Talk at Lisbon

29. 4. 20

Laughlin states on Riemann surfaces

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Based on

S.K. “Lauhglin states on higher genus Riemann surfaces”, Commun. Math. Phys. 2019

S.K, X.Ma, G. Marinescu, P. Wiegmann, “Quantum Hall effect and Quillen metric”, Commun. Math. Phys. 2017

S.K., P. Wiegmann, “Geometric adiabatic transport in QH states”, Phys. Rev. Lett. 2015

F.Ferrari, S.K., “FQHE on curved backgrounds and free fields”, JHEP 2014

S.K., D. Zvonkine, in preparation

Laughlin state

$$\Psi_L(z_1, \dots, z_N) = C \cdot \prod_{n<m}^{N} (z_n - z_m)^\beta \cdot e^{-\frac{B}{4} \sum_{n=1}^{N} |z_n|^2}$$

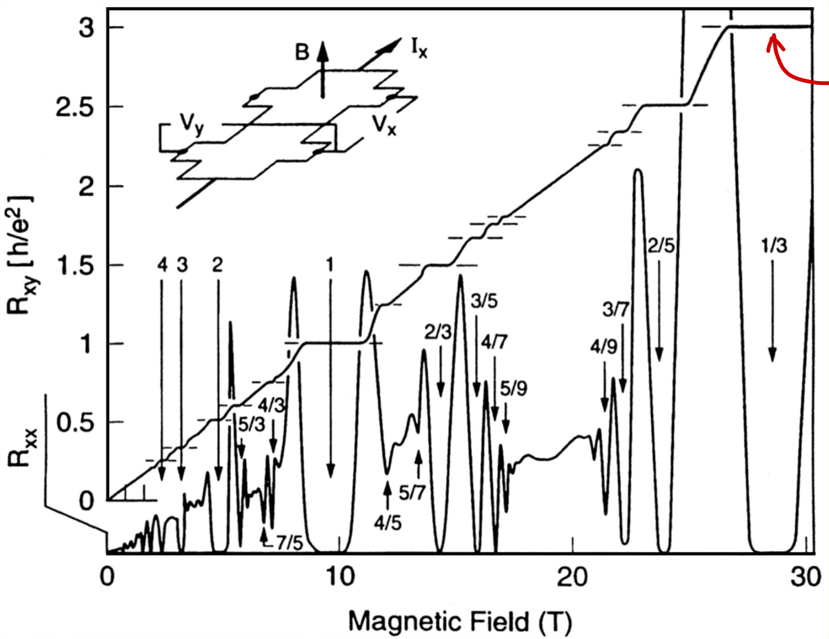
$\{z_n\} \in \mathbb{C}^N$

$\beta \in \mathbb{Z}_+$ $\quad$ $1/\beta$ is "filling fraction"

$B > 0$ $\quad$ "magnetic field"

Quantum Hall effect (QHE)

Precise quantization of Hall conductance $\sigma_H = \frac{1}{R_{xy}}$



Laughlin state corresponds to this plateau

$\sigma_H = 1/\beta$

Fractional QHE

Strongly-interacting (via Coulomb forces) system.

Laughlin 1983: Assign a trial wave function ("state") to each plateau.

$$\Psi_L(z_1, \dots, z_N) = c \cdot \prod_{n<m}^{N} (z_n - z_m)^\beta \cdot e^{-\frac{B}{4}\sum_{n=1}^{N}|z_n|^2}$$

* holomorphic
* vanishing conditions

$\Psi = 0$

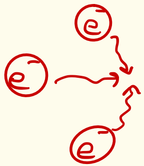
Another famous QHE state

Moore-Read 1991:

$$\Psi_{MR}(z_1, \dots, z_N) = c \cdot \mathrm{Pf}\left(\frac{1}{z_n - z_m}\right) \cdot \prod_{n<m}^{N} (z_n - z_m) \cdot e^{-\frac{B}{4} \sum_n |z_n|^2}$$

Pfaffian of anti-sym. matrix

$$\sigma_H = 5/2$$



$\Psi = 0$

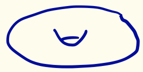
Goal

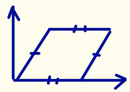
Define QH states (e.g. Laughlin state) on compact Riemann surfaces and study how do they depend on geometric data (genus, metric, moduli...) for large N.

Haldane 1983	$g=0$
Haldane-Rezayi 1985	$g=1$
Wen-Niu 1991	$g>1$
Avron-Seiler-Zograf 1995,	$\beta=1$, $\mathrm{Jac}(\Sigma)$, $\mathcal{M}_{1,1}$
N. Read 2009	$\beta>1$, $\mathcal{M}_{1,1}$

* Haldane-Rezayi (85)

β- degeneracy of Laughlin states on torus



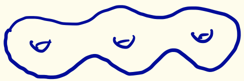


translational symmetry breaking

* Wen-Niu (89)

<u>Topological degeneracy</u> on genus-g

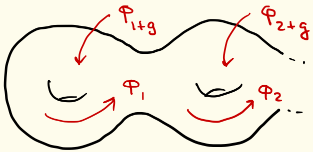
β^g Laughlin states for $\sigma_H = 1/\beta$ (conjecture).



Topological phases of matter

* Avron-Seiler-Zograf (1995)

Hall conductance from transport on Jacobian



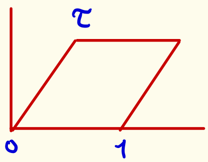
$$I_k = i \sum_{j=1}^{2g} \omega_{kj} \dot{\varphi}_j \qquad \varphi_j \in [0, 2\pi]^{2g} = \mathrm{Jac}(\Sigma)$$

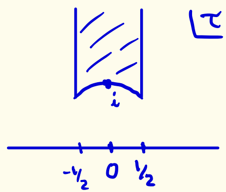
$$\omega = \sum_{j=1}^{g} d\varphi_j \wedge d\varphi_{j+g}$$

* Avron-Seiler-Zograf (1995), Read (2008)

transport on $\mathcal{M}_{1,1}$

("geometric adiabatic transport")





Hall viscosity -

Chern number on the moduli space of complex structures (τ)

$$c_1 = \frac{N_\Phi}{4} - \frac{c_H}{24} \chi(\Sigma)$$

SK-Wiegmann'15

Mathematically, QHE wave functions define sequences of probability measures on $\mathbb{C}^N$ (actually, $\mathbb{C}^N/S_N$)

$$\mu_N := \frac{1}{N!} \left|\Psi(z_1, \ldots, z_N)\right|^2 \cdot \prod_{n=1}^{N} d^2 z_n$$

Total mass of μ_N is the L^2- norm of Ψ. For Laughlin:

$$Z = \frac{1}{N!} \int_{\mathbb{C}^N} \exp\left[-\frac{B}{2} \sum_n |z_n|^2 + \beta \sum_{n \neq m} \log |z_n - z_m|\right] \prod_{n=1}^{N} d^2 z_n$$

2D Coulomb gas partition function.

More generally,

$$Z = \int_{\mathbb{C}^N} \exp\left\{ -N \sum_{n=1}^{N} V(z_n, \bar{z}_n) + \beta \sum_{n \neq m} \log |z_n - z_m| \right\} \cdot \prod_{n=1}^{N} d^2 z_n$$

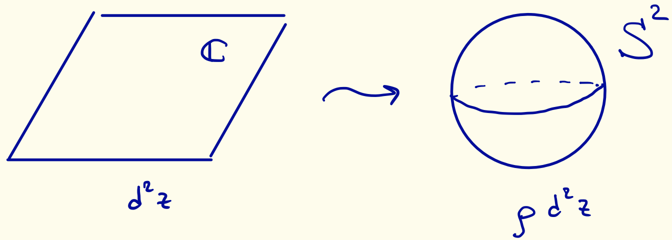
grav. "spin" ($s=1$ in pure Coulomb gas)

$$V = \psi(z, \bar{z}) - \frac{1-s}{N} \log \sqrt{g}(z, \bar{z})$$

"magnetic potential"

$B = \Delta \psi$

volume form $\sqrt{g}\, d^2 z$ on $\mathbb{C}$



Thm (Leblé-Serfaty 2017)

$$\log Z = -\beta N^2 I_V(\mu_V) + \frac{\beta}{2} N \log N - N\, C(\beta) - N\left(1-\frac{\beta}{2}\right)\int_{\mathbb{C}} \mu_V \log \mu_V + o(N)$$

where $I_V = -\iint_{\mathbb{C}\times\mathbb{C}} \log|z-w|\, d\mu(z)\, d\mu(w) + \int_{\mathbb{C}} V\, d\mu$

and μ_V its unique minimizer ("equilibrium measure")

[in fact, corollary of a stronger large deviations result]

$\mathcal{O}(1)$ term

Can, Laskin, Wiegmann 2014
F. Ferrari, SK 2014

$$\log Z = -\beta N^2 I_v(\mu_v) - N\left(s - \frac{\beta}{2}\right)\int_\Sigma \mu_v \log \mu_v$$

$$- \frac{c_H}{12} \int_\Sigma \left(|\partial \log \mu_v|^2 - 2\partial\bar\partial \log \mu_v \right) + \text{const} + R_{1/N}$$

remainder terms

$$c_H = 1 - 3\left(\sqrt{\beta} - \frac{2s}{\sqrt{\beta}}\right)^2 \qquad (s = 1 \text{ in pure Coulomb gas})$$

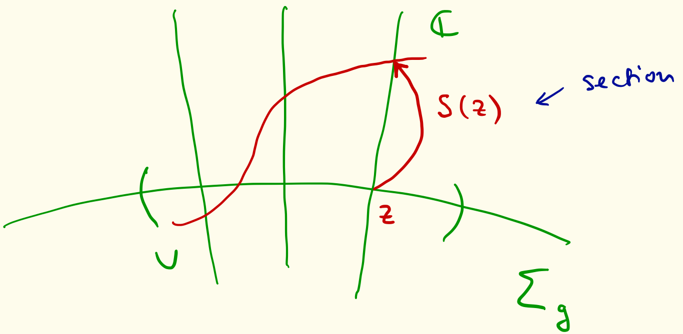
Coefficients in this expansion are of interest.

$R_{1/N}$ is a local funct. of B and curvature R of g.

QH states on Riemann surface

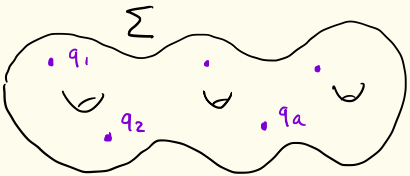
Consider genus-g Riemann surface (Σ, g, J).

Polynomials in $z \in \mathbb{C}$ $\longleftrightarrow$ sections $s(z)$ of a holomorphic line bundle L.



+ holomorphic transition functions t_{UV} on $U \cap V$

Line bundles L of $\deg L = N_\phi$ $\leftrightsquigarrow$ divisors D



$$D = \sum_{a=1}^{N_\phi} q_a$$

$$s(z) = (z - q_1) \cdots (z - q_{N_\phi})$$

$H^0(\Sigma, L)$ - vector space of holomorphic sections

<u>Riemann-Roch</u> : $\dim H^0(\Sigma, L) = N_\phi + 1 - g$

(for N_ϕ suff. large)

Hermitian metric h on L,

curvature $-i \partial \bar{\partial} \log h = F$ magnetic field

Definition of Laughlin states for $g > 1$

One may choose

$$|\Psi_L(z_1, \dots, z_N)|^2 = e^{-\beta \sum_{n<m} G(z_n, z_m)}$$

Locally looks like $\prod_{n<m} |z_n - z_m|^{2\beta}$,

but no determinant representation at $\beta = 1$

$$e^{-\sum_{n<m} G(z_n, z_m)} \neq \det S_n(z_m)$$

$N-1$ zeroes in z_1

$N-1+g$ zeroes in z_1

(Riemann - Roch thm)

Laughlin states on Σ_g

$$\Psi(z_1,\dots,z_N) = \prod_{n<m} (z_n - z_m)^\beta \, e^{-\frac{B}{4}\sum_n |z_n|^2}$$

$\beta \in \mathbb{Z}_+$

Def Consider (Σ, g, J) and holomorphic line bundle (L, h) of degree N_ϕ. Consider $\Sigma^N = \Sigma \times \dots \times \Sigma$

$$N = \frac{1}{\beta} N_\phi + 1 - g \qquad (\text{assume } N_\phi \mid \beta)$$

$*$ $\pi_n \Psi \in H^0(\Sigma, L)$ (restriction to n-th factor in $\Sigma \times \dots \times \Sigma$)

$*$ $\pi_{nm} \Psi \simeq (z_n - z_m)^\beta$, near diagonal $(z_n \sim z_m)$

$$\hat{H} = \sum_{n=1}^{N} \bar\partial_L^\dagger \bar\partial_L + \sum_{n \neq m} V(z_n, z_m)$$

Laughlin state at $\beta = 1$

$N = \dim H^0(\Sigma, L) = N_\phi + 1 - g$

$\{s_n(z)\}$ basis of $H^0(\Sigma, L)$

$$\Psi(z_1, \dots, z_N) = \det\, s_n(z_m)\Big|_{n,m=1}^{N}$$

Its L^2-norm is given by

$$Z = \int_{\Sigma^N} \| \det\, s_n(z_m) \|_h^2 \cdot \prod_{n=1}^{N} \sqrt{g}\, d^2 z_n$$

Consider a basis $\{s_n\} \in H^0(\Sigma, L)$, L^2-normalized

for some (h_0, g_0) : $\langle (s_n, s_m)_{h_0} \rangle_{g_0} = \delta_{nm}$

Then for any $h = h_0 e^{-N_\phi \psi}$ and $g = g_0 + \partial\bar\partial \phi$:

Thm | SK 2013, SK-Ma-Marinesu-Wiegmann 2017 — Asymptotic large N f-la for $\log Z$

$$\log Z = -N_\phi^2 S_2(\psi) + \frac{1}{2} N_\phi S_1(\phi, \psi) + \frac{1}{6} S_L(\phi) + O(1/N_\phi)$$

$$S_2(\psi) = \frac{1}{2\pi} \int_\Sigma |\partial \psi|^2 + \psi g_0$$

$$S_L(\phi) = \frac{1}{2\pi} \int \left(\left| \partial \log \frac{g}{g_0} \right|^2 + R_0 \log \frac{g}{g_0} \right)$$

$$S_1(\phi, \psi) = \frac{1}{2\pi} \int_\Sigma \left(-\frac{1}{2} \psi R_0 + F \log \frac{g}{g_0} \right)$$

<u>Proof</u> Variational f-la

$$\delta \log Z = -\frac{1}{2\pi}\int_\Sigma \left(N_\phi B_{N_\phi}\cdot \delta\psi - \frac{1}{2}\Delta B_{N_\phi}\cdot\delta\phi \right)\sqrt{g}\, d^2 z$$

where Bergman kernel for the $H^0(\Sigma, L)$

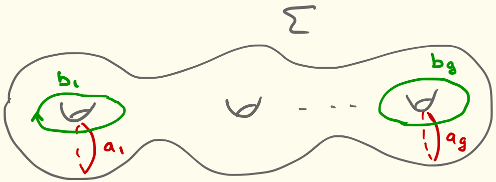
$$B_{N_\phi}(z,\bar z) = \sum_{n=1}^{N} \| s_n(z)\|_h^2 \simeq N_\phi + \frac{1}{2}R(g) + \mathcal{O}(1/N_\phi)$$

Complete asymptotic expansion at large N_ϕ

(Boutet de Monvel - Sjostrand, Zelditch, Catlin, ...)

$\square$

Laughlin states for $\beta > 1$, $g > 0$



- Canonical basis of holomorphic diff's

$$\omega_j \in H^1(\Sigma, \mathbb{Z}) \quad , j = 1, \dots, g$$

- Period matrix $\quad \tau_{ij} = \int_{b_i} \omega_j$

- Abel map $\quad \Sigma \to \mathrm{Jac}(\Sigma) = \mathbb{C}^g / \Lambda \qquad z \to \int^z \omega_j$

$$\Lambda = \{ m + m' \tau , \; m, m' \in \mathbb{Z}^g \}$$

Prop SK 2019 Basis of β^g Laughlin states (Wen-Niu conjecture)

$r = (1, \dots \beta)^g$

$$\boxed{\Psi_r = \Theta\begin{bmatrix} r/\beta \\ 0 \end{bmatrix}\left(\beta \sum_{n=1}^{N} z_n - \beta\Delta_0 - D,\ \beta\tau\right) \cdot \prod_{n<m} E(z_n, z_m)^{\beta} \cdot \prod_{n=1}^{N} \sigma(z_n)^{\frac{1}{g}\deg L - \beta}}$$

Here $E(z_n, z_m) \simeq \dfrac{z_n - z_m}{\sqrt{dz_n}\sqrt{dz_m}}$ is Prime form on $\Sigma \times \Sigma$

$\Theta\begin{bmatrix} a \\ b \end{bmatrix}(e, \tau)$ – Riemann theta, $\sigma(z) \sim$ Fay's $\frac{g}{2}$-diff.

$a, b \in \mathbb{R}^g$, $e \in \mathrm{Jac}(\Sigma)$

Completeness of this basis: Zvonkine-SK in preparation

Comments on proof

Prime-form on $\Sigma \times \Sigma$

$$E(z,y) = \frac{\theta\left(\int_z^y \omega - \delta, \tau\right)}{\sqrt{\omega_\delta(z)}\sqrt{\omega_\delta(y)}} \simeq \frac{z-y}{\sqrt{dz}\sqrt{dy}}$$

$\exists\ \delta \in (\mathbb{Z}/_2\mathbb{Z})^g$ (non-singular θ-characteristic), s.t.

$\operatorname{div} \omega_\delta = 2 D_{g-1}$

There are β^g independent order-β Θ-functions

$$\Theta\begin{bmatrix} r/\beta \\ 0 \end{bmatrix}(\beta Y, \beta\tau) \qquad Y \in \mathbb{C}^g/\Lambda$$

$$r = (1, \dots, \beta)^g$$

$$\Psi_r \sim \prod_{n<m}^{N} \big(E(z_n, z_m)\big)^\beta \cdot \Theta\begin{bmatrix} r/\beta \\ 0 \end{bmatrix}\Big(\beta \sum z_n - \beta\Delta - \beta D, \beta\tau\Big)$$

for $N \geq g$ (by Jacobi thm. $I(D_g) = \mathrm{Jac}(\Sigma)$)

So far Ψ_L is a degree $-\frac{1}{2}\beta(N-1)$ form on each Σ.

$$G(p) = \exp\left(- \sum_{j=1}^{g} \int_{a_k} w_k(z) \log E(z,p) \right)$$

$\frac{g}{2}$ - form with no poles/zeroes

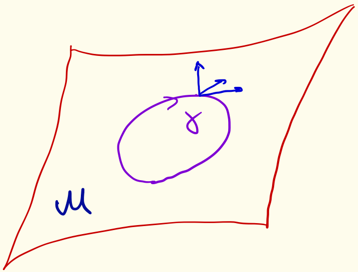
$G(p)$ lives on X_0

$$\boxed{\Psi_r = \Theta\begin{bmatrix} r/\beta \\ 0 \end{bmatrix}\left(\beta \sum_{n=1}^{N} z_n - \beta\Delta - \beta D, \beta\tau \right) \cdot \prod_{n<m}^{N} E(z_n, z_m)^{\beta} \cdot \prod_{n=1}^{N} G(z_n)^{\frac{1}{g}\beta(N-1)}}$$

Geometric adiabatic transport

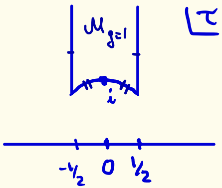
QHE wave functions are typically degenerate (β^g Laughlin states on genus-g surface) and depend on parameter spaces $\mathcal{M}$ (e.g. moduli space $\mathcal{M}_{g,n}$)

Thus we have a Hilbert bundle $V_{QH} \to \mathcal{M}$



adiabatic transport:

$$\Psi_r \to \underbrace{U_{rr'}(\gamma)}_{\text{holonomy matrix}} \Psi_{r'}$$



Conjecture N.Read 2008 (for $g > 0$) "holonomy equals monodromy"

V_{QH} is projectively flat (at least as $N \to \infty$)

(i.e. Berry curvature is $\mathcal{R} = c \cdot \mathbb{1}$, or equivalently adiabatic transport is independent of the path in $\mathcal{M}$, up to $U(1)$ phase)

Berry connection $\nabla : \Gamma(V) \to \Omega^1(V)$

$$\partial_y \langle \psi , \psi' \rangle_{L^2} = \langle \nabla \psi, \psi' \rangle_{L^2} + \langle \psi, \nabla \psi' \rangle_{L^2} \qquad y \in \mathcal{M}.$$

Projective flatness in CFT : Axelrod-della Pietra-Witten'90
Hitchin'90

* Laughlin and Pfaffian states are projectively flat on $\mathcal{M}_{1,1}$

$$\nabla^H \Psi_L = 0 \qquad \nabla^H = 4\pi i N_\phi \partial_\tau - \sum_{n=1}^{N} \partial^2_{z_n} + 2\beta(\beta-1) \sum_{n<m} \wp(z_n - z_m)$$

w/ N. Nemkov

* Is $H^0(\Sigma, L^d)$ projectively flat over $\mathcal{M}_{g,n}$, $\mathrm{Pic}_d(\Sigma)$?

Transport on moduli space of complex structures $\mathcal{M}_g$

SK-Ma-Marinescu-Wiegmann

Deformations of complex structure on Σ

$$g_{z\bar z}\,|dz|^2 \longrightarrow g_{z\bar z}\,|dz+\mu\, d\bar z|^2$$, where μ is $(1,-1)$-differential

$$\mu = g_{z\bar z}^{-1}\sum_{k=1}^{3g-3}\eta_k\,\delta y_k$$, where $\{\eta_k\}$ is a basis of holom. quadratic diffs.

* $\beta=1$ Laughlin state ("determinant line bundle" over $\mathcal{M}_g$)

Berry curvature $$R = i\,\partial_y\bar\partial_y \log Z$$

$$= i\,\partial_y\bar\partial_y \log \underbrace{\frac{Z}{\det'\bar\partial_L^{+}\bar\partial_L}}_{\text{Quillen metric}} + \underbrace{i\,\partial_y\bar\partial_y \log \det'\bar\partial_L^{+}\bar\partial_L}_{\to\, 0 \text{ as } \deg L\to\infty} =$$

$$= \left(\frac{1}{4}N_\phi + \frac{1}{12}\chi(\Sigma)\right)\Omega_{WP} + \ldots$$

Weil-Peterson (1,1) form

$$\Omega_{WP} = \int_\Sigma |\mu|^2 R\sqrt{g}\, d^2z$$

$\beta > 1$ Laughlin state

SK-Wiegmann 2015

For Laughlin states (rank - β^g vector bundle over $\mathcal{M}_g$)

$$\mathrm{Tr}\, R = \beta^g \left(\frac{1}{4} N_\phi - \frac{c_H}{24} \chi(\Sigma) \right) \Omega_{WP} + O\left(1/N_\phi \right)$$

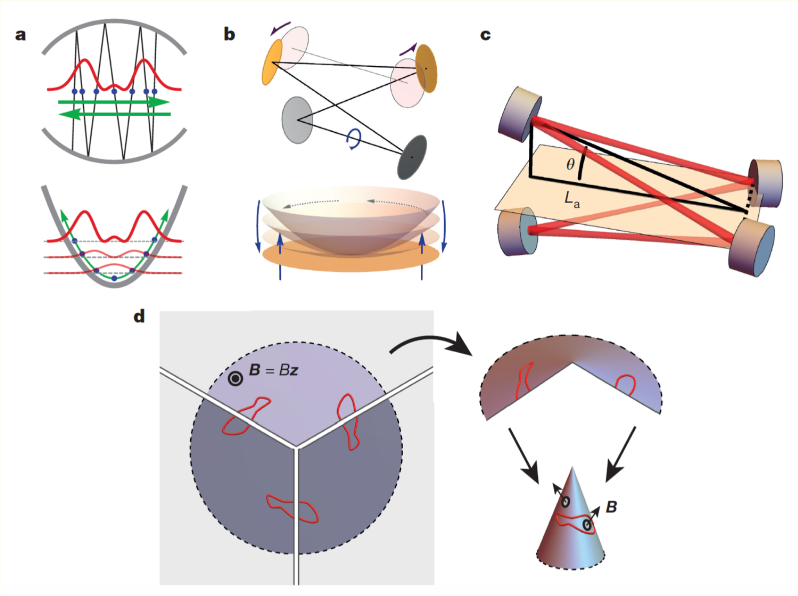
$$\boxed{c_H = 1 - 3 \left(\sqrt{\beta} - \frac{2s}{\sqrt{\beta}} \right)^2}$$

The End

Synthetic Landau levels for photons

Schine et. al., Nature 2016, 2018

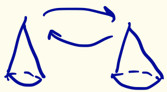
Experimental test of QHE in curved space



Anyon braiding

$$\Psi \to e^{i\frac{\pi}{\beta}} \Psi$$

Cone ("genon") braiding



$$\Psi \to e^{i\pi \frac{C_H}{12} \alpha_1 \alpha_2} \Psi$$