On an old conjecture of Almgren

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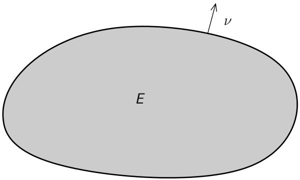
The problem

For $g : \mathbb{R}^d \to \mathbb{R}$ convex and coercive, $\Phi : \mathbb{R}^d \to \mathbb{R}^+$ convex one-homogeneous i.e. $\Phi(\lambda x) = |\lambda|\Phi(x)$, with $\Phi > 0$ on $\mathbb{R}^d \backslash \{0\}$ and $V > 0$ we consider

$$\min_{|E|=V} \int_{\partial E} \Phi(\nu) d\mathcal{H}^{d-1} + \int_E g dx \qquad (P_V)$$



Conjecture (Almgren): every minimizer is **convex**.

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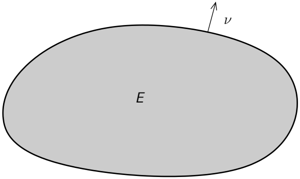

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To avoid some complications, we will assume g **strictly convex**.

Motivation and variant

This models equilibrium shapes of liquid drops/ crystals in the presence of an external field ($-\nabla g$). See e.g. Herring

We also consider the problem without volume constraint:

$$\min_E \int_{\partial E} \Phi(\nu) d\mathcal{H}^{d-1} + \int_E g dx \qquad (P)$$

which appears in Almgren-Taylor-Wang scheme for MCF.

Notation

We let

$$P_\Phi(E) = \int_{\partial E} \Phi(\nu) d\mathcal{H}^{d-1}.$$

When $\Phi = |\cdot|$ this is just the perimeter (denoted by $P(E)$). We also define

$$\mathcal{G}(E) = \int_E g dx.$$

$\Longrightarrow$ we minimize $\mathcal{F}(E) = P_\Phi(E) + \mathcal{G}(E)$.

The case $g = 0$

Let

$$W = \{x \in \mathbb{R}^d \ : \ x \cdot \nu \leq \Phi(\nu) \quad \forall |\nu| = 1\}.$$

This is the Wulff shape associated to Φ.

Theorem (Wulff, Dinghas, Taylor, Fonseca, Müller)

Up to translation W is the unique minimizer of

$$\min_{|E|=|W|} P_\Phi(E).$$

In particular: W is convex

Idea of proofs

Various proofs:

- symmetrization for $P_\Phi = P$: De Giorgi
- Brunn-Minkowski: Dinghas, Taylor, Fonseca
- Optimal transport: Gromov, Figalli-Maggi-Pratelli
- Second variation: Barbosa-Do Carmo
- ...

No 'direct' proof of convexity.

The case $\Phi = 0$

For $V > 0$, let t_V be such that $|\{g \leq t_V\}| = V$ then the **convex** set $\{g \leq t_V\}$ minimizes

$$\min_{|E|=V} \int_E g dx$$

Scaling

For $\lambda > 0$,

$$\mathcal{F}(\lambda E) = \lambda^{d-1} P_\Phi(E) + \lambda^d \int_E g(\lambda x) dx$$

- for $V \ll 1$, $P_\Phi(E)$ dominates and $E \sim W$
- for $V \gg 1$, $\mathcal{G}(E)$ dominates and $E \sim \{g \leq t\}$.

Terminology

- If W is polyhedral we say that Φ is crystalline
- We say that $\Phi \in C^2(\mathbb{R}^d \setminus \{0\})$ is uniformly elliptic if

$$\langle D^2\Phi(\nu)\xi, \xi\rangle \geq |\xi - \langle \nu, \xi\rangle \nu|^2 \qquad \forall |\xi| = |\nu| = 1.$$

In this case W is uniformly convex and C^2.

General properties of minimizers

Recall: $\Phi \gtrsim |\cdot|$ convex one-homogeneous and g convex coercive, $\mathcal{F} = P_\Phi + \mathcal{G}$ with $P_\Phi(E) = \int_{\partial E} \Phi(\nu)$ and $\mathcal{G}(E) = \int_E g$

$$\min_{|E|=V} \mathcal{F}(E) \qquad (P_V) \qquad \text{and} \qquad \min_E \mathcal{F}(E) \qquad (P)$$

- There always exist minimizers for (P_V) and (P). Every such minimizer is bounded.
- Every minimizer satisfies densities estimates $\Longrightarrow E \sim \mathring{E}$.
- $P_\Phi(F) \geq P_\Phi(E)$ if $F \supset E$ (E is outward minimizing the perimeter).
- If $\Phi \in C^{3,\alpha}$ is uniformly elliptic and $g \in C^{1,\alpha}$, then $\exists\ \Sigma$ with $\mathcal{H}^{d-3}(\Sigma) = 0$ and such that $\partial E \backslash \Sigma$ is $C^{3,\alpha}$

Notation from differential geometry

For a smooth $(d-1)-$manifold $M \subset \mathbb{R}^d$,

- D is the gradient in $\mathbb{R}^d$ and ∇ is the tangential gradient
- A is the second fondamental form
- $H^\Phi = \text{div}_M(D\Phi(\nu)) = tr(D^2\Phi(\nu)A)$ is the anisotropic mean curvature
- if $\Phi = |\cdot|$, $H^\Phi = H$ is the classical mean curvature.

First and second variation for (P)

If E is a minimizer of (P) and Φ is uniformly elliptic then

1) First variation: $H^{\Phi} + g = 0$ on $\partial E \setminus \Sigma$
2) Second variation: for $\varphi \in C^1_c(\partial E \setminus \Sigma)$,

$$\int_{\partial E \setminus \Sigma} \langle D^2\Phi(\nu)\nabla\varphi, \nabla\varphi\rangle - tr(D^2\Phi(\nu)A^2)\varphi^2 + D_\nu g\, \varphi^2 \geq 0 \quad (1)$$

If E satisfies (1) for every $\varphi \in C^1_c(\partial E \setminus \Sigma)$, we say that it is **stable**.

Rk: if $\Phi = |\cdot|$, then (1) reads:

$$\int_{\partial E \setminus \Sigma} |\nabla\varphi|^2 - tr(A^2)\varphi^2 + D_\nu g\, \varphi^2 \geq 0.$$

First and second variation for (P_V)

If E is a minimizer of (P_V) and Φ is uniformly elliptic then

1) First variation: $\exists \mu \in \mathbb{R}$ s.t. $H^\Phi + g = \mu$ on $\partial E \setminus \Sigma$
2) Second variation: for $\varphi \in C^1_c(\partial E \setminus \Sigma)$ with $\int_{\partial E} \varphi = 0$,

$$\int_{\partial E \setminus \Sigma} \langle D^2\Phi(\nu)\nabla\varphi, \nabla\varphi\rangle - tr(D^2\Phi(\nu)A^2)\varphi^2 + D_\nu g\, \varphi^2 \geq 0$$

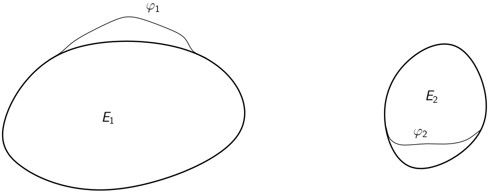
A first remark

If E is minimizing (P) or (P_V),

$P_\Phi(F) \geq P_\Phi(E)$ if $F \supset E \Longrightarrow E$ is mean convex i.e. $H^\Phi \geq 0$.

A second (important) remark

If E is a minimizer of (P_V) with **disconnected** boundary, then at least one connected component of ∂E is **stable**.



Otherwise $\exists\ \varphi_1, \varphi_2$ with $\int_{\partial E_1} \varphi_1 = -\int_{\partial E_2} \varphi_2 > 0$ s.t. $\varphi = \varphi_1 + \varphi_2$ satisfies

$$\int_{\partial E \setminus \Sigma} \langle D^2\Phi(\nu)\nabla\varphi, \nabla\varphi\rangle - tr(D^2\Phi(\nu)A^2)\varphi^2 + D_\nu g\, \varphi^2 < 0$$

Convexity for $d = 2$

Theorem (McCann, Okikiolu)

If $d = 2$, for every g, Φ and $V > 0$, every minimizer of (P_V) can be decomposed as $E = \cup_{i=1}^{N} E_i$ where $|E_i| = m_i$ with $m_i \neq m_j$ and E_i is **convex** *and is the* **unique** *minimizer of*

$$\min_{|K|=m_i,\ K \ convex} \mathcal{F}(E).$$

Idea of proof: since E mean convex $\Longrightarrow$ E_i convex. Uniqueness follows from OT argument (displacement convexity)

The case $V \ll 1$

Theorem (Figalli-Maggi+Figalli-Zhang)

Assume $V \ll 1$ then

i) *E is* **connected** *and $E \sim W$. In particular if $d = 2$ and $V \ll 1 \implies E$* **convex** *and* **unique** *(cf McCann).*

ii) *if Φ is crystalline $\implies E$ convex polytope with sides parallel to W.*

iii) *$g \in C^1$, $\Phi \in C^{2,\alpha}$ unif. elliptic $\implies E$ is* **convex**.

Rk: Result more quantitative. Proof relies on quant. isoper. inequality. Quantitative version of iii) uses second variation argument inspired by Barbosa-Do Carmo.

The case $V \gg 1$

Theorem (Caselles-Chambolle)

$\forall$ Φ and g, if $V \gg 1$ the minimizer of

$$\min_{|E|=V} \int_{\partial E} \Phi(\nu) + \int_E g \qquad (P_V)$$

is **unique and convex**.

Rk: this Theorem is not explicitely stated.

Idea of proof

- for every $t \in \mathbb{R}$, every minimizer E_t of

$$\min_E P_\Phi(E) + \int_E (g - t) \qquad (P^t)$$

  is a minimizer of (P_V) for $V = |E_t|$.
- If u is the local minimizer of

$$\int \Phi(Du) + \frac{1}{2} \int (u - g)^2,$$

  then it is **unique** and **convex**. Convexity follows from Alvarez-Lasry-Lions.
- For every $t \in \mathbb{R}$, $E_t = \{u < t\}$ is the unique solution of (P^t).

$\Longrightarrow$ minimizer of (P_V) is unique and convex for $V > |\{\min u\}|$.

- Observe that the proof of Caselles-Chambolle works in the regime where $(P_V) \rightsquigarrow (P)$.
- Also proves that solutions of (P) are generically unique and convex.

The case $V \sim 1$

Theorem (G.-De Philippis)

If $\Phi \in C^{3,\alpha}$ uniformly elliptic and $g \in C^{1,\alpha}$ then for every minimizer E of (P_V), ∂E is **connected**.

Combining with mean convexity (cf McCann)

Corollary

If $d = 2$, $\Phi \in C^{3,\alpha}$ uniformly elliptic and $g \in C^{1,\alpha}$, then E is **convex** *and* **unique**.

Also

Theorem (G.-De Philippis)

If $\Phi \in C^{3,\alpha}$ uniformly elliptic and $g \in C^{1,\alpha}$ then every minimizer of (P) is **convex**.

Idea of the proof:

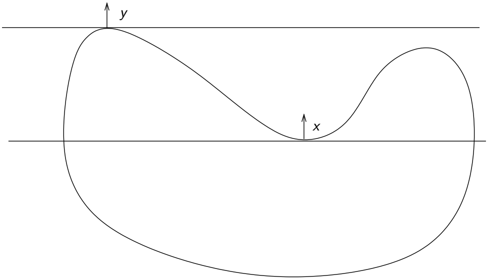
The idea is to consider the two-point function

$$S(x,y) = \langle \nu(x), y - x \rangle \qquad x \in \partial E \backslash \Sigma, y \in \partial E$$

and then

$$S(x) = \sup_{\partial E} S(x,y) = \sup_{\partial E} \langle \nu(x), y - x \rangle.$$

We have $S \geq 0$ and $S \equiv 0 \iff E$ convex.



Similar (but different) two-point functions introduced by Andrews to show preservation of interior ball condition by MCF, see also solution of Lawson's conjecture by Brendle.

Also reminiscent of doubling of the variable trick for viscosity solutions.

The Jacobi operator

Let

$$L_\Phi \varphi = \mathrm{div}_{\partial E}(D^2\Phi(\nu)\nabla\varphi) + Tr(D^2\Phi(\nu)A^2)\varphi$$

so that stability rewrites as

$$\int_{\partial E \setminus \Sigma} (-L_\Phi \varphi)\varphi + D_\nu g \varphi^2 \geq 0$$

Aim: prove that for minimizers of (P) or (P_V), S gives a negative variation i.e.

$$\int_{\partial E \setminus \Sigma} (-L_\Phi S)S + D_\nu g S^2 < 0$$

unless $S \equiv 0$.

Main Lemma

Recall: $H^\Phi = \mathrm{div}_{\partial E}(D\Phi(\nu)) = tr(D^2\Phi(\nu)A)$.

Lemma

If E is a minimizer of (P) or (P_V) then for $\bar{x} \in \partial E \backslash \Sigma$ if $S(\bar{x}) = S(\bar{x}, \bar{y})$, $\bar{y} \in \partial E \backslash \Sigma$ and

$$L_\Phi S(\bar{x}) \geq H^\Phi(\bar{x}) - H^\Phi(\bar{y}) + \langle \nabla H^\Phi(\bar{x}), \bar{y} - \bar{x} \rangle \tag{2}$$

in the viscosity sense.

Rk: minimality only used to get $\bar{y} \in \partial E \backslash \Sigma$. Under this condition (2) always holds.

Idea of proof

For $\varphi \in C^2(\partial E \backslash \Sigma)$ s.t. $\varphi(x) - S(x) \geq \varphi(\bar{x}) - S(\bar{x})$, the function

$$G(x,y) = \varphi(x) - S(x,y)$$

is minimal at $(\bar{x}, \bar{y})$. Indeed, since $S(x) = \sup_y S(x,y)$,

$$\begin{aligned} G(x,y) = \varphi(x) - S(x,y) \geq \varphi(x) - S(x) \\ \geq \varphi(\bar{x}) - S(\bar{x}) = \varphi(\bar{x}) - S(\bar{x},\bar{y}) = G(\bar{x},\bar{y}) \end{aligned}$$

Use 1st and 2nd order optimality conditions i.e. $\nabla_x G(\bar{x},\bar{y}) = \nabla_y G(\bar{x},\bar{y}) = 0$ and $\nabla^2_{x,y} G(\bar{x},\bar{y}) \geq 0$ and in particular

$$(\nabla^i_x + \nabla^i_y)(\nabla^i_x + \nabla^i_y) G(\bar{x},\bar{y}) \geq 0.$$

From Lemma to negative variation

Recall: E critical point $\implies H^\Phi + g = \mu$ for $\mu \in \mathbb{R}$.
Differentiating we get $\nabla H^\Phi(\bar{x}) = -\nabla g(\bar{x})$, Lemma $\implies$

$$\begin{aligned} L_\Phi S(\bar{x}) &\geq H^\Phi(\bar{x}) - H^\Phi(\bar{y}) + \langle \nabla H^\Phi(\bar{x}), \bar{y} - \bar{x} \rangle \\ &= g(\bar{y}) - g(\bar{x}) - \langle \nabla g(\bar{x}), \bar{y} - \bar{x} \rangle \end{aligned}$$

Since $D_\nu g(\bar{x}) S(\bar{x}) = D_\nu g(\bar{x}) \langle \nu, \bar{y} - \bar{x} \rangle$ and $Dg = \nabla g + (D_\nu g)\nu$,

$$L_\Phi S(\bar{x}) - D_\nu g(\bar{x}) S(\bar{x}) \geq g(\bar{y}) - g(\bar{x}) - \langle Dg(\bar{x}), \bar{y} - \bar{x} \rangle \geq 0$$

where last line used convexity of g.

Multiply by $(-S)$ and integrate to get negative second variation.

Conclusion of the proof:

- If E minimizes (P) then E is stable and we can use S as test for the stability to get $S \equiv 0 \implies E$ convex.
- If E minimizes (P_V), E must not be stable and S is not admissible (since $\int_{\partial E} S \neq 0$). However, if ∂E disconnected, one of its component must be stable $\implies E$ is convex $\implies \partial E$ is connected.

Rk: Minimality just used for regularity issues. If E is a smooth stable critical point then same proof applies.

Question about uniqueness

Question: For $d = 2$ and for $V \gg 1$, minimizers are unique, is it the case for every $V > 0$?

If YES then for Φ uniformly elliptic maybe convexity can be obtained by a continuity argument.

Some related problems:

- Sessile drops (Taylor-Almgren): $g = x_d$, $\sigma > 0$

$$\min_{|E|=V, E\subset \mathbb{R}^d_+} \int_{\partial E\cap \mathbb{R}^d_+} \Phi(\nu) d\mathcal{H}^{d-1} + \int_E x_d dx + \sigma \mathcal{H}^{d-1}(\partial E \cap \{x_d = 0\})$$

- Isoperimetric problem inside convex bodies: Ω convex, $g = I_\Omega$

$$\min_{|E|=V, E\subset\Omega} P_\Phi(E)$$

  If Ω bounded then $V \ll 1$ minimizer = Wulff shape, if $V \gg 1$ minimizer is unique and convex (Alter-Caselles-Chambolle).

- Relative isoperimetric problem inside convex bodies: Ω convex

$$\min_{|E|=V} \int_{\partial E\cap\Omega} \Phi(\nu) d\mathcal{H}^{d-1}.$$

  If Ω bounded $\implies \partial E$ connected (Sternberg-Zumbrun via stability). Conjecture: $\partial E \cap \Omega$ smooth (see S-Z, Jerison).

Semi-linear elliptic PDE

For $\Omega_0 \subset \Omega_1$ convex sets, two-point (or more) functions have been used in Korevaar, Caffarelli-Spruck ... to study level-set convexity of solutions to

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega_1 \backslash \Omega_0 \\ u = 1 & \text{on } \partial\Omega_0 \\ u = 0 & \text{on } \partial\Omega_1. \end{cases} \tag{3}$$

In particular, Weinkove used $S(x,y) = \langle Du(x) - Du(y), x - y \rangle$ restricted to $\{u(x) = u(y)\}$.

In general, by Hamel-Nadirashvili-Sire, solutions of (3) are not level-set convex but

Conjecture : if u is a solution of (3) which is stable i.e.

$$\int_{\Omega_1 \backslash \Omega_0} |\nabla \varphi|^2 + f'(u)\varphi^2 \geq 0 \qquad \forall \varphi \in C_c^1(\Omega_1 \backslash \Omega_0)$$

then u is **level-set convex** (see Cabré-Chanillo).

Thank you for your attention.