Hydrodynamics for SSEP with non-reversible slow boundary dynamics: the critical regime and beyond

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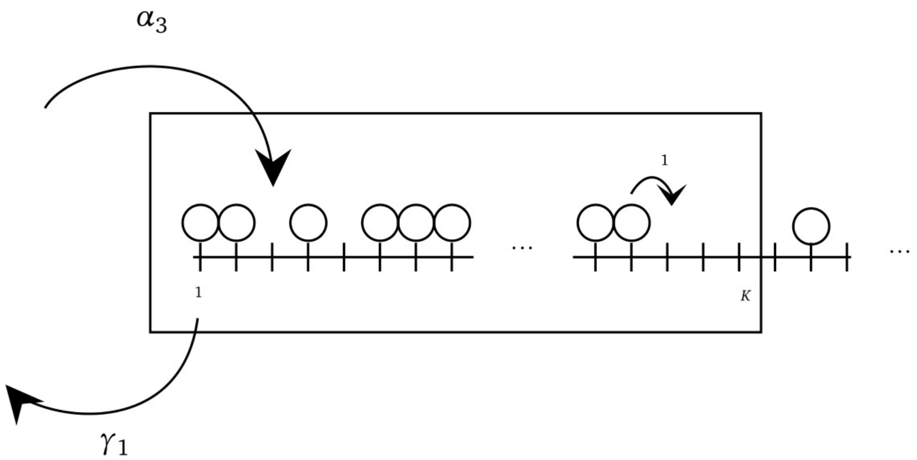
Outline



- Lattice: $\Lambda_N = \{1, \ldots, N-1\}$;
  - Site: $x \in \Lambda_N$;
  - Bond: $\{x, y\}$ with $x, y \in \Lambda_N$;
- Process: $\eta = (\eta(1), \ldots, \eta(N-1))$;
- State Space: $\Omega_N = \{0, 1\}^{\Lambda_N}$;
- "Boundary": $I_- = \{1, \ldots, K\}$ and $I_+ = \{N-1-K, \ldots, N-1\}$ for $K \geq 1$.

Formal Description

Poisson Clocks

- $N_{x,x+1}$, for $x \in \{1, \dots, N-2\}$ (Exclusion rule):
  - Poisson Process associated to bond $\{x, x+1\}$, with parameter $\eta(x)(1-\eta(x+1)) + \eta(x+1)(1-\eta(x))$;
- $N_{0,j}$, for $j \in I_-$ (Creation/Anihilation for the left):
  - Poisson process associated to the bond $\{0, j\}$, with parameter $\boldsymbol{\alpha}_j(\eta)(1-\eta(j)) + \boldsymbol{\gamma}_j(1-\eta)\eta(j)$, where

$$\boldsymbol{\alpha}_j(\eta) := \eta(1)\dots\eta(j-1)\alpha_j,$$
$$\boldsymbol{\gamma}_j(1-\eta) := (1-\eta(1))\dots(1-\eta(j-1))\gamma_j$$

Time Evolution

Starting from a random initial configuration η_0, the (jump) process η evolves according to the Poisson clocks. For example,

- $N_{x,x+1}$ rings at time t, then $x, x+1$ are flipped:
  - $\eta_{t-}(x) = 1, \eta_{t-}(x+1) = 0 \implies \eta_t(x) = 0, \eta_t(x+1) = 1$;
  - $\eta_{t-}(x) = 0, \eta_{t-}(x+1) = 1 \implies \eta_t(x) = 1, \eta_t(x+1) = 0$;
  - Otherwise, we see nothing since the particles are unlabbeled.

We can show that η is a *Markov Process*.

Generator description

Definition (Generator)

Let $\mathcal{L}_N = \mathcal{L}_{N,0} + N^{-\theta}\mathcal{L}_{N,b}$ act on functions $f\colon \Omega_N \to \mathbb{R}$ be defined by

Generator

$$(\mathcal{L}_{N,0}f)(\eta) = \sum_{x=1}^{N-2}\{f(\eta^{x,x+1}) - f(\eta)\},$$

$$(\mathcal{L}_{N,b}f)(\eta) = (\mathcal{L}_{N,-}f)(\eta) + (\mathcal{L}_{N,+}f)(\eta),$$

where

$$(\mathcal{L}_{N,\pm}f)(\eta) = \sum_{I_\pm} c_x^{\pm}(\eta)\{f(\eta^x) - f(\eta)\}$$

Flip

$$\eta^{x,y}(z) = \begin{cases} \eta(z), & z \neq x, y \\ \eta(y), & z = x \\ \eta(x), & z = y \end{cases}$$

$$\eta^{x}(z) = \begin{cases} \eta(z), & z \neq x, \\ 1-\eta(x), & z = x \end{cases}$$

Rates

$$c_x^-(\eta) = \boldsymbol{\alpha}_x(\eta)(1-\eta(x)) + \boldsymbol{\gamma}_x(1-\eta)\eta(x), \qquad c_x^+(\eta) = (1-\eta(x))\boldsymbol{\beta}_x(\eta) + \eta(x)\boldsymbol{\delta}_x(1-\eta).$$

Context

- On [4] the authors introduced the dynamics with $\alpha_x = \delta_x = j$ constant, and $\gamma_x = \beta_x = 0$ for $\theta = 1$;
  - On [6] the authors show the Hydrodynamic Limit and Fick's Law, and on [5] the Hydrostatic Limit. These results are in some form consequence of the Propagation of Chaos property: particles become independent as the size of the system increases.
- On [1] the authors show the Hydrodynamic and Hydrostatic for $\theta \geq 0$ and $K = 1$.
- On [8] is developed a method that encompasses the case $\theta = 0$ and $K \geq 1$, based on a work on [9].
- On [7] we show the Hydrodynamic Limit and the Hydrostatic Limit for $K = 2$ by adapting the method above, which relies on duality arguments and correlation estimates, for $\theta \in (0, 1)$;
- For the moment we have no information regarding Large Deviations neither equilibrium or non-equilibrium Fluctuations (behavior around the expected value).
- The Matrix Product Ansatz and Bethe Ansatz do not work, and on an ongoing work we are investigating the extension for the Matrix Product Ansatz for similar models with $K = 2$.

Scalling Limit

To study the Local density we shall consider the accelerated process $\{\eta_{N^2 t}\}_{t\geq 0}$. This scale is achived by considering the generator $\mathcal{L} := N^2 \mathcal{L}_N$.

Definition (Empirical measure)

$$\pi^N(\eta, du) = \frac{1}{N-1} \sum_{x \in \Lambda_N} \eta(x) \delta_{\frac{x}{N}}(du),$$

and its time evolution by $\pi_t^N(du) := \pi^N(\eta_{N^2 t}, du)$.

Definition (Associated profile)

We say that a sequence of probability measures $\{\mu_N\}_{N\geq 1}$ on Ω_N is associated with a profile $\rho_0 : [0,1] \to [0,1]$ if for any continuous function $G : [0,1] \to \mathbb{R}$ and every $\delta > 0$

$$\lim_{N\to\infty} \mu_N\Big(\eta \in \Omega_N : \big|\langle \pi^N, G\rangle - \langle G, \rho_0\rangle\big| > \delta\Big) = 0.$$

The *Hydrodynamic Limit* extends the initial association to all (bounded) times. But before that, we need to introduce the Hydrodynamic Equations (HDE).

Hydrodynamic Equation

Consider the Heat Equation on $[0,1]$

$$\begin{cases} \partial_t \rho_t(u) = \partial_u^2 \rho_t(u), & (t,u) \in [0,T] \times (0,1), \\ \rho(0,\cdot) = f_0(\cdot), \end{cases}$$

and the boundary conditions

$\theta = 1$

$$\begin{cases} \partial_u \rho_t(0) = -D_{\alpha,\gamma} \rho_t(0), & t \in [0,T], \\ \partial_u \rho_t(1) = D_{\beta,\delta} \rho_t(1), & t \in [0,T], \end{cases}$$

$\theta > 1$

$$\begin{cases} \partial_u \rho_t(0) = 0, & t \in [0,T], \\ \partial_u \rho_t(1) = 0, & t \in [0,T], \end{cases}$$

where for $\lambda = (\lambda_1, \dots, \lambda_K), \sigma = (\sigma_1, \dots, \sigma_K)$ and $f\colon [0,1] \to \mathbb{R}$ we defined

$$(D_{\lambda,\sigma} f)(u) := \sum_{x=1}^{K} \{\lambda_x (1 - f(u)) f^{x-1}(u) - \sigma_x f(u)(1 - f(u))^{x-1}\}.$$

Weak Formulation

Definition

For $\theta \geq 1$ and measurable $f_0 \in [0,1]$ define

$$F_\theta(\rho, G, t) = \langle \rho_t, G_t \rangle - \langle f_0, G_0 \rangle - \int_0^t \langle \rho_s, \left(\partial_u^2 + \partial_s\right) G_s \rangle \, ds$$
$$+ \int_0^t \left\{ \rho_s(1)\partial_u G_s(1) - \rho_s(0)\partial_u G_s(0) \right\} ds$$
$$- \mathbb{1}_{\theta=1} \left(\int_0^t G_s(1)(D_{\beta,\delta}\rho_s)(1) \, ds + \int_0^t G_s(0)(D_{\alpha,\gamma}\rho_s)(0) \, ds \right).$$

We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the heat equation (with boundary conditions depending on θ) if

1. $\rho \in L^2(0,T;\mathcal{H}^1)$,
2. ρ satisfies the weak formulation $F_\theta(\rho, G, t) = 0$ for all $t \in [0,T]$ and function $G \in C^{1,2}([0,T] \times [0,1])$.

Remark

Note that we do not ask (weak) time-differentiability of the solution.

Hydrodynamic Limit

Hypotesis

The (finite) sequences α, γ, β and δ are non-increasing, (H0)

Lemma (Uniqueness)

The weak solution of the Heat Equation with Neumann b.c. is unique. Assuming (H0)*, the weak solution of the Heat Equation with nonlinear Robin b.c. is unique.*

Theorem (Hydrodynamic Limit)

Let $f_0 : [0,1] \to [0,1]$ be a measurable function and let $\{\mu_N\}_{N\geq 1}$ be a sequence of probability measures in Ω_N associated with f_0. Then, for any $t \in [0, T]$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\Big(\big|\langle \pi_t^N, G\rangle - \langle G, \rho_t\rangle\big| > \delta\Big) = 0,$$

where $\rho_t(\cdot)$ is the unique weak solution of the Heat Equation with Neumann b.c. ($\theta > 1$) or nonlinear Robin b.c. (under (H0) *and for $\theta = 1$).*

Some tools

Dynkin's Martingale

Let $\{X_t\}_{t\geq 0}$ be a Markov process with generator $\mathcal{L}$ and countable state space E, and $f\colon \mathbb{R}^+ \times E \longrightarrow \mathbb{R}$ bounded *with some regularity assumptions*. For all $t \geq 0$ let

$$M_t(f) := f(t, X_t) - f(0, X_0) - \int_0^t (\partial_s + \mathcal{L}) f(s, X_s)\, ds,$$

$$B_t(f) := \mathcal{L} f(t, X_t)^2 - 2 f(t, X_t) \mathcal{L} f(t, X_t),$$

$$N_t(f) := (M_t(f))^2 - \int_0^t B_s(f)\, ds.$$

Then $\{M_t(f)\}_{t\geq 0}$ and $\{N_t(f)\}_{t\geq 0}$ are martingales w.r.t. the natural filtration of $\{X_t\}_{t\geq 0}$. In particular, $[M(f)]_t := \int_0^t B_s(f)\, ds$ is the quadratic variation of $M_t(f)$.

Remark

Note that the absence of time derivatives of ρ comes from the formula for $M_t(f)$. Computing $M_t(f)$ allow us to "guess" the Hydrodynamic Equation. Having an explicit formula for the quadratic variation allow us to control it.

Dynkin's formula

$$M_t^N(G) = \langle \pi_t^N, G_t\rangle - \langle \pi_0^N, G_0\rangle - \int_0^t \langle \pi_s^N, (\partial_s + \Delta_N) G_s\rangle \, ds$$
$$- \int_0^t \Big\{ \nabla_N^+ G_s(0)\eta_{sN^2}(1) - \nabla_N^- G_s(1)\eta_{sN^2}(N-1)\Big\} ds$$
$$- \frac{N^2}{N^\theta}\int_0^t \Big\{ \langle \pi^N(D_{\alpha,\gamma}^{N,-}\eta_{sN^2}, \cdot), G_s\rangle + \langle \pi^N(D_{\beta,\delta}^{N,+}\eta_{sN^2}, \cdot), G_s\rangle \Big\} ds,$$

with $(D_{\lambda,\sigma}^{N,-} f)(x) = 1_{x\in I_-}\{\lambda_x(1-f(x))\prod_{y=1}^{x-1} f(x) - \sigma_x f(x)\prod_{y=1}^{x-1}(1-f(x))\}$. More precisely,

$$\frac{N^2}{N^\theta}\langle \pi^N(D_{\alpha,\gamma}^{N,-}\eta, \cdot), G_s\rangle = \frac{1}{N^\theta}\sum_{x\in I_-} D_{\alpha,\gamma}^{N,-}\eta(x)\nabla G_s(0) + O(\tfrac{K^2}{N})$$

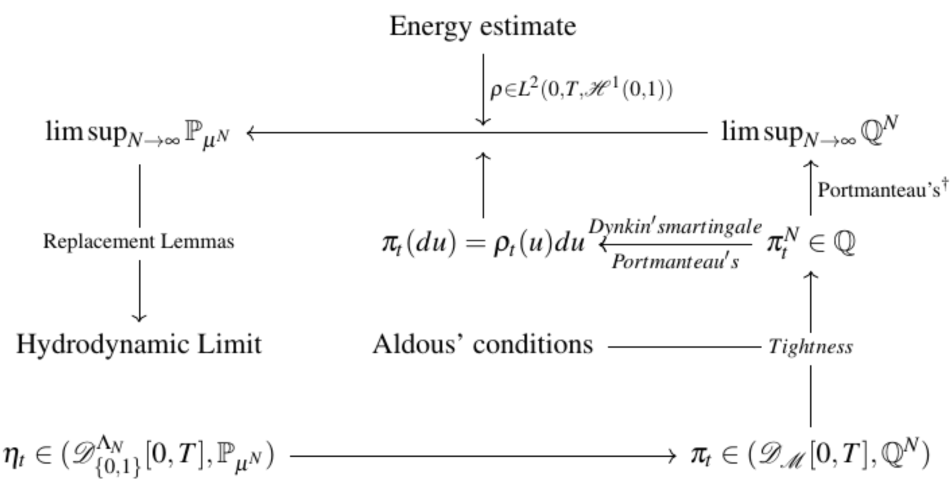
Remark (Main technical issue ($\theta = 1$))

While for $\theta > 1$ the correlation terms vanish as $N \to \infty$, for $\theta = 1$ we need to show

$$\frac{N^2}{N^\theta}\langle \pi^N(D_{\alpha,\gamma}^{N,-}\eta_{sN^2}, \cdot), G_s\rangle \xrightarrow{\mathbb{P}_{\mu_N}} (D_{\alpha,\gamma}\rho_s)(0)\, G_s(0).$$

Strategy

- Convergence in subsequences:
  - Prokhorov's Theorem + Aldous' criterion for tightness;
- Characterization of the Limit points:
  - Absolute continuity: $\pi_t(du) = \rho_t(u)du$;
  - Existence of solutions via microscopic system;
  - Replacement Lemmas (mean field estimates to control correlation terms) [1, 3];
- Uniqueness of the Limit (PDE's problem):
  - Choice of test function (backwards heat equation) [2]



Replacement Lemma

Box average

$$\overrightarrow{\eta}_s^{\varepsilon N}(1) := \frac{1}{\varepsilon N}\sum_{x=2}^{1+\varepsilon N}\eta_s(x), \quad \overleftarrow{\eta}_s^{\varepsilon N}(N-1) := \frac{1}{\varepsilon N}\sum_{x=N-2}^{N-1-\varepsilon N}\eta_s(x).$$

For N sufficiently large, $\overrightarrow{\eta}_s^{\varepsilon N}(1) \sim \rho_s(0)$ (resp. $\overrightarrow{\eta}_s^{\varepsilon N}(N-1) \sim \rho_s(1)$).

Lemma (Replacement Lemma)

Let $\psi : \Omega \to \Omega$ be a positive and bounded function which satisfies $\psi(\eta) = \psi(\eta^{z,z+1})$ for any $z = x+1, \cdots, x+\varepsilon N - 1$. For any $t \in [0, T]$ and $x \in \Lambda_N$ such that $x \in \{1, \cdots, N - \varepsilon N - 2\}$ we have that

$$\limsup_{\varepsilon\to 0}\limsup_{N\to+\infty}\mathbb{E}_{\mu_N}\left[\left|\int_0^t \psi(\eta_{sN^2})(\eta_{sN^2}(x) - \overrightarrow{\eta}_{sN^2}^{\varepsilon N}(x))\,ds\right|\right] = 0.$$

Idea of the proof

Definition (Bernoulli product measure)

Let $\alpha : [0,1] \to [0,1]$ be a measurable profile and $\nu_\alpha^N(\eta : \eta(x) = 1) = \alpha(\frac{x}{N})$.

Definition (Dirichlet form/Carré du champ)

Let μ be probability measure on Ω_N and $f : \Omega_N \to \mathbb{R}$ a density w.r.t. μ.

- Dirichlet form: $\langle \sqrt{f}, -\mathcal{L}_N \sqrt{f} \rangle_\mu$;
- Carrè du champ: $D_N(\sqrt{f}, \mu) := D_{N,0}(\sqrt{f}, \mu) + D_{N,b}(\sqrt{f}, \mu)$, where

$$D_{N,0}(\sqrt{f}, \mu) := \sum_{x=1}^{n-2} \int_{\Omega_N} \left[\sqrt{f(\eta^{x,x+1})} - \sqrt{f(\eta)} \right]^2 d\mu,$$

$$D_{N,\pm}(\sqrt{f}, \mu) = \sum_{x \in I_\pm^K} \int c_x^\pm(\eta) \left[\sqrt{f(\eta^x)} - \sqrt{f(\eta)} \right]^2 d\mu.$$

Using $ab - b^2 = -\frac{1}{2}(a-b)^2 + \frac{1}{2}(a^2 - b^2)$ they are related by

$$\langle \sqrt{f}, -\mathcal{L}_N \sqrt{f} \rangle_\mu = \frac{1}{2} D_N(\sqrt{f}, \mu) - \frac{1}{2} E_\mu \left[(\mathcal{L}_N f)(\eta) \right]$$

Idea of the proof

1. We are able to reduce the problem to that of estimating

$$\frac{H(\mu|\nu_\alpha^N)}{NB} + t \sup_{f \text{ density}} \left\{ \langle \psi(\eta)(\eta(x) - \bar{\eta}^{\epsilon N}(x)), f\rangle_{\nu_\alpha^N} + \frac{N}{B}\langle \mathcal{L}_N \sqrt{f}, \sqrt{f}\rangle_{\nu_\alpha^N} \right\}$$

2. We now see that for $\alpha(x/N) = \alpha$,

$$\left| \langle \varphi(\eta)(\eta(x) - \eta(y)), f\rangle_{\nu_\alpha^N} \right| \leq \tfrac{C_1}{A} D_N(\sqrt{f}, \nu_\alpha^N) + C_2 A(y-x),$$

with $A, B > 0$ arbitrary, $C_1, C_2 > 0$ constants.

3. Since $H(\mu|\nu_\alpha^N) = O(N)$ and since $\theta \geq 1$, we have $E_\mu\left[(\mathcal{L}_N f)(\eta)\right] = O(N^{-\theta})$, and we may choose appropriate $A \equiv A(\theta, N)$, $B \equiv B(\theta, N)$.

Advantages

- Trade natural measure by a more suitable one $(H(\mu|\nu_\alpha^N))$;
- Trade time evolution by variational formula;
- Trade entropy control by controlling distance from equilibrium (μ invariant $\implies E_\mu\left[(\mathcal{L}_N f)(\eta)\right] = 0$).

Uniqueness

The proof of uniqueness of weak solutions for the Robin case relies on a choice of test function and monotonicity of the boundary operators. As test function we choose the backward heat equation with Robin b.c, similarly to [2].

Lemma

For any $t \in (0, T]$, *the following problem with Robin boundary conditions*

$$\begin{cases} \partial_s \varphi(s,u) + a\partial_u^2 \varphi(s,u) = \lambda \varphi(s,u), & (s,u) \in [0,t] \times (0,1), \\ \partial_u \varphi(s,0) = b(s,0)\varphi(s,0), & s \in [0,t), \\ \partial_u \varphi(s,1) = -b(s,1)\varphi(s,1), & s \in [0,t), \\ \varphi(t,u) = h(u), & u \in (0,1), \end{cases}$$

with $h(u) \in C_0^2([0,1])$, $\lambda \geq 0$, $0 < a(u,t) \in C^{2,2}([0,T] \times [0,1])$, *and for* $u \in \{0,1\}$, $0 < b(u,t) \in C^2[0,T]$, *has a unique solution* $\varphi \in C^{1,2}([0,t] \times [0,1])$. *Moreover, if* $h \in [0,1]$ *then we have* $\forall (s,u) \in [0,t] \times [0,1]$:

$$0 \leq \varphi(s,u) \leq e^{-\lambda(t-s)}.$$

Insipired by [6], we can show:

Lemma

Let $\lambda = (\lambda_1, \dots, \lambda_K)$ *and* $\sigma = (\sigma_1, \dots, \sigma_K)$ *and recall that*

$$D_{\lambda,\sigma} f := \sum_{x=1}^{K} \{\lambda_x (1-f) f^{x-1} - \sigma_x f (1-f)^{x-1}\}.$$

Then for $f_i : \mathbb{R} \to \mathbb{R}$ *with* $i = 1, 2$, *we have*

$$D_{\lambda,\sigma} f_1 - D_{\lambda,\sigma} f_2 = -(f_1 - f_2)\, V_{\lambda,\sigma}(f_1, f_2),$$

where $V_{\lambda,\sigma}(f_1, f_2) = V_\lambda(f_1, f_2) + V_\sigma(1 - f_1, 1 - f_2)$ *with the operator* V_ϕ, *for any sequence* $\phi = (\phi_x)_{1 \le x \le K}$, *acting on functions* $(f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$, *as*

$$V_\phi(f_1, f_2) = \sum_{\substack{x=1 \\ \phi_{K+1} := 0}}^{K} (\phi_x - \phi_{x+1}) \sum_{i=0}^{x-1} f_1^{x-1-i} f_2^{i}.$$

In particular, if λ *and* σ *are non-negative, non-increasing and* $f_i \ge 0$ *for* $i = 1, 2$ *then there is a constant* $\underline{v}_K(\lambda, \sigma)$, *such that*

$$V_{\lambda,\sigma} f \ge \underline{v}_K(\lambda, \sigma) > 0.$$

Letting $w = \rho_1 - \rho_2$ with ρ_1, ρ_2 solutions starting from the same initial data, we have

$$\langle w_t, G_t\rangle = \int_0^t \langle w_s, \Big(\partial_u^2 + \partial_s\Big) G_s\rangle\, ds + \int_0^t w_s(0)\Big(\partial_u G_s(0) - G_s(0)\, V_{\alpha,\gamma}(0,s)\Big)\, ds$$
$$- \int_0^t w_s(1)\Big(\partial_u G_s(1) + G_s(1)\, V_{\beta,\delta}(1,s)\Big)\, ds,$$

with $V_{\lambda,\sigma}(\rho_s^{(1)}, \rho_s^{(2)})(v,v) := w_s(v)\, V_{\lambda,\sigma}(v,s)$.

- For $G = \varphi$, recall that $\partial_u \varphi(s,0) = b(s,0)\varphi(s,0)$ and $\partial_u \varphi(s,1) = -b(s,1)\varphi(s,1)$.
- We can regularize $b(s,0)$ (resp. $b(s,1)$) and approximate it to $V_{\alpha,\gamma}(s,0)$ (resp. $V_{\beta,\delta}(s,1)$), *but only since* $V > 0$;
- We then choose a, h, λ accordingly to show $w^+ < \epsilon$, then repeat the proof for $w^- < \epsilon$.

Definition (Local current)

- $J_t^N(x)$ - conservative current through bond $\{x, x+1\}$ up to time t :
  - counts the number of particles that jumped from the site x to the site $x+1$ minus the number of particles that jumped from the site $x+1$ to the site x.
- $K_t^N(x)$ - non-conservative current at the site x up to time t :
  - counts the number of particles that have been created minus the number of particles that have been removed of the system at site x.

Definition (Empirical current)

The empirical measure associated with the conservative/non-conservative current:

$$J_t^N := \frac{1}{N^2}\sum_{x=1}^{N-2} J_t^N(x)\delta_{x/N}, \quad K_t^N := \frac{1}{N}\sum_{x\in I_+^K\cup I_-^K} K_t^N(x)\delta_{x/N}.$$

Remark (On some technical details)

To use the same approach we need to consider instead the joint generator of the processes (η, J^N) and (η, K^N), which gets a bit "messy" [?], but identified the PDE through Dynkin's martingale, the L.L.N. for the current becomes a corollary of the Hydrodynamic Limit.

Main result

Theorem (Law of large Numbers for the current)

For any $t \in [0, T]$ *and* $f \in C^1([0,1])$,

$$\lim_{N\to+\infty} \mathbb{P}_{\mu_N}\left[\left|\langle J_t^N, f\rangle - \int_0^t \int_0^1 f(u)\,\partial_u \rho_s(u)\,du\,ds\right| > \delta\right] = 0\,,$$

$$\lim_{N\to+\infty} \mathbb{P}_{\mu_N}\left[\left|\langle K_t^N, f\rangle - \mathbf{1}_{\{\theta=1\}} \int_0^t f(0)(D_{\alpha,\gamma}\rho_s)(0) + f(1)(D_{\beta,\delta}\rho_s)(1)\,ds\right| > \delta\right] = 0\,,$$

where $\rho_t(\cdot)$ *is the unique weak solution of* $(HDE)_\theta$.
In other words, writing $j_t^N = J_t^N + K_t^N$, *we have that* j^N *converges weakly to* $j du$, *where* j *is a weak solution to*

$$j = -\nabla \rho.$$

Hydrostatic Limit

- The *Hydrodynamic Limit* states that starting from some measure, the local density of particles is associated (in some sense) to the solution of a Hydrodynamic Equation.
- Unsurprisingly, taking the stationary measure, we are associated to the stationary solution of the Hydrodynamic Equation. This association takes the name of *Hydrostatic Limit*.
- The proofs usually rely on estimating correlations w.r.t. the stationary measure, or derive (in particular cases) the Hydrostatic as a limiting case of the Hydrodynamic.
- Only in 2018, [11], based on [10] the Hydrostatic Limit was shown, *in general*, to be consequence of the Hydrodynamic Limit.
  - The main argument comes from a concentration result for classical solutions of the HDE (which is easily adapted to weak solutions)
  - Uniqueness of stationary solutions (may be relaxed if the measure is concentrated on a particular solution, in the vein of []) and convergence to the stationary solution.

Hypotesis

$$\text{The (finite) sequences } \alpha,\ \gamma,\ \beta \text{ and } \delta \text{ are non-increasing,} \tag{H0}$$

$$\delta_1 \leq \alpha_1 \quad \text{and} \quad \beta_1 \leq \gamma_1, \tag{H1}$$

$$\alpha + \beta \quad \text{and} \quad \gamma + \delta \quad \text{are non increasing.} \tag{H2}$$

Theorem (Hydrostatic Limit)

For $\theta = 1$, *assuming* (H1) *there exists a unique stationary solution* ρ^* *of the HDE, and assuming also* (H0), μ_N^{ss} *is associated with it:*

$$\lim_{N\to\infty} \mu_N^{ss}\Big(\big|\langle \pi^N, G\rangle - \langle G, \rho^*\rangle\big| > \delta\Big) = 0.$$

For $\theta > 1$, *assuming* (H2) *there exists a unique constant* $m^* \in [0,1]$, *such that* μ_N^{ss} *is associated with the constant profile* $\rho^* \equiv m^*$. *More precisely, letting* $\mathfrak{i} = \alpha + \beta$ *and* $\mathfrak{o} = \gamma + \delta$, *we have* $m^* = \lim_{t\to\infty} m(t)$ *where*

$$m(t) = m_0 + \int_0^t (D_{\mathfrak{i},\mathfrak{o}}\, m)(s)\, ds.$$

Set of weak stationary solutions to the $(HDE)_\theta$

$$\mathcal{E}_\theta := \left\{\pi \in \mathcal{M}^+ : \pi(du) = \rho^*(u)du \ , \ F_\theta(\rho^*, G, t) = 0, \ \forall t \in [0, T], \ \forall G \in C^{1,2}([0, T] \times [0, 1])\right\}.$$

Let $\mathcal{P}_N := \mu_N^{ss} \circ (\pi^N)^{-1}$ be the distribution of the stationary empirical measure.

Proposition (Analogous to [10])

$\{\mathcal{P}_N\}_{N\in\mathbb{N}}$ *is concentrated in* $\mathcal{E}$, *i.e.*, $\forall \delta > 0$,

$$\lim_{N\to\infty} \mathcal{P}_N \left(\pi \in \mathcal{M}^+ : \inf_{\tilde{\pi}\in\mathcal{E}} d(\pi, \tilde{\pi}) \geq \delta \right) = 0.$$

To prove this, one needs two ingredients:

1. The empirical measure macroscopically governed by a HDE;
2. The existence of a "unique" solution of the HDE and its convergence, w.r.t. the $\mathbb{L}^2$ norm, as time goes to infinity, to a stationary solution.

$\theta = 1$

Proof follows directly from the previous concentration result. The main difficulty is showing uniqueness of/convergence to stationary solutions.

Stationary solution

Under assumptions (H0),(H1), $\mathcal{E} = \{\rho^*(u)du\}$ where

$$\rho^*(u) = (1-u)\rho^*(0) + u\rho^*(1)$$

with its value at the boundary determined by the unique solution of the nonlinear system of equations $\rho^*(1) - \rho^*(0) = -D_{\alpha,\gamma}\rho^*(0) = D_{\beta,\delta}\rho^*(1)$.

Remark (on uniqueness)

Although $D_{\alpha,\gamma}, D_{\beta,\delta}$ induces a $K-$degree polynomial, we are able to guarantee uniqueness on $[0,1]$ thanks to:

- D is Lipschitz monotone decreasing: $D_{\lambda,\sigma}f_1 - D_{\lambda,\sigma}f_2 = -(f_1 - f_2)V_{\lambda,\sigma}(f_1, f_2)$, with $0 < V_{\lambda,\sigma}(f_1, f_2) < \infty$;
- $-\sigma_1 = D_{\lambda,\sigma}\mathbf{1} \leq D_{\lambda,\sigma}f \leq D_{\lambda,\sigma}\mathbf{0} = \lambda_1$;
- Intermediate Value Theorem.

with no need to resort to some maximum principle.

Convergence to steady state

Motivation

Proceeding with an energy estimate approach, we want to take $w := \rho - \rho^*$ as a test function to obtain that $F(\rho_t, w, T) - F(\rho^*, w, T) = 0$, which implies

$$0 \geq \frac{1}{2}\frac{d}{dt}\|w_t\|_{L^2}^2 + C\|w_t\|_{L^2}^2 + w_t^2(0)\,V_{\alpha,\gamma}(\rho_t, \rho^*)(0,0) + w_t^2(1)\,V_{\beta,\delta}(\rho_t, \rho^*)(1,1),$$

then use that $V > 0$ and conclude with Gronwall inequality that

$$\|w_t\|_{L^2} = O(e^{-2Ct}), \qquad C > 0.$$

Main issue

We cannot take w as test function, since we do not know if ρ has weak time-derivatives. To solve this issue, we show that ρ is weakly continuous w.r.t. time.

For that, we relate the weak and mild formulations.

Definition (Mild solution, [6])

We call mild solution of the $(HDE)_1$ any function $\rho : [0, T] \times [0,1] \to [0,1]$ satisfying $M(\rho, t) := \rho_t - S\rho_t = 0$, with

$$S\rho_t(u) = \int_0^1 P_t(u,v) f_0(v)\, dv + \int_0^t \Big\{ P_{t-s}(u,0)(D_{\alpha,\gamma}\rho_s)(0) + P_{t-s}(u,1)(D_{\beta,\delta}\rho_s)(1) \Big\} ds,$$

where P_t is the density kernel generated by the Laplacian ∂_u^2 on $[0,1]$ with reflecting Neumann boundary conditions.

The following result is insipired by [10]

Proposition

If $\rho : [0, T] \to [0,1]$ is a weak solution , then ρ also satisfies $M(\rho, t) = 0$ a.e. $\forall t > 0$. Moreover, if $\rho : [0, T] \times [0,1]$ satisfies $\langle M(\rho, t), G\rangle = 0$ for any $G \in C^{1,2}([0, T] \times [0,1])$, then $F(S\rho, G, t) = 0$.

Approach

$$F(\rho_t, G, t) = 0 \implies \langle \rho_t - S\rho_t, G\rangle = 0 \implies F(S\rho, S\rho, t) = 0$$
$$\implies F(S\rho, S\rho, t) - F(\rho^*, S\rho, t) = 0 \implies \|S\rho_t - \rho^*\|_{L^2} = O(e^{-2Ct}).$$

$\theta > 1$

Strategy

For the Neumann case we do not have uniqueness of stationary solutions, since Neumann Laplacian has no global atractor, hence any constant is solution. What we do instead is follow a similar approach to [11].

- Show L.L.N. for the total mass under $N^{1+\theta}$ time-scale;
- Show that the mass converges to a constant as $t \to \infty$;
- Show that the stationary measure is concentrated on this particular constant, under the N^2 time-scale.

To relate the configurations under different time-scalles we take advantage of the measure being stationary.

Definition (Mass of the system)

$$m_t^N = \frac{1}{N-1} \sum_{x \in \Lambda_N} \eta_{tN^{1+\theta}}^N(x),$$

We study m_t^N by following the same *Dynkin+Tightness+Characterization* approach.

Let $\mathcal{Q}_N := \mu_N^{ss} \circ (m^N)^{-1}$ be the distribution of the trajectories on the Skorokhod space started from the stationary distribution μ_N^{ss}.

1. Dynkin: $m_t^N = m_0^N + M_t^N + \int_0^t \{\sum_{x \in I_-} (D_{\alpha,\gamma}^{N,-}\eta_s)(x) + \sum_{x \in I_+} (D_{\beta,\delta}^{N,+}\eta_s)(x)\} ds$;
2. Tightness: $\mathcal{Q}^* := \lim_{N\to\infty} \mathcal{Q}_N$ exists;
3. Characterization of the Limit points:
   - Adapt Replacement Lemmas:
   $$\limsup_{N\to+\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t \eta_{sN^{1+\theta}}(z)(\eta_{sN^{1+\theta}}(x) - m_s^N)\, ds\right|\right] = 0.$$
   - Concentration of the trajectories: $\mathcal{Q}^*\left(m(\cdot) : m(t) = m_0 + \int_0^t (D_{\alpha+\beta,\gamma+\delta} m)(s)\, ds\right) = 1$;
   - Uniqueness of solutions and convergence to $m^* := m(\infty)$:
     - Both consequence of D being Lipschitz and monotone decreasing.
   - Relationship with Hydrodynamic Limit:
     - Consequence of

$$\lim_{N\to\infty} \mathcal{P}_N\left(\pi \in \mathcal{M}^+ : \inf_{\tilde{\pi}\in\mathcal{E}} d(\pi,\tilde{\pi}) \geq \delta\right) = 0, \quad \lim_{N\to\infty} \mathcal{P}_N\left(\pi \in \mathcal{M}^+ : \inf_{\tilde{\pi}\in\mathcal{E}} d(\pi,\tilde{\pi}) \geq \delta\right) = 0.$$

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