Pattern Formation through Spatial Segregation

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joint works with M. Conti, D. De Silva, B. Noris, N. Soave, H. Tavares, G. Verzini and A. Zilio

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Table of contents

Complex patterns

My researches focus on nontrivial solutions of systems of differential equations characterized by strongly nonlinear interactions.

In these cases, the configuration space is typically multi-dimensional or even infinite-dimensional, and we are interested in the effect of the nonlinearities on the emergence of non trivial self-organized structures. Such patterns correspond to selected solutions of the differential system possessing special symmetries or shadowing particular shapes.

- We want to understand, from the mathematical point of view, what are the main mechanisms involved in the aggregation process in terms of the global structure of the problem.

Therefore we will consider cases where

(a) the interaction becomes the prevailing mechanism,
(b) the equations are very far from being solved explicitly,
(c) the problems can not be seen in any extent as perturbations of simplersystems.

Following this common thread, we deal with a number of different type of strong interactions.

Repulsive interactions

As in competition-diffusion systems, where pattern formation is driven by strongly repulsive forces. Our ultimate goal is to capture the geometry and analysis of the phase segregation, including its asymptotic aspects and the classification the solutions of the related PDE's. We deal with elliptic, parabolic and hyperbolic systems of differential equations with strongly competing interaction terms, modeling both the dynamics of competing populations (Lotka-Volterra systems) and other relevant physical phenomena, among which the phase segregation of solitary waves of Gross-Pitaevskiĭ systems arising in the study of multicomponent Bose-Einstein condensates.

We approach all these different problems with the same basic methodology which relies on the following steps

- asymptotic analysis
- analysis of special self-similar simple solutions
- interface analysis
- gluing techniques to build complex solutions.

Basic methodology

Asymptotics

Asymptotic analysis. The study of the effect of singularities (or singular limits) on the profiles of the solution shows striking similarities between classical and quantum systems and free boundary problems, and it draws, in the essential points, the most crucial elements of the classical theory of minimal surfaces. The monotonicity formulæ, adjusted for the different cases, the blow-up analysis, the classification of the limiting (conic) solutions equivariant by dialation, along with the appropriate tools of dimensional reduction, underpin the asymptotic analysis of solutions.

Basic methodology

Special solutions

Entire solutions. Equilibrium configurations, of course, play a fundamental role. Other simple, yet nontrivial, patterns also appear naturally as symmetric extremals of the associated energies. Symmetries are the key tool for this exploration. On the other hand, entire solutions also carry transitions from one configuration to another. Entire solutions also heavily enter in the blow-up analysis, as they represent the limiting profiles in some scaling process.

Basic methodology
Building complex solutions

Interface analysis. Asymptotic limiting profiles arise as blow-ups at different scales. They may show sharp transitions of the gradients, obeying different refraction of reflection rules. Here we shall take advantage of tools from free boundary theory in order to describe the geometric features of the interface.

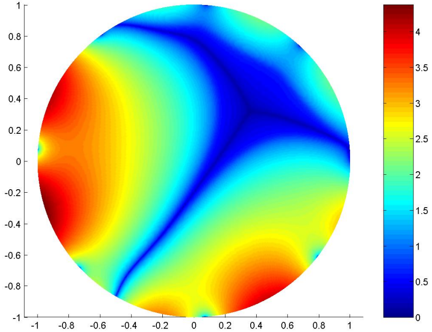
Gluing techniques. Having gathered different types of elementary solutions, the next step consists of gluing them to build more complex patterns. Gluing can be done, once more, using global variational techniques, or other methods.

Table of contents

Competition diffusion systems with Lotka-Volterra interactions: symmetric competition rates

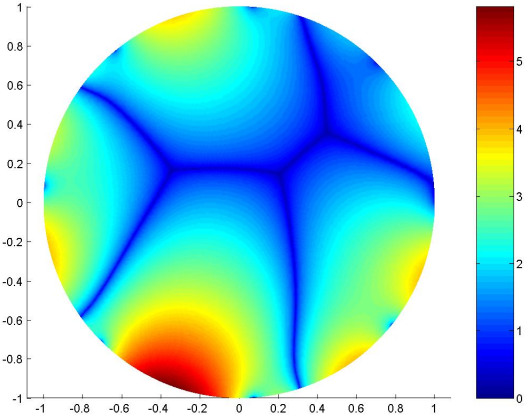
With large and symmetric interspecific competition rates $\beta_{ij} = \beta_{ji}$ and three populations:

$$\frac{\partial u_i}{\partial t} - \mathrm{div}(d_i \nabla u_i) = f_i(u_i) - u_i \sum_{\substack{j=1 \\ j\neq i}}^{k} \beta_{ij} u_j \text{ in } \Omega,$$



Competition diffusion systems with Lotka-Volterra interactions : asymmetric competition rates

With large and symmetric interspecific competition rates $\beta_{ij} = \beta_{ji}$ and five populations:



Competition diffusion systems with Lotka-Volterra interactions: symmetric competition rates

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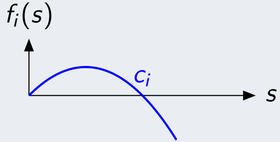
$$\overbrace{\frac{\partial u_i}{\partial t}}^{\text{evolution}} - \overbrace{\mathrm{div}(d_i \nabla u_i)}^{\text{diffusion}} = \overbrace{f_i(u_i)}^{\text{reaction}} - \overbrace{u_i \sum_{\substack{j=1 \\ j \neq i}}^{k} \beta_{ij} u_j}^{\text{competition}} \text{ in } \Omega,$$

u_i is the density of the ith population,
$d_i > 0$ diffusion rates,
$\beta_{i,j}$ interspecific competition rates,
$f_i(s) = u(c_i - u)$ internal forces (logistic)



Energy minimizing configurations of Bose–Einstein condensates in multiple spin–states with repulsive interaction potentials

$$\mathcal{E}(\psi_1, \cdots, \psi_k) = \int_\Omega \overbrace{\sum_i^h \frac{1}{2}|\nabla \psi_i|^2 + F_i(|\psi_i|^2)}^{\text{internal energy}} + \overbrace{\sum_{j\neq i}^k \beta_{ij}|\psi_i|^2|\psi_j|^2}^{\text{interaction energy}}$$

$$\int_\Omega |\psi|_i^2 = m_i \ , \qquad i = 1, \ldots, k$$

- Defocusing: S.M. Chang, C.S. Lin, T.C. Lin, and W.W. Lin, *Phys. D* **196**, 341–361 (2004)
- Focusing: Conti M., Terracini S., Verzini G., *J. Functional Analysis*, 198 (2003) 160-196

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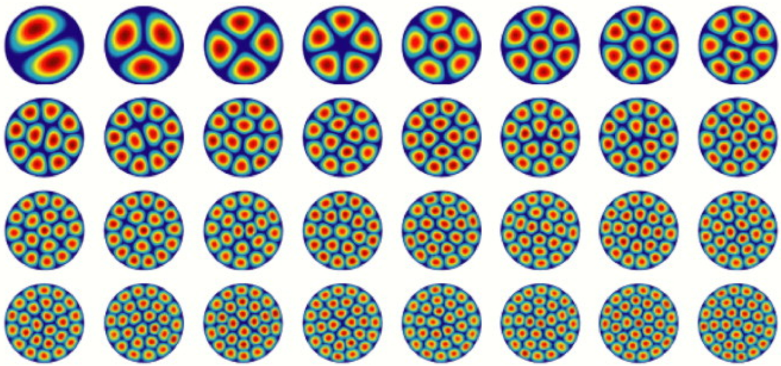
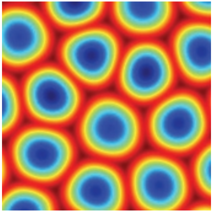


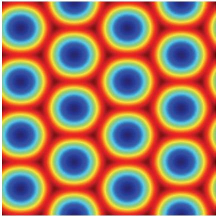
Optimal partition problems for Dirichlet eigenvalues

$$\min\left\{\sum_{i=1}^{h}\lambda_1^p(\omega_i)\ :\ (\omega_1,\cdots,\omega_k)\in\mathfrak{B}_h(\Omega))\right\}$$

where we consider the set of open h-partitions:

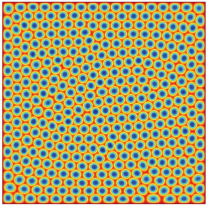
$$\mathfrak{B}_h=\{(\omega_1,\ldots,\omega_h):\ \omega_i \text{ open},\ |\omega_i\cap\omega_j|=0 \text{ for } i\neq j \text{ and } \cup_i\omega_i\subseteq\Omega\}.$$





B. Bourdin, D. Bucur, and . Oudet, Optimal Partitions for Eigenvalues, *SIAM J. Sci. Comput.* 31, 2009/10 pp. 4100-4114

With more and more nodal components:



With higher eigenvalues:

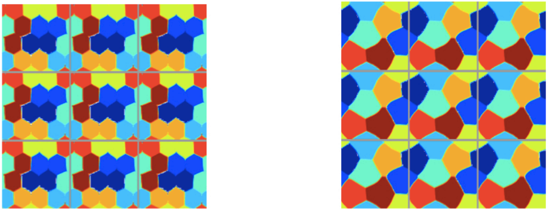


FIG. 3.7. *Optimal partitions of the sum of the second (left) and third (right) eigenvalues of the Dirichlet Laplacian for* $n = 8$ *cells. The periodicity is highlighted by repeating the unit cell 9 times on a two dimensional lattice.*

Diffusion vs strong competition

For the the sake of simplicity, we consider only stationary cases, with all equal diffusions, namely we deal with the semilinear ellipitc system:

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j \neq i}^{k} a_{ij} u_j \qquad \text{in } \Omega, \quad +\text{B.C.}, \quad i = 1, \dots, k, \ \text{(P)}$$

subject to diffusion, reaction and competitive interaction $(a_{ij}, \beta > 0)$.

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- What is the regularity of the common nodal set?

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Table of contents

Gause's law

Two species competing for the same limiting resource cannot coexist at constant population values. When one species has even the slightest advantage over another, the one with the advantage will dominate in the long term. This leads either to the extinction of the weaker competitor or to an evolutionary or behavioral shift toward a different ecological niche.

Gause, Georgii Frantsevich (1934). The Struggle For Existence (1st ed.). Baltimore: Williams & Wilkins. Archived from the original on 2016-11-28

If similar competing species cannot coexist, then how do we explain the great patterns of diversity that we observe in nature? If species living together cannot occupy the same niche indefinitely, then how do competitors coexist?

Mimura'sresult

M. Mimura, Asymptotic behaviors of a parabolic system related to a planktonic prey and predator model, SIAM J. Appl. Math. 37(3) (1979) 499-512

Mimura considered predator-prey system with no flux boundary condition in a bounded set:

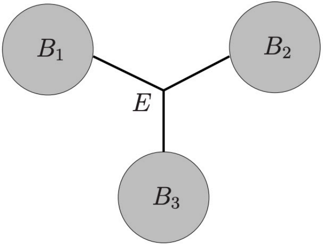
$$\begin{cases} u_t = d_1 \Delta u + f(u)u - uv \\ v_t = d_2 \Delta v + g(u)u + uv \end{cases}$$

Showing that if $f'(u) \leq 0$ and $g'(v) \geq 0$ for $u \geq 0$, $v \geq 0$, and if there is a positive, spatially constant, steady state then every uniformly bounded, nonnegative solution becomes spatially homogeneous as $t \to +\infty$.

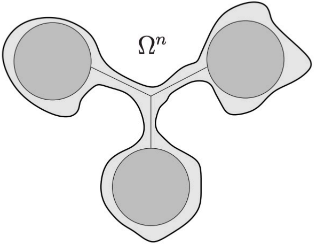
Berestycki and A. Zilio, Predators-prey models with competition I-IV

Ecological niches

Coexistence needs complex geometries, that allow the presence of niches, or strongly inhomogenous environments (modeled by strongly varying diffusion functions d_i).



(a) the set $\Omega^0 = B_1 \cup B_2 \cup B_3$ and segments E joining the balls



(b) sets Ω obtained by small perturbation of Ω^0.

Felli, V. and Conti, M. (2008). Coexistence and segregation for strongly competing species in special domains. INTERFACES AND FREE BOUNDARIES, 10(2), 173-195.

Table of contents

Segregation phenomena

We say that a family of solutions $\{\mathbf{u}_\beta\}_\beta$ segregates if

$$u_{i,\beta} \to u_i, \qquad u_i \cdot u_j \equiv 0, \qquad \text{a.e. as } \beta \to +\infty$$

for nontrivial limits (with some abuse, we talk about "disjoint supports").

Several questions to be addressed:

- what kind of convergence (in terms of function spaces);
- regularity properties of the limiting profiles;
- geometry of the nodal set $\Gamma = \{x \ : \ u_i(x) = 0, \ \forall i = 1, \ldots, k\}$.

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For standard diffusions, this has been studied in a number of papers by different teams:

- M. Conti, B. Noris, H. Tavares, S. Terracini, G. Verzini, N. Soave, A. Zilio
- J. Wei, T. Weth
- L.A. Caffarelli, A. Karakhanyan, F. Lin, JM. Roquejoffre, V. Quitalo, S. Patrizi
- E.N. Dancer, Y. Du, K. Wang, Z. Zhang
- A.R. Domingos, B. Noris, M. Ramos

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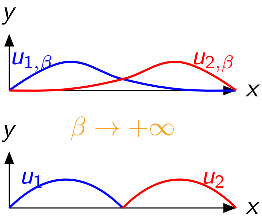
Also the one-dimensional problem is significant:

$$\begin{cases} -u_1'' = f(u_1) - \beta u_1 u_2 & \text{in } (a,b) \\ -u_2'' = f(u_2) - \beta\gamma u_1 u_2 & \text{in } (a,b) \\ u_i(a) = u_i(b) = 0 \end{cases}$$



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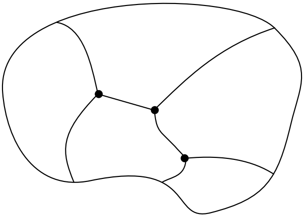
Even though, only in dimension $N \geq 2$ "true" free boundaries arise.



We consider different models:

Symmetric quadratic interactions (Lotka-Volterra)

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j\neq i} a_{ij} u_j, \qquad i = 1, \dots, k,$$

with

$$a_{ij} = a_{ji}$$

We consider different models:

Symmetric cubic interactions (Groß-Pitaevskii energies)

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j \neq i} a_{ij} u_j^2, \qquad i = 1, \dots, k,$$

Variational structure iff

$$a_{ij} = a_{ji}$$

We consider different models:

Asymmetric quadratic interactions (Lotka-Volterra)

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j \neq i} a_{ij} u_j, \qquad i = 1, \dots, k,$$

with

$$a_{ij} \neq a_{ji}$$

We consider different models:

Anomalous diffusions $s \in (0,1)$ (all range of exponents)

$$-(\Delta)^s u_i = f_i(x, u_i) - \beta u_i^p \sum_{j \neq i} a_{ij} u_j^q, \qquad i = 1, \dots, k,$$

Variational structure iff

$$a_{ij} = a_{ji} \qquad p = q - 1$$

We consider different models:

Interaction at a distance (different range of exponents)

$$-\Delta u_i = f_i(x, u_i) - \beta u_i^p \sum_{j \neq i} a_{ij}(\mathbb{1}_{B_1} \star u_j^q), \qquad i = 1, \dots, k,$$

Variational structure iff

$$a_{ij} = a_{ji} \qquad p = q - 1$$

Basic questions:

1. does the particular expression of the interaction matter? (quadratic vs cubic interactions)
2. is there an underlying variational principle for the limiting profiles? (symmetric vs asymmetric interactions)
3. do the diffusion rules matter? (standard $s = 1$ vs anomalous diffusion $0 < s < 1$)
4. what is the role of distance? (pointwise interaction vs interaction at a distance)

Table of contents

Energy minimizing segregated configurations

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ and let us call segregated state a k–uple $U = (u_1, \ldots, u_k) \in (H^1(\Omega))^k$ where

$$u_i(x) \cdot u_j(x) = 0 \qquad i \neq j, \text{ a.e. } x \in \Omega$$

We define the internal energy of U as

$$J_\infty(U) = \sum_{i=1,\cdots,k} \left\{ \int_\Omega \frac{1}{2} \underbrace{d_i^2(x)}_{d_i \equiv 1 \;, i=1,\ldots,k} |\nabla u_i(x)|^2 - \underbrace{F_i(x, u_i(x))}_{F_i \equiv 0 \;, i=1,\ldots,k} dx \right\},$$

Our goal is to minimize J_∞ among a class of segregated states subject to some boundary and positivity conditions.

A weak reflection law

Theorem (M. Conti, S. T. Terracini, G. Verzini 2005, L. Caffarelli, F. Lin 2008, Tavares, S.T. 2012)

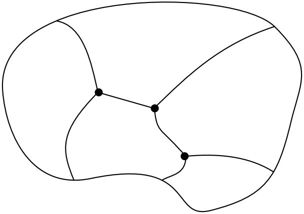
For reasonable F's, the minimization problem has a (unique) segregated stationary configuration U, which is Lipschitz. Let Γ_U its nodal set, Then, there exists a set $\Sigma_U \subseteq \Gamma_U$ the regular part, relatively open in Γ_U, such that

- $\mathcal{H}_{dim}(\Gamma_U \setminus \Sigma_U) \leq N-2$, *and if $N=2$ then actually $\Gamma_U \setminus \Sigma_U$ is a locally finite set;*
- Σ_U *is a collection of hyper-surfaces of class $C^{1,\alpha}$ (for every $0<\alpha<1$).*

Furthermore for every $x_0 \in \Sigma_U$

$$\lim_{x \to x_0^+} |\nabla U(x)| = \lim_{x \to x_0^-} |\nabla U(x)| \neq 0,$$

where the limits as $x \to x_0^{\pm}$ are taken from the opposite sides of the hyper-surface. Furthermore, if $N = 2$ then Σ_U consists in a locally finite collection of curves meeting with equal angles at singular points.



Theorem (Donnelly and Fefferman 1988, Lin 1991, Han, Hardt and Lin 1998, Garofalo and Lin 1986)

Let $u : \Omega \to \mathbb{R}$ be a solution of a stationary Schrödinger equation of type

$$-\Delta u + V(x)u = 0$$

for some L^∞ potential V, let $\Gamma_u = u^{-1}(0)$ and $\Sigma_u \subseteq \Gamma_u$ the set of regular points (i.e. $\nabla u(x) \neq 0$). Then

- $\mathcal{H}_{dim}(\Gamma_u \setminus \Sigma_u) \leq N - 2$, *and if $N = 2$ then actually $\Gamma_u \setminus \Sigma_u$ is a locally finite set;*
- Σ_u *is a collection of hyper-surfaces of class $C^{1,\alpha}$ (for every $0 < \alpha < 1$).*

A penalized functional

Let $\beta > 0$. We consider the minimization of the functional

$$J_\beta(U) = \sum_{i=1}^{k} \int_\Omega \left(\frac{1}{2} d_i^2(x) \, |\nabla u_i(x)|^2 - F_i(x, u_i(x)) \right) dx$$
$$+\beta \sum_{1 \le i < j \le k} \int_\Omega u_i^2(x) u_j^2(y) \, dx \, dy$$

in the set H of all configurations with fixed boundary data, and we take the singular limit:

$$\beta \to +\infty \, .$$

With respect to the search of a minimizer for $\inf_{H_\infty} J_\infty$, the advantage stays in the fact that we can get rid of the infinite dimensional constraint $u_i u_j = 0$ for $i \neq j$, and we can easily show that a minimizer for J_β in H does exists, and satisfies an Euler-Lagrange equation.

Theorem (B. Noris, H. Tavares, S.T. and G. Verzini, CPAM 2010)

For every $\beta > 0$, there exists at least one minimizer $U_\beta = (u_{1,\beta}, \dots, u_{k,\beta})$ for $\inf_H J_\beta$, which is a solution of

$$\begin{cases} -\Delta u_i = -\beta u_i \sum_{j\neq i} u_j^2 & \text{in } \Omega \\ u_i > 0 & \text{in } \Omega \\ u_i = f_i & \text{in } \partial\Omega. \end{cases} \tag{1}$$

The family $\{U_\beta : \beta > 0\}$ is uniformly bounded in $H^1(\Omega, \mathbb{R}^k) \cap \mathcal{C}^{0,\alpha}$, and there exists $U = (u_1, \dots, u_k) \in H$ such that:

1. *$U_\beta \to U$ strongly in $\mathcal{C}^{0,\alpha} \cap H^1(\Omega)$ as $\beta \to +\infty$, up to a subsequence;*
2. *$u_i(x)u_j(x) \to 0$ for every $i \neq j$, so that $U \in H_\infty$;*
3. *for every $i \neq j$,*

$$\lim_{\beta \to +\infty} \beta \iint_\Omega u_{i,\beta}^2(x) u_{j,\beta}^2(y)\, dx\, dy = 0$$

4. *U is a minimizer for $\inf_{H_\infty} J_\infty$.*

Conclusions

All the properties and those of the nodal set extend to limits, as $\beta \to +\infty$, of sequences of critical points of the penalized functional, satisfying a uniform-in-β H^1 bound.

A remarkable aspect of competition-diffusion systems with standard laplacians is that the basic rules of spatial segregation do not depend on the particular form of the interaction.

Table of contents

Competition in the Lotka-Volterra model

We consider the semilinear system:

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j \neq i} a_{ij} u_j \qquad \text{in } \Omega, \qquad i = 1, \ldots, k, \qquad \text{(LV)}$$

where $u_i \geq 0$, $\beta > 0$, $a_{ij} > 0$ (+ boundary conditions).

(LV) is the stationary version of the
competition-diffusion system with Lotka-Volterra interactions:

$$\partial_t u - \Delta u_i = f_i(u_i) - \beta u_i \sum_{j \neq i} a_{ij} u_j.$$

(LV) is never variational. It can be
either symmetric ($a_{ij} = a_{ji}$) or asymmetric ($a_{ij} \neq a_{ji}$).

Table of contents

The symmetric case for $k \geq 3$ populations

We assume $a_{ij} = a_{ji} (= 1$ w.l.o.g.). The system becomes

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j \neq i} u_j \qquad \text{in } \Omega, \qquad i = 1, \dots, k,$$

Theorem (Conti, Terracini, Verzini '05)

Let U_β be a family of H^1–bounded solutions. For every $\alpha < 1$ there exists $L_\alpha > 0$ such that

$$\sup_{x,y \in \Omega} \frac{|u_{i,\beta}(x) - u_{i,\beta}(y)|}{|x - y|^\alpha} < L_\alpha$$

for all $i = 1, \dots, k$ and for all $\beta > 0$.

This allows to pass to the limit as $\beta \to +\infty$.

Optimal uniform Lipschitz bounds have been obtained
[Soave-Zilio, *ARMA* 2015]

Segregation limit in the symmetric case

Theorem (Conti, Terracini, Verzini '05)

Let $U_\beta = (u_{1,\beta}, ..., u_{k,\beta})$ be a solution of the system at fixed β, and $\beta \to \infty$. There exists U such that, for all $i = 1, \dots, k$:

1. *up to subsequences, $u_{i,\beta} \to u_i$ strongly in H^1 and in $\mathcal{C}^\alpha$, for any $\alpha \in (0,1)$*
2. *if $i \neq j$ then $u_i \cdot u_j = 0$ a.e. in Ω*
3. $-\Delta u_i \leq f(x, u_i)$
4. $$-\Delta \left(u_i - \sum_{j \neq i} u_j \right) \geq f(x, u_i) - \sum_{j \neq i} f(x, u_j)$$
5. *the segregated limiting profiles are Lipschitz.*

This agrees with the case $k = 2$, which reads

$$-\Delta(u_1 - u_2) \geq 0, \qquad -\Delta(u_2 - u_1) \geq 0.$$

The class $\mathcal{S}$

Define

$$\widehat{u}_i = u_i - \sum_{j \neq i} u_j$$

and similarly

$$\widehat{f}(x, \widehat{u}_i) = \begin{cases} f_i(x, u_i) & \text{if } x \in \operatorname{supp}(u_i) \\ -f_j(x, u_j) & \text{if } x \in \operatorname{supp}(u_j),\ j \neq i. \end{cases}$$

Then the segregation limits belong to the class

$$\mathcal{S} = \left\{ (u_1, \cdots, u_k) : \begin{array}{l} u_i \geq 0,\ u_i \cdot u_j = 0 \text{ if } i \neq j \\ -\Delta u_i \leq f(x, u_i) \\ -\Delta \widehat{u}_i \geq \widehat{f}(x, \widehat{u}_i),\ \forall i \end{array} \right\}$$

(+ boundary conditions)

Basic properties in $\mathcal{S}$

The multiplicity of a point $x \in \Omega$ is

$$m(x) = \sharp \{ i : meas\left(\{u_i > 0\} \cap B(x,r) \right) > 0 \ \forall \, r > 0 \} \ .$$

Proposition

Let $x_0 \in \Omega$:

(a) If $m(x_0) = 0$, then there is $r > 0$ such that $u_i \equiv 0$ on $B(x,r)$, for every i.

(b) If $m(x_0) = 1$, then there are i and $r > 0$ such that $u_i > 0$ and

$$-\Delta u_i = f_i(x, u_i) \quad \text{on } B(x,r).$$

(c) If $m(x_0) = 2$, then are i, j and $r > 0$ such that $u_k \equiv 0$ for $k \neq i, j$ and

$$-\Delta(u_i - u_j) = g_{ij}(x, u_i - u_j) \quad \text{on } B(x,r),$$

where $g_{i,j}(x,s) = f_i(x, s^+) - f_j(x, s^-)$.

Multiple junctions of nodal lines

We wish to analyze the structure of the zero set of a k-tuple $U \in \mathcal{S}$, i.e. the set

$$\mathcal{Z} = \{x : u_i(x) = 0 \text{ for every } i = 1, \dots, k\}.$$

Such set naturally splits into the union of the regular part $\mathcal{Z}_2 = \{x \in \mathcal{Z} : m(x) = 2\}$, which is itself the union of the interfaces

$$\Gamma_{ij} = \partial\omega_i \cap \partial\omega_j \cap \mathcal{Z}_2,$$

and of the singular part

$$\mathcal{W} = \mathcal{Z} \setminus \mathcal{Z}_2.$$

With this respect, the sets ω_i and ω_j are said to be adjacent whenever $\Gamma_{ij} \neq \emptyset$.

Structure of the nodal set

Theorem (Conti-Terracini-Verzini '05, Caffarelli-Karakanyan-Lin '08, Tavares-Terracini '12)

Let U be in the class $\mathcal{S}$, and let $\mathcal{Z} = \{x \in \Omega \ : \ U(x) = 0\}$. Then, there exists a set $\mathcal{Z}_2 \subseteq \mathcal{Z} =$ the regular part, relatively open in $\mathcal{Z}$, such that

- *$\mathcal{Z}_2$ is a collection of hyper-surfaces of class $C^{1,\alpha}$ (for every $0 < \alpha < 1$). Furthermore for every $x_0 \in \mathcal{Z}_2$*

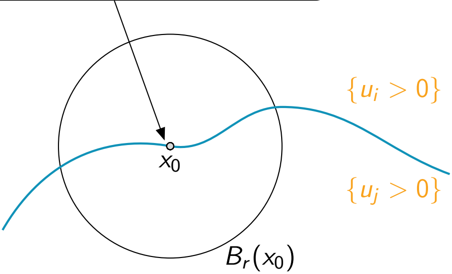
$$\lim_{x \to x_0^+} |\nabla U(x)| = \lim_{x \to x_0^-} |\nabla U(x)| \neq 0,$$

  where the limits as $x \to x_0^{\pm}$ are taken from the opposite sides of the hyper-surface;
- *$\mathcal{H}_{dim}(\mathcal{Z} \setminus \mathcal{Z}_2) \leq N - 2$, and $\lim_{x \to x_0} |\nabla U(x)| = 0$.*

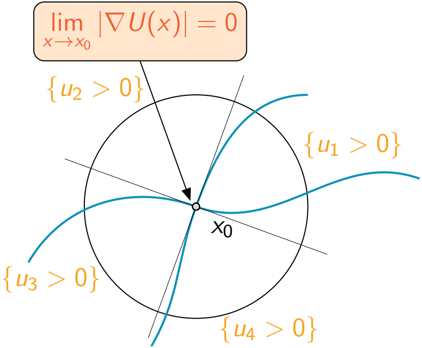
Furthermore, if $N = 2$ then $\mathcal{Z}$ consists in a locally finite collection of curves meeting with equal angles at a locally finite number of singular points.

Nodal set: regular points

$$\lim_{\substack{x\to x_0\\ x\in\{u_i>0\}}} \nabla u_i(x) = -\lim_{\substack{x\to x_0\\ x\in\{u_j>0\}}} \nabla u_j(x)$$



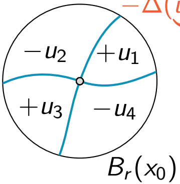
Nodal set: singular points ($N = 2$)



Asymptotic expansion near multiple points

An heuristic argument without reactions:

$$-\Delta(\underbrace{u_1 - u_2 + u_3 - u_4}_{w}) = f_1 - f_2 + f_3 - f_4$$

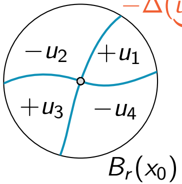


Then $w(r,\vartheta) = \sum_{k\in\mathbb{Z}} [a_k \cos(k\vartheta) + b_k \sin(k\vartheta)]\, r^k$

Asymptotic expansion near multiple points

An heuristic argument without reactions:

$$-\Delta\underbrace{(u_1 - u_2 + u_3 - u_4)}_{w} = f_1 - f_2 + f_3 - f_4$$



Then $w(r,\vartheta) = \sum_{k\in\mathbb{Z}} [a_k \cos(k\vartheta) + b_k \sin(k\vartheta)]\, r^k$ and

- $a_k^2 + b_k^2 = 0$ for $k < 0$ as w is not singular in x_0,
- $a_k^2 + b_k^2 = 0$ for $k = 0, 1$ as $m(x_0) = 4$,

$$w(r,\vartheta) = r^2 \cos(2\vartheta + \vartheta_0) + o(r^2) \text{ as } r \to 0.$$

In general, $w \sim r^{m(x_0)/2}$, also in the odd case.

Table of contents

The limiting profiles

Back to the original problem

$$-\Delta u_i = f_i(x,u_i) - \beta u_i \sum_{j\neq i} a_{ij} u_j \qquad \text{in } \Omega, \qquad i = 1,\ldots,k,$$

assume now $a_{ij} \neq a_{ji}$

- **Passing to the limit as $\beta \to \infty$ we find a new class $\mathcal{S}$:**

Define, for every $i = 1,\ldots,k$,

$$\widehat{u}_i := u_i - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} u_j,$$

and $\widehat{f}_i$ accordingly. The differential inequalities take the usual form

$$-\Delta \widehat{u}_i \geq \widehat{f}_i(x,\widehat{u}_i) \qquad \text{in } \Omega.$$

Asymptotics and nodal set

- What doesn't changes:
  - equi–hölderianity w.r.t. β
  - proportional gradients at points x_0 with $m(x_0) = 2$:

$$\lim_{\substack{x \to x_0 \\ x \in \{u_i > 0\}}} a_{ji} \nabla u_i(x) = - \lim_{\substack{x \to x_0 \\ x \in \{u_j > 0\}}} a_{ij} \nabla u_j(x)$$

  - vanishing of the gradient at points x_0 with $m(x_0) \geq 3$
- What changes:
  - local expansion at multiple points (in dimension $N = 2$).

Asymptotics and nodal set

- What doesn't changes:
  - equi–hölderianity w.r.t. β
  - proportional gradients at points x_0 with $m(x_0) = 2$:

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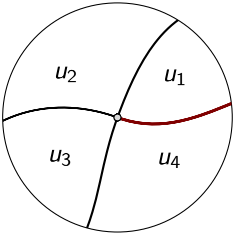
  - vanishing of the gradient at points x_0 with $m(x_0) \geq 3$
- What changes:
  - local expansion at multiple points (in dimension $N = 2$).

Near an isolated point x_0 with (e.g.) $m(x_0) = 4$ we have that

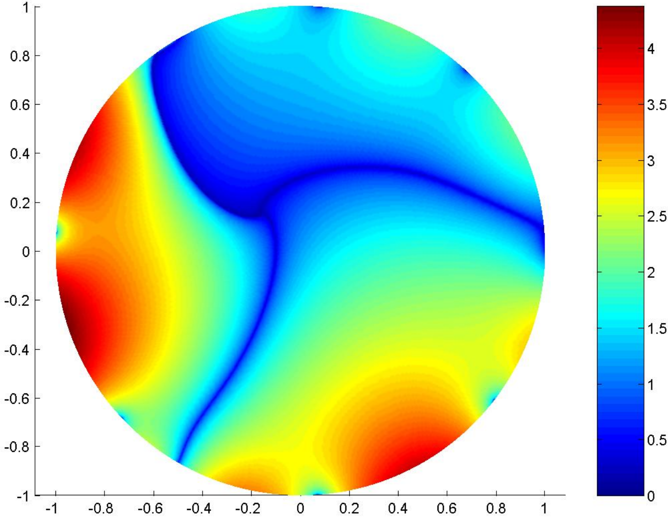
$$w = u_1 - \frac{a_{12}}{a_{21}} u_2 + \frac{a_{12}a_{23}}{a_{21}a_{32}} u_3 - \frac{a_{12}a_{23}a_{34}}{a_{21}a_{32}a_{43}} u_4$$

satisfies

$$-\Delta w = 0 \qquad \text{in } B_{r_0}(x_0) \setminus \underbrace{\left(\overline{\{u_1 > 0\}} \cap \overline{\{u_4 > 0\}} \right)}_{\tilde{\Gamma}}.$$



Asymptotics and nodal set



Conclusion

Let $(u_1, \ldots, u_k)$ be a segregated limiting profile in the asymmetric case.

Theorem (S. T. , G. Verzini, A. Zilio, CPAM 2019)

Let $\mathcal{Z}$ be a compact connected component of $\{x : m(x) \geq 3\}$. Then $\mathcal{Z} = \{x_0\}$.

Theorem (S. T. , G. Verzini, A. Zilio, CPAM 2019)

Let $x_0 \in \Omega$ with $m(x_0) = h \geq 3$. Then there exists $\alpha \in \mathbb{R}$ and ϑ_0 such that

$$w(r,\vartheta) = Cr^{h/2+2\alpha^2/h} \exp(\alpha\theta) \cos\left(\frac{h}{2}\theta - \alpha \log r + \vartheta_0\right) + o(r^{h/2+2\alpha^2/h})$$

as $r \to 0$, where (r,θ) denotes a system of polar coordinates about x_0 and $\tilde{U}$ is a suitably weighted sum of the components u_i.

Remarks

In the asymmetric case, the nodal partition determined by the supports of the components can not be optimal with respect to any Lagrangian energy. Indeed, it is known that boundaries of optimal partitions share the same nodal properties of the energy minimizing configurations. Hence they can not exhibit logarithmic spirals. This fact is in striking contrast with the picture for symmetric inter-specific competition rates: indeed, in such a case, we know that solutions are unique, together with their limit profiles in the class $\mathcal{S}$. Hence, though system does not possess a variational nature, it fulfills a minimization principle in the segregation limit, while this is impossible in the asymmetric setting.

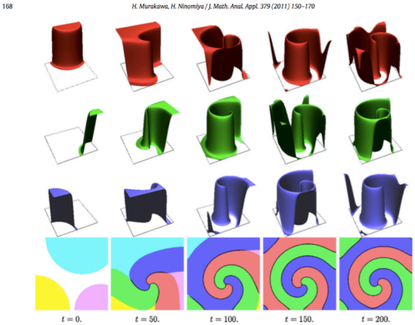
Nevertheless, even in the asymmetric case, functions in the class $\mathcal{S}$ still share with the solutions of variational problems, including harmonic functions, the following fundamental features:

- singular points are isolated and have a finite vanishing order;
- the possible vanishing orders are quantized;
- the regular part is smooth.

The parabolic problem

With asymmetric interspecific competition rates $\beta_{i,j} \neq \beta_{j,i}$ large and three populations:

$$\frac{\partial u_i}{\partial t} - \Delta u_i = f_i(u_i) - u_i \sum_{\substack{j=1 \\ j \neq i}}^{h} \beta_{i,j} u_j \text{ in } \Omega,$$



H. Murakawa and H. Ninomiya, *Fast reaction limit of a three-component reaction-diffusion system*. J. Math. Anal. Appl. 379 (2011), no. 1, 150-170,

The spiralling wave ansatz in two-dimension

$$\frac{\partial u_i}{\partial t} - \Delta u_i = f_i(u_i) - \beta u_i \sum_{\substack{j=1 \\ j\neq i}}^{h} a_{i,j} u_j \text{ in } \mathbb{C},$$

Ansatz:

$$u_i(t,x) = v_i\left(e^{i\omega t}x\right) , \qquad x \in \mathbb{C}$$

The spiralling wave ansatz in two-dimension

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Ansatz:

$$u_i(t,x) = v_i\left(e^{i\omega t}x\right) , \qquad x \in \mathbb{C}$$

Then $(v_1, \ldots, v_i)$ solve

$$\omega x^{\perp} \cdot \nabla v_i - \Delta v_i = f_i(v_i) - \beta v_i \sum_{\substack{j=1 \\ j\neq i}}^{h} a_{i,j} v_j \text{ in } \mathbb{C}.$$

Spiralling limiting profiles:

$$\omega x^{\perp} \cdot \nabla v_i - \Delta v_i = f_i(v_i) - \beta v_i \sum_{\substack{j=1 \\ j\neq i}}^{h} a_{i,j} v_j \text{ in } \mathbb{C}, \qquad (*)$$

Next we pass to the limit as $\beta \to +\infty$.

Theorem (Salort,Terracini,Verzini, Zilio 2019)

For every ω, for a codimension two set of boundary traces, there exists a unique solution in the class $\mathcal{S}$ associated with $(*)$ in the unit disk. Furthermore, there exists $\alpha \in \mathbb{R}$ and θ_0 such that

$$\tilde{V}(r,\theta) = Cr^{h/2} \exp(\alpha\vartheta) |\cos\left(\frac{h}{2}\vartheta - \alpha \log r + \vartheta_0\right)| + o(r^{h/2})$$

as $r \to 0$, where (r,θ) denotes a system of polar coordinates about 0 and $\tilde{U}$ is a suitably weighted sum of the components v_i.

Some numerical simulations (by courtesy of Alessandro Zilio)

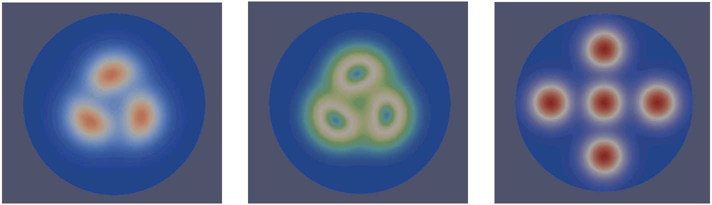
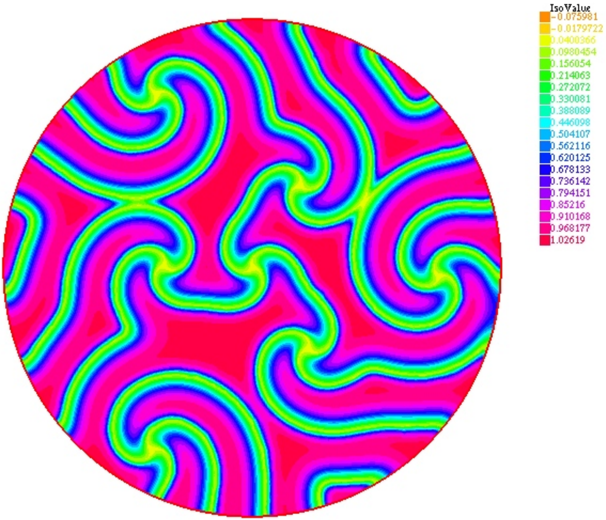


Figure 1 Figure 2 Figure 3

Final remarks

- Wave spirals appear in reaction-diffusion systems related with ventricular fibrillation.
- Work on the dynamics of spiral waves by Björn Sandstede, Arnd Scheel, Claudia Wulff (no singular limit).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0.156054
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0.388089
0.446098
0.504107
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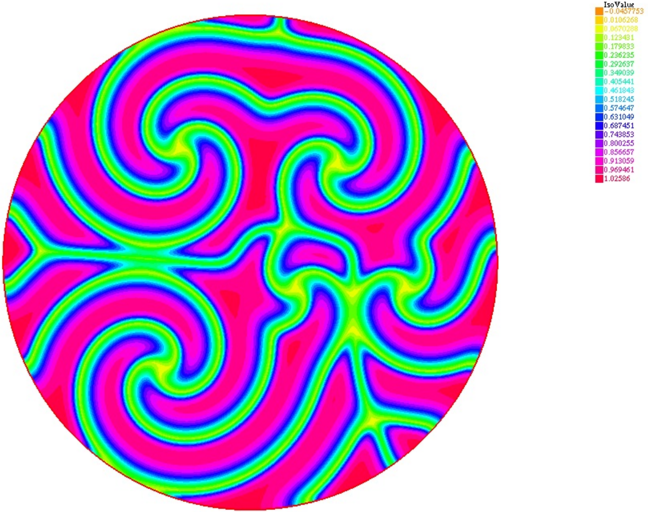
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0.461843
0.518245
0.574647
0.631049
0.687451
0.743853
0.800255
0.856657
0.913059
0.969461
1.02586

Table of contents

Generalizations

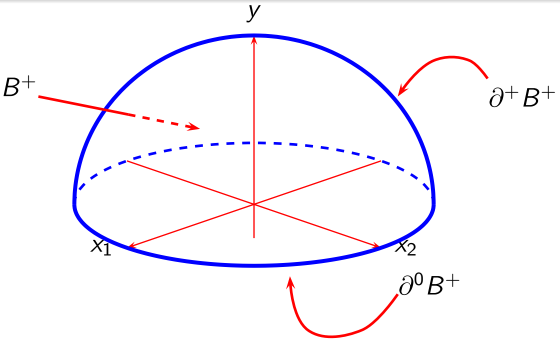
Other classes of interactions, including non variational ones, like Lotka-Volterra, share the main phenomenology. This is not the case for different type of diffusions, as, for examples, anomalous ones associated with fractional laplacians.

Anomalous diffusions arise in models of population dynamics: for instance, while the standard laplacian seems well suited to describe the diffusion of predators in presence of an abundant prey, when the prey is sparse observations suggest that fractional laplacians give a more accurate model. The square root of the laplacian is of interest in relativistic quantum electrodynamics.

The $(P)_\beta$ problem

In view of the local realization of the square root of the laplacian, we state our main results for harmonic functions satisfying a nonlinear Neumann boundary condition. Consider the following problem.

$$(P)_\beta \qquad \begin{cases} -\Delta v_{i,\beta} = 0 & \text{in } B^+ \\ \partial_\nu v_{i,\beta} = f_{i,\beta}(v_{i,\beta}) - \beta v_{i,\beta} \sum_{j\neq i} v_{j,\beta}^2 & \text{on } \partial^0 B^+ \end{cases}$$



Local uniform Hölder bounds

Theorem (S.T., G. Verzini, A. Zilio JEMS 2016)

Let

- $f_{i,\beta}$ *be continuous and uniformly bounded (w.r.t. β) on bounded sets,*
- $\{\mathbf{v}_\beta\}_\beta$ *be solutions to* $(P)_\beta$ *with* $\|\mathbf{v}_\beta\|_{L^\infty(B_1^+)} \leq M$,

with M independent of β. Then for every $\alpha \in (0, 1/2)$ there exists $C = C(M, \alpha)$, not depending on β, such that

$$\|\mathbf{v}_\beta\|_{C^{0,\alpha}(\overline{B^+_{1/2}})} \leq C(M, \alpha).$$

Furthermore, $\{\mathbf{v}_\beta\}_\beta$ is precompact in $H^1 \cap C^{0,\alpha}(\overline{B^+_{1/2}})$, for every $\alpha < 1/2$.

- if $\beta \leq \bar{\beta} < \infty$, regularity theory for non linear Steklov problems ensure that $\mathbf{v}_\beta \in C^\infty(\overline{B_r^+})$ for all $r < 1$ and the regularity in $\overline{B^+}$ is only limited by the regularity of the boundary data on $\partial^+ B^+$.

Optimal Regularity of the Limiting Profile

Theorem (S.T., G. Verzini, A. Zilio JEMS 2016)

Under the above assumptions, assume furthermore that

- $f_{i,\beta}$ *are locally Lipschitz,* $f_i(s) = f_i'(0)s + O(|s|^{1+\varepsilon})$ *as* $s \to 0$,
- $f_{i,\beta} \to f_i$, *uniformly on compact sets.*

Then there exists a limiting profile $\mathbf{v} \in C^{0,1/2}(\overline{B^+_{1/2}})$ *such that, up to subsequences,*

$$\mathbf{v}_\beta \to \mathbf{v} \quad \text{in } H^1 \cap C^{0,\alpha},\ \alpha < 1/2,$$

and

$$\begin{cases} -\Delta v_i = 0 & \text{in } B^+_{1/2} \\ v_i \partial_\nu v_i = f_i(v_i) v_i \quad \text{and} \quad v_i v_j = 0\ \forall i \neq j & \text{on } \partial^0 B^+_{1/2}. \end{cases}$$

- for solutions of the above system, $C^{0,1/2}$ is the optimal regularity, as one can see choosing

$$v_1(x,y) = \Re(x_1 + iy)^{1/2}, \quad v_2(x,y) = \Im(x_1 + iy)^{1/2}.$$

Global uniform Hölder bounds

Theorem (S.T., G. Verzini, A. Zilio JEMS 2016)

Let $\{\mathbf{u}_\beta\}_\beta$ *be a family of* $H^{1/2}(\mathbb{R}^N)$ *solutions to the problems*

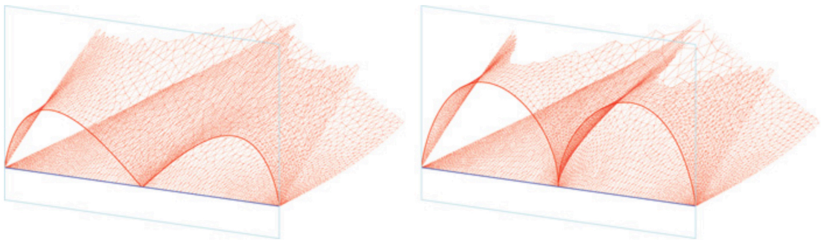
$$\begin{cases} (-\Delta)^{1/2} u_i = f_{i,\beta}(u_i) - \beta u_i \sum_{j\neq i} u_j^2 & \text{on } \Omega \\ u_i \equiv 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω *is a bounded domain of* $\mathbb{R}^N$*, with sufficiently smooth boundary. Let us assume that* $\|\mathbf{u}_\beta\|_{L^\infty(\Omega)} \leq M$ *. Then for every* $\alpha \in (0, 1/2)$

$$\|\mathbf{u}_\beta\|_{C^{0,\alpha}(\mathbb{R}^N)} \leq C(M, \alpha).$$

1. for mixed boundary conditions, the maximal regularity expected at β fixed is already $C^{0,1/2}$. [Shamir, *Israel J. Math.* 1968]
2. analogous results hold for the spectral fractional laplacian with homogeneous Dirichlet boundary conditions.
3. L^∞ bounds can be derived from $H^{1/2}$ ones, by a Brezis-Kato argument.

Limiting profiles when $s = 1/2$: the type of interaction does matter!



On the left, a numerical approximation of a limiting profile for problem (P) with quadratic (Lotka-Volterra) competition $p = q = 1$ and $s = 1/2$, for which Lipschitz continuity of the segregated traces holds. On the right, the simulation for the analogous problem with cubic (Gross-Pitaevskii) competition $p = 1$, $q = 2$, which optimal regularity is only $\mathcal{C}^{0,1/2}$ [Verzini and Zilio, CPDE 2014].

Segregated minimal configurations

We are concerned with

1. The class of energy minimizing configurations segregating only at the characteristic hyperplane $\{y=0\}$, that is solutions to

$$\min\left\{\sum_{i=1}^k \int_{B^+} |\nabla u_i|^2 dz \ : \ \begin{array}{l} u_i(x,0)\cdot u_j(x,0)\equiv 0 \ \partial^0 B^+\text{-a.e. forall } \ i\neq j, \\ u_i=\varphi_i, \text{on } \partial^+ B^+ \text{ for } \quad i=1,\dots,k, \end{array}\right\}$$

   where the φ_i's are nonnegative $H^{1/2}$-boundary data which are segregated on the hyperplane $\{y=0\}$.

2. Our theory applies also to to segregated minimizing configurations involving non local energies, like, for instance the solutions to the following problem (when $s=1/2$):

$$\min\left\{\sum_{i=1}^k \int_{\mathbb{R}^{2n}} \frac{|u_i(x)-u_i(y)|^2}{|x-y|^{n+2s}} \ : \ \begin{array}{l} u_i(x,0)\cdot u_j(x,0)\equiv 0 \text{ a.e. in } \mathbb{R}^n \text{ for } \ i\neq j \\ u_i\equiv\varphi_i, \text{on } \mathbb{R}^n\setminus\tilde{\Omega} \text{ for } \quad i=1,\dots,k, \end{array}\right.$$

   where φ_i are nonnegative $H^{1/2}(\mathbb{R}^n)$ data which are segregated themselves.

Structure of the nodal set

It is immediate to check that the mimizing limiting profiles given by our a priori bounds in Hölder spaces are indeed minimizing critical profiles in the sense above. For such class of solutions, we are going to prove a theorem on the structure of the nodal set $\mathcal{N}(\mathbf{u})$, which is the perfect counterpart of the results by Caffarelli-Lin for the standard laplacian.

Theorem (Structure of the nodal set of minimizing critical configurations, De Silva-T Calc. Var. PDE 2019)

Let $\Omega \subset \mathbb{R}^{n+1}$, with $n \geq 2$ and let $\mathcal{N}(\mathbf{u}) = \{x \in \tilde{\Omega} : \ \mathbf{u}(x,0) = 0\}$. Assume $\mathbf{u}$ is locally a minimizer of the energy within segregated states on $\tilde{\Omega}$. Then, $\mathcal{N}(\mathbf{u})$ is the union of a relatively open regular part $\Sigma_{\mathbf{u}}$ and a relatively closed singular part $\mathcal{N}(\mathbf{u}) \setminus \Sigma_{\mathbf{u}}$ with the following properties:

1. *$\Sigma_{\mathbf{u}}$ is a locally finite collection of hyper-surfaces of class $C^{1,\alpha}$ (for some $0 < \alpha < 1$).*
2. *$\mathcal{H}_{dim}(\mathcal{N}(\mathbf{u}) \setminus \Sigma_{\mathbf{u}}) \leq n-2$ for any $n \geq 2$. Moreover, for $n = 2$, $\mathcal{N}(\mathbf{u}) \setminus \Sigma_{\mathbf{u}}$ is a locally finite set.*

As already pointed out, our results extend those by Caffarelli and Lin to the fractional case, or, equivalently, to the case when the phase segregation takes place only on the characteristic hyperplane.
It is worthwhile noticing that, in case of the standard diffusion, the nodal set of the segregated critical configurations shares the same measure theoretical features with the nodal set of harmonic functions; this is not the case of the fractional diffusion; indeed, as shown by Sire, Tortone and Terracini, the stratified structure of the nodal set of s-harmonic functions is far more complex than that of the segregated critical configurations.
The asymptotics and properties of limiting profiles of competition diffusion systems with quadratic (Lotka-Volterra) mutual interactions have been investigated by Verzini and Zilio; as discussed there, the free boundary, in the Lotka-Volterra case, resembles the nodal set of s-harmonic functions with some important differences however, enlightened in that paper.

Table of contents

Interaction at a distance

We consider the minimization problem

$$\inf_{U \in H_\infty} J_\infty(U),$$

where the set H_∞ and the functional J_∞ are defined by

$$H_\infty = \left\{ U = (u_1, \dots, u_k) \in H^1(\Omega, \mathbb{R}^k) \;\middle|\; \begin{array}{c} \mathrm{dist}(\mathrm{supp}\, u_i, \mathrm{supp}\, u_j) \geq 1 \\ \forall i \neq j \;,\; u_i = f_i \text{ a.e. in } \partial\Omega \end{array} \right\},$$

and

$$J_\infty(U) = \sum_{i=1}^{k} \int_\Omega |\nabla u_i|^2.$$

A penalized functional

Let $\beta > 0$. We consider the minimization of the functional

$$J_\beta(U) = \sum_{i=1}^{k} \int_\Omega |\nabla u_i|^2 + \beta \sum_{1 \le i < j \le k} \iint_{\Omega_1 \times \Omega_1} \mathbb{1}_{B_1}(x-y) u_i^2(x) u_j^2(y)\, dx\, dy$$

in the set H.

We see the solutions of our problem as singular limits as

$$\beta \to +\infty\,.$$

The we have the following existence theorem.

Theorem (N. Soave, H. Tavares, S.T., A. Zilio, ARMA 2017)

For every $\beta > 0$, there exists a minimizer $U_\beta = (u_{1,\beta}, \dots, u_{k,\beta})$ for $\inf_H J_\beta$, which is a solution of

$$\begin{cases} -\Delta u_i = -\beta u_i \sum_{j \neq i} (\mathbb{1}_{B_1} \star u_j^2) & \text{in } \Omega \\ u_i > 0 & \text{in } \Omega \\ u_i = f_i & \text{in } \Omega_1 \setminus \Omega. \end{cases} \tag{2}$$

The family $\{U_\beta : \beta > 0\}$ is uniformly bounded in $H^1(\Omega_1, \mathbb{R}^k) \cap L^\infty(\Omega_1)$, and there exists $U = (u_1, \dots, u_k) \in H$ such that:

1. $U_\beta \to U$ *strongly in* $H^1(\Omega)$ *as* $\beta \to +\infty$, *up to a subsequence;*
2. $\mathrm{dist}(\mathrm{supp}\, u_i, \mathrm{supp}\, u_j) \geq 1$ *for every* $i \neq j$, *so that* $U \in H_\infty$;
3. *for every* $i \neq j$,

$$\lim_{\beta \to +\infty} \beta \iint_{\Omega_1 \times \Omega_1} \mathbb{1}_{B_1}(x - y) u_{i,\beta}^2(x) u_{j,\beta}^2(y) \, dx \, dy = 0$$

4. U *is a minimizer for* $\inf_{H_\infty} J_\infty$.

Lemma (N. Soave, H. Tavares, S.T., A. Zilio, ARMA 2018)

Let $U = (u_1, \dots, u_k)$ be any minimizer of J_∞ in H_∞. Denote $S_i = \{x \in \Omega : \ u_i > 0\}$, for every $i = 1, \dots, k$. Then:

1. *Subsolution in Ω: We have that $-\Delta u_i \leq 0$*
2. *Solution in S_i: We have that $-\Delta u_i = 0$ in $int(S_i)$,*
3. *Exterior sphere condition for the positivity sets: S_i satisfies the 1-uniform exterior sphere condition in Ω, in the following sense: for every $x_0 \in \partial S_i \cap \Omega$ there exists a ball B with radius 1 which is exterior to S_i and tangent to S_i at x_0, i.e.*

$$S_i \cap B = \emptyset \quad \text{and} \quad x_0 \in \overline{S_i} \cap \overline{B}.$$

Moreover, in $B \cap B_1(x_0)$ we have $u_j \equiv 0$ for every $j = 1, \dots, k$ (including $j = i$).

Basic regularity of minimizers

Theorem (N. Soave, H. Tavares, S.T., A. Zilio, ARMA 2018)

Let $U = (u_1, \dots, u_k)$ be any minimizer of J_∞ in H_∞.

1. *Lipschitz continuity:* u_i *is Lipschitz continuous in* Ω*, and in particular* S_i *is an open set, for every* i*.*
2. *Lebesgue measure of the free-boundary: the free-boundary* $\partial\{u_i > 0\}$ *has zero Lebesgue measure, and its Hausdorff dimension is strictly smaller than* N*.*
3. *Exact distance between the supports: for every* $x_0 \in \partial S_i \cap \Omega$ *there exists* $j \neq i$ *such that*

$$\overline{B_1(x_0)} \cap \partial\, \mathrm{supp}\, u_j \neq \emptyset.$$

Extremality Conditions at regular points

Next, we establish a relation involving the normal derivatives of two "adjacent components" on the regular part of the free boundary. In what follows, for each i, $\nu_i(x)$ will denote the exterior normal at a point $x \in \partial S_i$ (at points where such a normal vector does exist).

Assumptions

Let $x_0 \in \partial S_i \cap \Omega$, and let us assume that $\Gamma_i^R := \partial S_i \cap B_R(x_0)$ is a smooth hypersurface, for some $R > 0$. By the 1-uniform exterior sphere condition, we know that the principal curvatures of ∂S_i in x_0, denoted by $\chi_h^i(x_0)$, $h = 1, \dots, N-1$, are smaller than or equal to 1 (where we agree that outward is the positive direction). We further suppose that the strict inequality holds, that is there exists $\delta > 0$ such that

$$\chi_1^i(x_0), \dots, \chi_{N-1}^i(x_0) \leq 1 - \delta.$$

We know that there exists $j \neq i$ and a unique $y_0 \in \partial \operatorname{supp} u_j$ such that $|x_0 - y_0| = 1$.

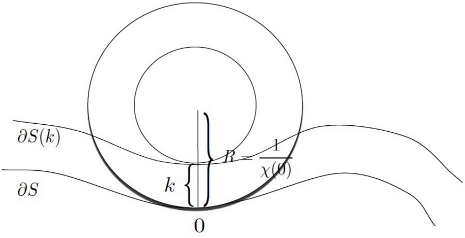
Curvature relations

In this circumstances, the curvature radii are related by

$$R(y_0) = 1 + R(x_0)$$

and the principal curvatures satisfy the relations:

$$\chi_h^i(y_0) = \frac{-\chi_h^i(x_0)}{1 - \chi_h^i(x_0)} \qquad \text{(CR)}$$



Theorem (N. Soave, H. Tavares, S.T., A. Zilio, ARMA 2018)

Let $\mathbf{u} = (u_1, \ldots, u_k)$ *be any minimizer of* J_∞ *in* H_∞*. Under the previous assumptions, we have that* $y_0 = x_0 + \nu_i(x_0)$ *is the unique point in* $\bigcup_{k \neq i} \partial \operatorname{supp} u_k$ *at distance* 1 *from* x_0*. If* $y_0 \in \partial \operatorname{supp} u_j \cap \Omega$*, then* $\partial \operatorname{supp} u_j$ *is also smooth around* y_0*, and*

$$\frac{(\partial_\nu u_i(x_0))^2}{(\partial_\nu u_j(y_0))^2} = \begin{cases} \displaystyle\prod_{\substack{h=1 \\ \chi_h^i(x_0) \neq 0}}^{N-1} \left| \frac{\chi_h^i(x_0)}{\chi_h^j(y_0)} \right| & \text{if } \chi_h^i(x_0) \neq 0 \text{ for some } h, \\ 1 & \text{if } \chi_h^i(x_0) = 0 \text{ for all } h = 1, \ldots, N-1. \end{cases}$$

We stress that, since the sets S_i and S_j are at distance 1 from each other and (CR) holds, $\chi_h^i(x_0) \neq 0$ if and only if $\chi_h^j(y_0) \neq 0$, and hence the term on the right hand side is always well defined.

Connected results by L. Caffarelli, S. Patrizi and V. Quitalo, On a long range segregation model, JEMS, 2017 for the limiting profiles of a non variational singular limit problem with long range interactions of Lotka-Volterra type.
For such type of long range quadratic interactions, the free boundary condition becomes:

$$\frac{|\partial_\nu u_i(x_0)|}{|\partial_\nu u_j(y_0)|} = \begin{cases} \displaystyle\prod_{\substack{h=1 \\ \chi_h^i(x_0)\neq 0}}^{N-1} \left|\frac{\chi_h^i(x_0)}{\chi_h^j(y_0)}\right| & \text{if } \chi_h^i(x_0) \neq 0 \text{ for some } h, \\ 1 & \text{if } \chi_h^i(x_0) = 0 \text{ for all } h = 1, \dots, N-1. \end{cases}$$

Connected results by L. Caffarelli, S. Patrizi and V. Quitalo, On a long range segregation model, JEMS, 2017 for the limiting profiles of a non variational singular limit problem with long range interactions of Lotka-Volterra type.
While for the long range cubic interactions, the free boundary condition was:

$$\frac{(\partial_\nu u_i(x_0))^2}{(\partial_\nu u_j(y_0))^2} = \begin{cases} \displaystyle\prod_{\substack{h=1 \\ \chi_h^i(x_0)\neq 0}}^{N-1} \left|\frac{\chi_h^i(x_0)}{\chi_h^j(y_0)}\right| & \text{if } \chi_h^i(x_0) \neq 0 \text{ for some } h, \\ 1 & \text{if } \chi_h^i(x_0) = 0 \text{ for all } h = 1, \dots, N-1. \end{cases}$$

We see again an effect of the type of interaction (quadratic rather than cubic) on the geometry of the phase separation.

Table of contents

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