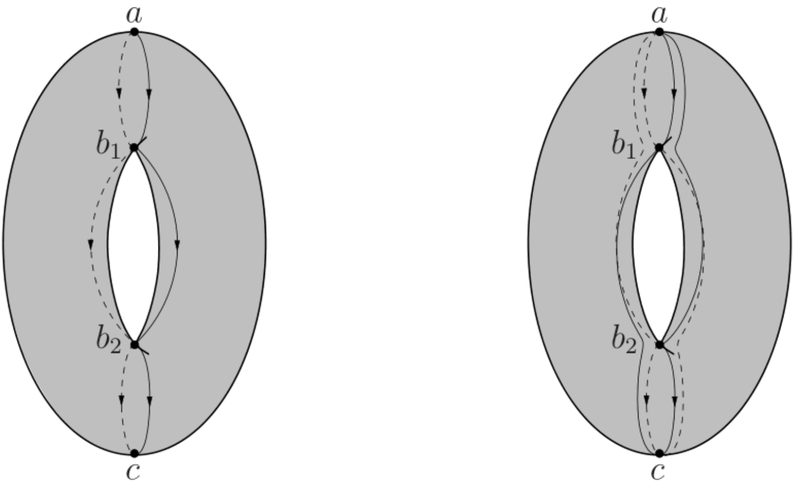
$$\dim \mathcal{L}(p,q) = \mathrm{ind}_f(p) - \mathrm{ind}_f(q) - 1.$$

In particular, if $\mathrm{ind}_f(p) = \mathrm{ind}_f(q) + 1$ then $\mathcal{L}(p,q)$ is a 0-manifold. One can show that it is compact, and thus a finite set.

Theorem

Let M be a closed manifold. For a generic choice of f and/or g the Morse-Smale condition holds.

Fixing the 2-torus



Showing $\partial^2 = 0$: gluing

Given $p \in \mathrm{Crit}_{k+1} f$:

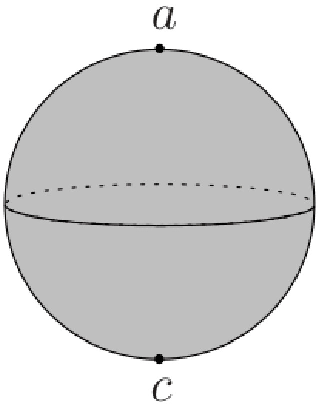
$$\partial^2 p = \sum_{r \in \mathrm{Crit}_{k-1} f} \left| \bigcup_{q \in \mathrm{Crit}_k f} \mathcal{L}(p,q) \times \mathcal{L}(q,r) \right| r.$$

It turns out that we can compactify $\mathcal{L}(p,r)$ as a 1-manifold with boundary:

$$\overline{\mathcal{L}(p,r)} = \underbrace{\mathcal{L}(p,r)}_{\text{interior}} \bigcup \underbrace{\bigcup_{q \in \mathrm{Crit}_k f} \mathcal{L}(p,q) \times \mathcal{L}(q,r)}_{\text{boundary}}$$

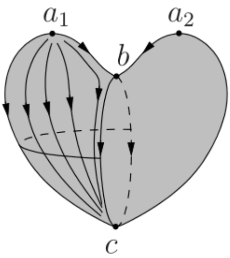
and $\partial^2 = 0$ since compact 1-manifolds are disjoint unions of arcs and circles.

Computations: round sphere



$$
\begin{array}{ccc}
k & C_k(M,f) & H_k(M;\mathbb{Z}/2) \\
2 & \mathbb{Z}/2\cdot a & \mathbb{Z}/2 \\
 & \downarrow & \\
1 & 0 & 0 \\
 & \downarrow & \\
0 & \mathbb{Z}/2\cdot c & \mathbb{Z}/2
\end{array}
$$

Computations: heart-shaped sphere



k	$C_k(M,f)$	$H_k(M;\mathbb{Z}/2)$
2	$\mathbb{Z}/2\cdot(a_1,a_2)$	$\mathbb{Z}/2$
	$\big\downarrow \times(1,1)$	
1	$\mathbb{Z}/2\cdot b$	0
	$\big\downarrow \times 2$	
0	$\mathbb{Z}/2\cdot c$	$\mathbb{Z}/2$

Generalizations

Morse homology can be defined on a Hilbert manifold M with a Riemannian metric G and a Morse function f such that:

1. f is bounded from below;
2. The Morse index of critical points of f is finite;
3. The manifold is complete;
4. f satisfies the **Palais-Smale condition**: any sequence $x_n \in M$ s.t. $f(x_n)$ is bounded and $\lim |\nabla f(x_n)| = 0$ has a convergent subsequence;
5. (f, G) satisfies the Morse-Smale condition.

Theorem

$$H_\bullet^{\text{Morse}}(M, f, G) \cong H_\bullet^{\text{sing}}(M)$$

Application to Lagrangian mechanics

Consider a "quadratic Lagrangian" such as

$$\begin{aligned} L\colon S^1 \times TQ &\to \mathbb{R} \\ (t,q,v) &\mapsto \frac{1}{2}|v|^2 - V(t,q). \end{aligned}$$

The critical points of the **Lagrangian action functional**

$$\begin{aligned} \mathcal{E}_L\colon W^{1,2}\left(S^1;M\right) &\to \mathbb{R} \\ x &\mapsto \int_{S^1} L(t,x(t),x'(t))\,\mathrm{d}t \end{aligned}$$

are the 1-periodic solutions of the Euler-Lagrange equations

$$\nabla_q L(t,x(t),x'(t)) = \frac{\mathrm{D}}{\mathrm{d}t}\nabla_v L(t,x(t),x'(t)).$$

For generic choices of potential V conditions 1–4 hold. For generic choices of Riemannian metric (uniformly equivalent to the $W^{1,2}$-metric) it satisfies condition 5.

The Morse complex $MC(M, \mathcal{E}_L, G)$ is generated by 1-periodic solutions of E-L and the same homology as $H_\bullet^{\text{sing}}(\mathcal{L}M)$.

Theorem (Gromoll-Meyer)

If the sequence $b_k(\mathcal{L}M)$ is unbounded then M has infinite geometrically distinct closed geodesics.

The actual Arnol′d conjecture

Conjecture (Arnol′d)

Let (M, ω) be a compact symplectic manifold, and let ϕ be an Hamiltonian symplectomorphism with non-degenerate fixed points. Then

$$\#\mathrm{Fix}(\phi) \geq \min_{f \in C^\infty(M) \text{ Morse}} \#\mathrm{Crit}(f).$$

For general Hamiltonian symplectomorphisms ϕ:

$$\#\mathrm{Fix}(\phi) \geq \min_{f \in C^\infty(M)} \#\mathrm{Crit}(f).$$

The Morse inequalities and the “weak” Arnol′d conjecture give a common lower bound for the above quantities.

Hamiltonian action functional

On $(\mathbb{R}^{2n}, \omega_0)$ we can define a functional

$$\mathcal{A}_H \colon \mathcal{L}(\mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R}$$
$$(q, p) \mapsto \int_{S^1} p(t) \cdot q'(t) - H(t, q(t), p(t)) \, \mathrm{d}t$$

with critical points the 1-periodic solutions of Hamilton's equation $\mathcal{P}(H)$!

However (for example on $W^{1,2}(S^1; \mathbb{R}^{2n})$):

- Morse index may be infinite;
- The Palais-Smale condition may fail.

Floer's approach

Study directly the space of "formal" negative L^2-gradient trajectories of $\mathcal{A}_H$: solutions $u \in C^\infty(\mathbb{R} \times S^1; \mathbb{R}^{2n})$ of **Floer's equation**

$$\partial_s u(s,t) - J_0\left(\partial_t u(s,t) - X_{H_t}(u(s,t))\right) = 0,$$

where

$$J_0 = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

This is a first-order elliptic system of partial differential equations: a zero-order perturbation of the Cauchy-Riemann equation

$$\partial_s u - J_0 \partial_t u = 0.$$

For general symplectic manifolds Floer's equation is

$$\partial_s u(s,t) - J_{u(s,t)}\left(\partial_t u(s,t) - X_{H_t}(u(s,t))\right) = 0,$$

where J is an **almost-complex structure** which is **compatible with** ω i.e.

$$J_p^2 = -\mathrm{id}_{T_pM}$$

and

$$\omega_p(v, J_p w) = g(v,w), \qquad v, w \in T_pM$$

defines a Riemannian metric.

The trajectory spaces $\mathcal{M}(x_-, x_+)$ for $x_\pm \in \mathcal{P}(H)$ consist of solutions u of Floer's equation such that

$$\lim_{s \to \pm\infty} u(s, \cdot) = x_\pm$$

in $C^\infty(S^1; M)$.

$\mathbb{R}$-action: $(\sigma \cdot u)(s, t) := u(s + \sigma, t)$

Moduli spaces $\mathcal{L}(x_-, x_+) = \mathcal{M}(x_-, x_+)/\mathbb{R}$.

Floer complex

Under the following topological assumption

$$\int_{S^2} u^* c_1(M) = 0, \qquad \forall u \in C^\infty(S^2; M)$$

we can associate to **contractible** 1-periodic solutions of Hamilton's equation their Conley-Zhender indices μ_{CZ}.

The Floer complex is then $FC(M, H, J; \mathbb{Z}/2) = (\mathbb{Z}/2 \cdot \mathcal{P}_0(H), \partial)$ with differential given by

$$\partial x = \sum_{\substack{y \in \mathcal{P}_0(H), \\ \mu_{CZ}(y) = \mu_{CZ}(x) - 1}} |\mathcal{L}(x, y)| y.$$

Theorem

If (M, ω) *is* **aspherical**; *i.e.*

$$\int_{S^2} u^*\omega = 0, \qquad u \in C^\infty(S^2; M)$$

then $FC(M, H, J; \mathbb{Z}/2)$ *is a well-defined chain complex for generic choices of* H *and* J.

Floer homology is the homology $FH(M, H, J; \mathbb{Z}/2)$ of this complex.

Floer theory

The powerful machinery of Morse theory/Morse homology no longer works and needs to be replaced. Just proving that the Floer complex is well-defined requires powerful analytic tools, such as:

- Regularity for elliptic PDE — Calderón-Zygmund inequality, freezing coefficients, elliptic bootstrapping ;
- Bubbling-off analysis — originating from Gromov compactness and using the removable singularity theorem;
- Differential geometry on Banach manifolds — implicit function theorem, Sard theorem, Thom transversality;
- Functional analysis — Banach open mapping theorem, Fredholm operators, Newton-Picard method;
- Continuation principles for "almost-holomorphic" maps — Carleman similarity principle.

Invariance of Floer homology on closed manifolds

Let (M, ω) be a closed symplectic manifold satisfying the topological assumptions.

Theorem

If (H_0, J_0) and (H_1, J_1) are data such that the Floer complex is well-defined then there is a canonical isomorphism

$$FH(M, H_0, J_0) \cong FH(M, H_1, J_1).$$

We can thus define $H_\bullet^{\text{Floer}}(M) = FH(M, H, J)$.

Computation and the Arnol′d conjecture

Theorem

There are compatible a.c.s J and C^2-small autonomous Hamiltonian H such that

$$FC_\bullet(M, H, J) \cong MC_{\bullet+n}(M, H, g).$$

In particular there is an isomorphism

$$H_\bullet^{\text{Floer}}(M) \cong H_{\bullet+n}^{\text{sing}}(M),$$

and the "weak" Arnol′d conjecture holds.

Legendre transformation

$$T^*Q \xleftrightarrow{\text{Legendre transformation}} TQ$$

$$\text{Momentum} \longleftrightarrow \text{Velocity}$$

$$\frac{1}{2}|p|^2 + V(t,q) = H \longleftrightarrow L = \frac{1}{2}|v|^2 - V(t,q)$$

$$\text{Hamilton's equations} \xleftrightarrow{(*)} \text{Euler-Lagrange equations}$$

The bijection $(*)$ induces a bijection between the generators of $FC(T^*Q, H, J)$ and $MC(\mathcal{L}Q, \mathcal{E}_L, G)$.

Theorem (Abbondandolo-Schwarz)

For generic compatible uniformly continuous J there is a chain isomorphism

$$FC_\bullet(T^*Q, H, J; \mathbb{Z}/2) \cong MC_\bullet(\mathcal{L}Q, \mathcal{E}_L, G; \mathbb{Z}/2).$$

Theorem (Viterbo)

Let Q be a closed manifold. Then

$$H_\bullet^{\text{Floer}}(T^*Q; \mathbb{Z}/2) \cong H_\bullet^{\text{sing}}(\mathcal{L}Q; \mathbb{Z}/2).$$

If Q is spin then

$$H_\bullet^{\text{Floer}}(T^*Q) \cong H_\bullet^{\text{sing}}(\mathcal{L}Q).$$

Ongoing work: S^1-equivariant version

$\mathcal{L}Q$ is an $\boldsymbol{S^1}$**-space**: $(\tau \cdot x)(t) := x(t+\tau)$. There is a suitable definition of S^1-equivariant Floer homology, due to Bourgeois and Oancea.

Is it the case that

$$FH_\bullet^{S^1}(T^*Q;\mathbb{Z}/2) \cong H_\bullet^{S^1}(\mathcal{L}Q;\mathbb{Z}/2)?$$

For $\mathbb{Z}$-coefficients is Q spin the appropriate condition?

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