Localization anisotropy and the complex geometry of 2-dimensional insulators

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QM³ QUANTUM MATTER MEETS MATH

Outline

Insulators and metals

The geometry of threading a flux through the system in the case of band insulators

The quantum metric in twist-angle space, the localization tensor and the complex structure τ

Relation to the low-energy theory near a quantum phase transition and the geometric character of τ

Example: A modified massive Dirac model

Conclusions and outlook

Insulators and metals

Insulators versus metals (band theory)

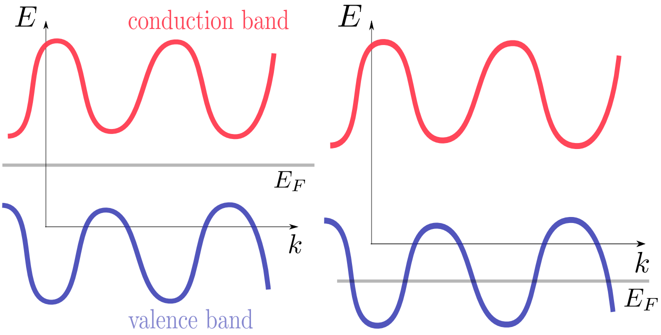

Figure: A band insulator (left) and metal (right). The ground state is obtained by filling all the states below E_F. In the insulator there is an energy gap to excite the system, while on the metal there isn't.

Phenomenology of the insulating state

- Intuitively, the insulating character of a system of electrons has to do with the way the electrons are organized in space.
- If the electrons are free to move in every direction, then by turning on an external electric field, we will generate a longitudinal current. This is the case of a metal.
- On the contrary, if the electrons are localized, applying an electric field will produce no longitudinal current.

Phenomenology of the insulating state (cont.)

- The property of a system of electrons being in an insulating state is related to the conductivity tensor σ^i_j.
- When we apply a uniform external electric field E^i, we have a current

$$\langle j^i \rangle = \sigma^i_j E^j + O(||E||^2).$$

- An insulator has a zero direct current (symmetric part of) conductivity at zero temperature, while a metal has a non-zero direct current conductivity.
- The anti-symmetric part of the conductivity describes the transverse conductivity and it is responsible for the anomalous Hall effect.

- The intuition concerning electron localization is right, as the conductivity tensor can be explicitly related to the localization tensor

$$G^{\mu\nu} = \langle X^{\mu} X^{\nu} \rangle - \langle X^{\mu} \rangle \langle X^{\nu} \rangle,$$

  where X^{μ}, $\mu = 1, ..., d$, denotes the center of mass position operator and $\langle \cdot \rangle$ is the ground state expectation value.
- The quantity $G^{\mu\nu}$ is finite for an insulator and it diverges for metals, in accordance with intuition.

The geometry of threading a flux through the system in the case of band insulators

Setup

- We have a system of fermions on a 2-dimensional lattice with periodic boundary conditions. The period on each direction is N.
- Later, we will take the $N \to \infty$ thermodynamic limit.

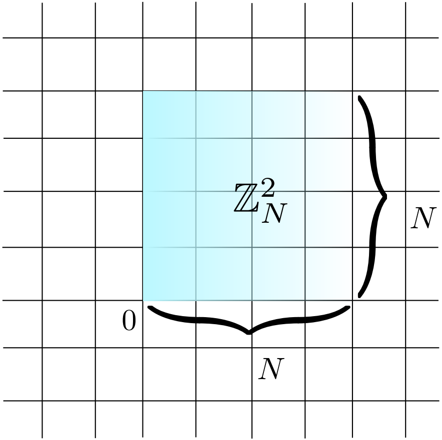

Figure: Position space.

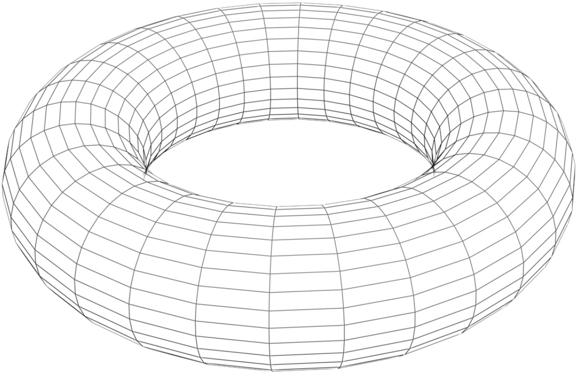

Figure: Topologically, the positions of the fermions take values in a two-torus.

- We can thread fluxes associated with the generators of the fundamental group of the torus, i.e., those loops which are not contractible to a point.
- It means that the fermions get phases when they are adiabatically moved around these loops.
- This does not introduce a magnetic field, it is equivalent to introducing "twisted" boundary condition on the wave-functions.

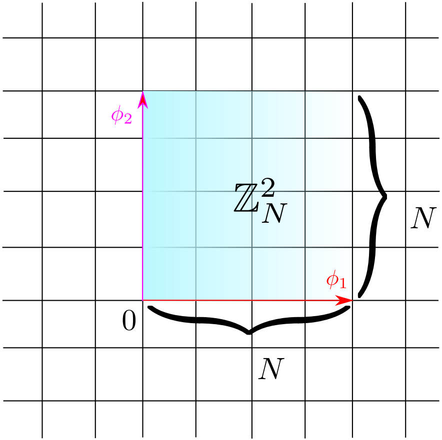

Figure: Twisted boundary conditions.

- In the thermodynamic limit the real space becomes the lattice $\mathbb{Z}^2$ and the allowed momenta live in the Brillouin zone, which itself is topologically a torus B.Z. $= \mathbb{R}^2/2\pi\mathbb{Z}^2$:

$$e^{i\mathbf{k}\cdot\mathbf{r}} = e^{i(\mathbf{k}+\mathbf{K})\cdot\mathbf{r}}, \text{ for } \mathbf{r} \in \mathbb{Z}^2 \text{ and } \mathbf{K} \in 2\pi\mathbb{Z}^2.$$

- With the standard trivial boundary conditions, the allowed momenta for the fermions are

$$\mathbf{k} = \frac{2\pi}{N}\mathbf{m}, \text{ with } \mathbf{m} \in \{0, ..., N-1\}^2,$$

which should be understood as taking values in B.Z. In some appropriate sense, we recover B.Z. as $N \to \infty$.

- When we thread a flux through the system,

$$\mathbf{k} = \frac{2\pi}{N}\mathbf{m} + \frac{\phi}{N}, \text{ with } \mathbf{m} \in \{0, ..., N-1\}^2.$$

- Again, we will recover the Brillouin zone when $N \to \infty$.

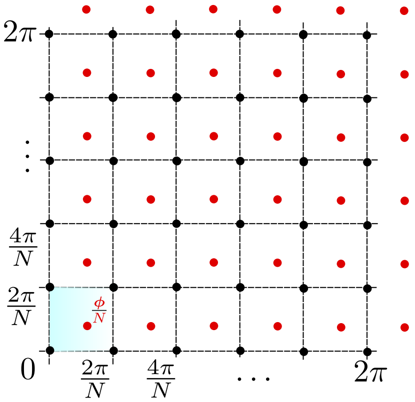

Figure: Observe that as we change the fluxes ϕ_1, ϕ_2 from 0 to 2π, we cover the whole Brillouin zone. Moreover, the allowed momenta for ϕ and $\phi + 2\pi(m, n)$ are the same for $m, n \in \mathbb{Z}$.

Tight binding free fermion models

- Simplest example: 1d chain with no internal degrees of freedom with nearest neighbour hoppings,

$$H = -t\sum_x (|x+1\rangle\langle x| + |x\rangle\langle x+1|) - \mu I$$
$$= \sum_k (-2t\cos(k) - \mu)|k\rangle\langle k|.$$

- In second quantization language this corresponds to

$$\mathcal{H} = \sum_k (-2t\cos(k) - \mu)\psi_k^\dagger \psi_k,$$

with $\psi_k, \psi_k^\dagger$ fermionic annihilation and creation operators.

Tight binding free fermion models (cont.)

- More generally, in the translation invariant, charge preserving setting, in 2 spatial dimensions,

$$\mathcal{H} = \sum_{\mathbf{k}} \psi_{\mathbf{k}}^{\dagger} H(\mathbf{k}) \psi_{\mathbf{k}},$$

  where now $\psi_{\mathbf{k}}^{\dagger} = [\psi_{\mathbf{k},1}^{\dagger} ... \psi_{\mathbf{k},n}^{\dagger}]$ is an array of fermion creation operators, accounting for internal degrees of freedom and $H(\mathbf{k})$ is an $n \times n$ Hermitian matrix yielding the action of $\mathcal{H}$ in the single particle sector.
- If the hoppings in real space decay fast enough, the map $B.Z. \ni \mathbf{k} \mapsto H(\mathbf{k})$ is smooth.

- Assume that we are in a band insulating state, so that there are bands below the Fermi level and bands above.
- The valence band projector

$$P(\mathbf{k}) = \Theta(E_F - H(\mathbf{k}))$$

  is smooth and defines a vector bundle, the Bloch bundle $E \to B.Z.$, over the Brillouin zone.
- Over each $\mathbf{k} \in B.Z.$, we take the vector space of eigenvectors with energy below E_F, i.e., $E_{\mathbf{k}} = \mathrm{Im} P(\mathbf{k})$.
- Smoothness of P guarantees smoothness of E.

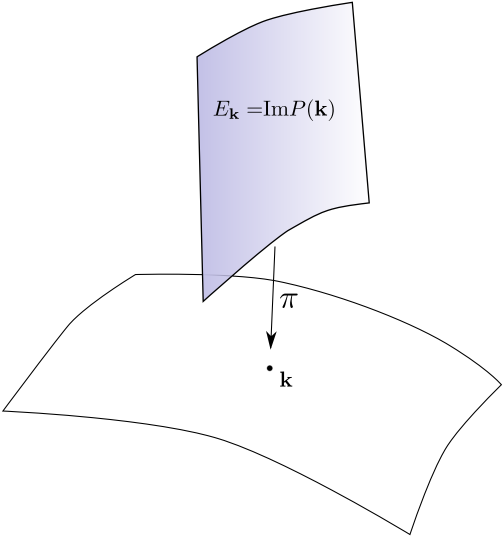

Figure: The Bloch bundle $E \to B.Z.$ defined by the valence band projector $\mathbf{k} \mapsto P(\mathbf{k})$. Notice that the $E_{\mathbf{k}}$ is naturally a subspace of a fixed vector space $\mathbb{C}^n$ since $H(\mathbf{k})$ is an $n \times n$ matrix.

Berryology: microscopic Berry connection

- Since each space $E_{\mathbf{k}} \subset \mathbb{C}^n$, we can define a parallel transportation rule.
- Namely, we have a connection/ covariant derivative on $E \to B.Z.$, $\nabla \Psi(\mathbf{k}) = P(\mathbf{k}) d\Psi(\mathbf{k})$, for single particle wave functions Ψ on E (sections of E).
- This connection is a microscopic Berry connection as it is seen by single particle wave-functions in the valence band in momentum space.

Berryology: microscopic Berry connection (cont.)

- Given a local orthonormal basis for E provided by wave functions $\{\Psi_i\}_{i=1}^r$, the associated $\mathrm{U}(r)$ gauge field, known as the Berry gauge field, is given by

$$A = [A_{ij}] = [\langle \Psi_i | d | \Psi_j \rangle] = A_\mu dk^\mu.$$

- The Berry curvature is given by

$$F = dA + A \wedge A = [F_{ij}] = \frac{1}{2} F_{\mu\nu} dk^\mu \wedge dk^\nu,$$
$$\text{with } F_{\mu\nu} = \frac{\partial A_\nu}{\partial k^\mu} - \frac{\partial A_\mu}{\partial k^\nu} + [A_\mu, A_\nu].$$

Ground state

- Now it we are considering the finite system with periodic boundary conditions, we are sampling $H(\mathbf{k})$ at points $\mathbf{k} = (2\pi/N)\mathbf{m}$, with $\mathbf{m} \in \{0, ..., N-1\}^2$.
- The ground state is obtained by filling the bands below E_F.
- This state is constructed as follows.

Ground state (cont.)

- Forgetting about the periodicity of $H(\mathbf{k})$ in $\mathbf{k}$, we obtain a family of matrices in $\mathbb{R}^2$. Since $\mathbb{R}^2$ is contractible, we can find global assignments

$$\mathbb{R}^2 \ni \mathbf{k} \mapsto s_i(\mathbf{k}) = (a_i^1(\mathbf{k}), ..., a_i^n(\mathbf{k})) \in \mathbb{C}^n, \ i = 1, ..., r,$$

  such that for each $\mathbf{k}$, they form an orthonormal basis of $E_{\mathbf{k}}$. The s_i's induce, generally, multivalued wave functions over the Brillouin zone.
- The s_i's give rise to creation operators (Bogoliubov-Valatin transformation)

$$\xi_{i,\mathbf{k}}^{\dagger} = \sum_{j=1}^{n} a_i^j(\mathbf{k}) \psi_{j,\mathbf{k}}^{\dagger}, \ i = 1, ..., r.$$

Ground state (cont.)

- The many-body ground state is at size for the finite size and periodic boundary conditions is then

$$|GS\rangle = \prod_{\mathbf{m}\in\{0,\dots,N-1\}^2} \prod_{i=1}^{r} \xi^\dagger_{i,\mathbf{k}=\frac{2\pi\mathbf{m}}{N}} |0\rangle.$$

- By threading a flux/twist-angle ϕ, we obtain a family of ground states

$$|GS(\phi)\rangle = \prod_{\mathbf{m}\in\{0,\dots,N-1\}^2} \prod_{i=1}^{r} \xi^\dagger_{i,\mathbf{k}=\frac{2\pi\mathbf{m}}{N}+\frac{\phi}{N}} |0\rangle$$

- Since the theory returns to itself ($H(\mathbf{k})$ is periodic) when $\phi \mapsto \phi + 2\pi\gamma$ with $\gamma \in \mathbb{Z}^2$, we get a family of ground states parametrized by the twist-angle torus T^2.

Berryology: macroscopic Berry connection

- Now at each flux $\phi \in T^2$ we attach the one-dimensional subspace of the many-body Hilbert space generated by $|GS(\phi)\rangle$. We get a line bundle $\mathcal{L} \to T^2$.
- The Berry connection (defined through projection) gives us a parallel transportation rule consistent with the adiabatic theorem of quantum mechanics. The associated gauge field is

$$\mathcal{A} = \langle GS(\phi)|d|GS(\phi)\rangle = \mathcal{A}_\mu d\phi^\mu.$$

- We refer to this connection as a macroscopic Berry connection as it is a property of the many-body ground state of the full theory.

Berryology: micro-to-macroscopic

- The remarkable consequence of the Slater determinant ground state is the relation between the macroscopic connection on $\mathcal{L} \to T^2$ and the microscopic connection $E \to B.Z. \cong T^2$, namely,

$$\mathcal{A}(\phi) = \frac{1}{N} \sum_{\mathbf{m}} \mathrm{Tr}\Big[A_\mu(\frac{2\pi\mathbf{m}}{N} + \frac{\phi}{N})\Big] d\phi^\mu,$$

and

$$\mathcal{F}(\phi) = \frac{1}{2N^2} \sum_{\mathbf{m}} \mathrm{Tr}\Big[F_{\mu\nu}(\frac{2\pi\mathbf{m}}{N} + \frac{\phi}{N})\Big] d\phi^\mu \wedge d\phi^\nu.$$

Observable consequences

- One can show that the transverse Hall conductivity is $\sigma_{xy} = ie^2 \mathcal{F}_{12}(\phi = 0)$.
- As we take the thermodynamic limit $N \to \infty$,

$$\mathcal{F}(\phi) = \frac{1}{2N^2} \sum_{\mathbf{m}} \mathrm{Tr}\Big[F_{\mu\nu}\big(\frac{2\pi \mathbf{m}}{N} + \frac{\phi}{N}\big)\Big] d\phi^\mu \wedge d\phi^\nu$$

$$\to \Big[\int_{B.Z.} \mathrm{Tr}\Big(\frac{F}{(2\pi)^2} \Big) \Big] d\phi^1 \wedge d\phi^2 = -\frac{ic_1}{2\pi} d\phi^1 \wedge d\phi^2,$$

  where c_1 is the 1st Chern number of E, so that $\sigma_{xy} = e^2 c_1 / 2\pi$, which is the famous result of Thouless for the integer plateaus in the Hall effect.
- So the curvature became constant in the thermodynamic limit, what about other quantities?

The quantum metric in twist-angle space, the localization tensor and the complex structure τ

Quantum metric in twist-angle space

- The family of quantum states $\{|GS(\phi)\rangle\}_{\phi\in T^2}$, provides a notion of infinitesimal distance between fluxes:

$$|\langle GS(\phi)|GS(\phi+\delta\phi)\rangle|^2 \approx 1 - G_{\mu\nu}(\phi)\delta\phi^\mu\delta\phi^\nu.$$

- The quantity $G = G_{\mu\nu}(\phi)d\phi^\mu d\phi^\nu$ is the pullback of the Fubini-Study metric on the projective space $\mathbb{P}\mathcal{H}$,

$$g_{FS} = |d|\Psi\rangle - \langle\Psi|d|\Psi\rangle|\Psi\rangle|^2.$$

Marzari-Vanderbilt theory

- Most remarkably, denoting $\langle \cdot \rangle = \langle GS(\phi)| \cdot |GS(\phi)\rangle$, we have
$$G_{\mu\nu}(\phi) = \langle X^\mu X^\nu \rangle - \langle X^\mu \rangle \langle X^\nu \rangle.$$
- The intuition behind the previous formula is that in the thermodynamic limit $X^\mu = i\partial/\partial k_\mu$, and
$$|GS(\phi + \delta\phi)\rangle = \exp(-i\delta\phi_\mu X^\mu)|GS(\phi)\rangle.$$
- Then
$$\begin{aligned} &\langle GS(\phi)|GS(\phi + \delta\phi)\rangle \\ &= \langle GS(\phi)| \exp(-i\delta\phi_\mu X^\mu)|GS(\phi)\rangle := Z(\delta\phi). \end{aligned}$$
- So
$$-\frac{\partial^2 \log Z}{\partial\delta\phi_\mu \partial\delta\phi_\nu}\bigg|_{\delta\phi=0} = \langle X^\mu X^\nu \rangle - \langle X^\mu \rangle \langle X^\mu \rangle.$$

- Thus, the quantum metric in twist-angle space is a measure of electron localization!
- For a family of quantum states, it is a known result that if the gap closes while changing some parameter, the quantum metric becomes singular.
- This is ultimately tied to the fact that if the family $\{|\Psi_0(x)\rangle\}_{x\in M}$ is the ground state of a family $\{H(x)\}_{x\in M}$ we have the formula

$$\text{quantum metric} = \sum_{j\neq 0} \frac{\langle\Psi_0|dH|\Psi_j\rangle\langle\Psi_j|dH|\Psi_0\rangle}{(E_j - E_0)^2},$$

  resembling perturbation theory (not a coincidence).
- As a consequence, if the system became metallic by tunning some external parameter, the localization tensor blows up.

Quantum metric in twist-angle space

- The Slater determinant induces a similar form for the quantum metric as it did for the Berry curvature:

$$G_{\mu\nu}(\phi) = \frac{1}{N^2} \sum_{\mathbf{m}} g_{\mu\nu}(\frac{2\pi\mathbf{m}}{N} + \frac{\phi}{N}) d\phi^\mu d\phi^\nu,$$

  where $g = g_{\mu\nu}(\mathbf{k}) dk^\mu dk^\nu = \mathrm{Tr}(PdPdP)$ is the quantum metric in momentum space.
- The thermodynamic limit now provides us with a flat metric:

$$G_{\mu\nu}(\phi) = \int_{B.Z.} \frac{d^2 k}{(2\pi)^2} g_{\mu\nu}(\mathbf{k}),$$

  clearly independent of ϕ.

- A flat metric in the two torus is described by the Riemannian volume $V = \int_{T^2} \sqrt{detG}\, d\phi_1 d\phi_2$ and by a complex parameter τ:

$$\tau = \frac{G_{12}}{G_{11}} + i\frac{\sqrt{\det G}}{G_{11}} \in \mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\},$$

so that

$$G = \frac{V}{(2\pi)^2 \mathrm{Im}(\tau)}(d\phi_1^2 + 2\mathrm{Re}(\tau) d\phi_1 d\phi_2 + |\tau|^2 d\phi_2^2).$$

- The latter determines a complex coordinate on the torus $\phi = \phi^1 + \tau\phi^2$, so that

$$G \propto |d\phi|^2 = d\bar{\phi} d\phi.$$

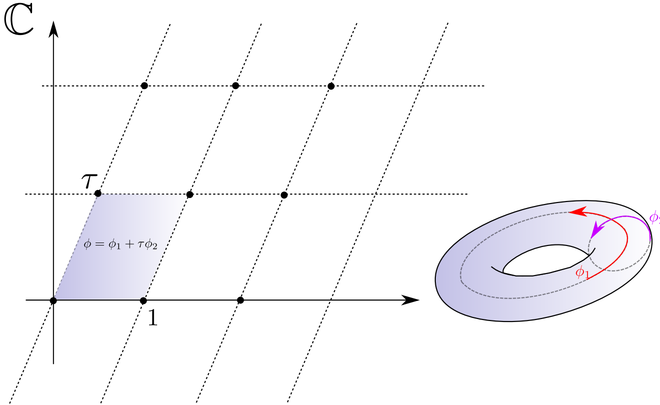

Figure: Illustration of the complex torus of twist-angles/fluxes.

Physical interpretation

- Notice that since $G^{\mu\nu} = \langle X^\mu X^\nu \rangle - \langle X^\mu \rangle \langle X^\nu \rangle$, τ is naturally related to the anisotropy of position correlations and the Riemannian volume the strength of these correlations.
- For instance, the standard $\tau = i$ corresponding to the lattice $\mathbb{Z}^2$, gives $G^{12} = 0$ for so that the directions X^1 and X^2 decouple $\langle X^1 X^2 \rangle = \langle X^1 \rangle \langle X^2 \rangle$.

Gauge ambiguity

- We have to identify τ and τ' such that

$$\tau' = \frac{a\tau + b}{c\tau + d}, \text{ with } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2;\mathbb{Z})$$

- The reason is that this corresponds to an observer choosing a different basis for the lattice $\mathbb{Z}^2$ in real space $X^\mu \mapsto A^\mu{}_\nu X^\nu$, with $A \in \mathrm{GL}(2;\mathbb{Z})$. Essentially, it is a gauge choice.
- From the math side, this corresponds to choosing a different element in the isomorphism class of the complex torus.
- Thus, in principle, by changing the insulator, we could move around the space of complex structures of the torus.

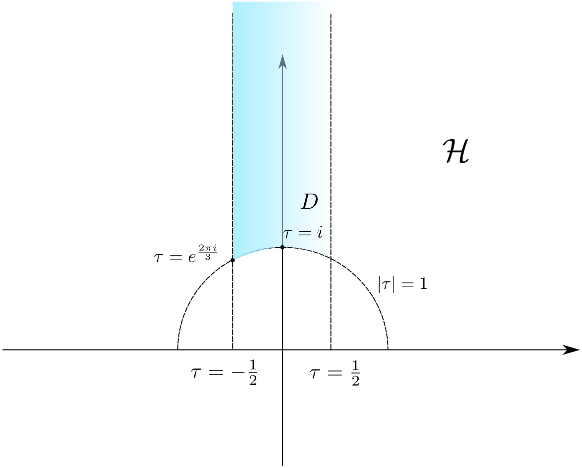

Figure: The quotient space $\mathcal{H}/\mathsf{SL}(2;\mathbb{Z})$ has a fundamental domain given by $D = \{\tau \in \mathcal{H} : |\tau| \geq 1 \text{ and } \mathrm{Re}(z) \leq 1/2\}$. The points $\tau, \tau' \in D$ with $\mathrm{Re}(\tau) = \pm 1/2$ and $\tau' = \tau \pm 1$ or $|\tau| = 1$ and $\tau' = -1/\tau$ are the same in the quotient space.

Relation to the low-energy theory near a quantum phase transition and the geometric character of τ

- Suppose we are given a single-particle Hamiltonian and that two levels cross generically at a critical momentum $\mathbf{k}_c$ by tuning a parameter M of the system to a value M_c.
- By a shift of the variables we can assume $\mathbf{k}_c = 0$ and $M_c = 0$.
- The two-level crossing can be described, in a neighborhood of $(\mathbf{k}, M) = 0$, by a 2×2 low energy Hamiltonian of the form

$$H(\mathbf{k}, M) \approx (ak_1 + bk_2)\sigma_1 + (ck_1 + dk_2)\sigma_2 + M\sigma_3,$$

  with $ad - bc \neq 0$.
- In the absence of symmetries this case is completely generic, in the sense that other types of crossings, like quadratic band crossing, can be be adiabatically connected to this one.

- Define new momenta

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = A \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, \text{ with } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

rendering this block an isotropic Dirac Hamiltonian,

$$q_1\sigma_1 + q_2\sigma_2 + M\sigma_3.$$

- The new momenta $\mathbf{q}$ explicitly violate the dual lattice $2\pi\mathbb{Z}^2$, since

$$\mathbf{k} + \mathbf{K} \mapsto \mathbf{q} + A\mathbf{K},$$

and if $\mathbf{K} \in 2\pi\mathbb{Z}^2$ we will have $A\mathbf{K} \in 2\pi\mathbb{Z}^2$ iff $A \in \mathrm{GL}(2;\mathbb{Z})$.

Quantum metric

- To compute the quantum metric in twist-angle space, $G_{\mu\nu} = \int_{B.Z.} g_{\mu\nu}(\mathbf{k}) d^2k/(2\pi)^2$, we need to compute the quantum metric in momentum space, $g_{\mu\nu}(\mathbf{k})$, which near the critical point $\mathbf{k} = 0$, assumes the form in the $\mathbf{q}$ coordinates:

$$\begin{bmatrix} \frac{q_1^2+M^2}{(\mathbf{q}^2+M^2)^2} & -\frac{q_1 q_2}{(\mathbf{q}^2+M^2)^2} \\ -\frac{q_1 q_2}{(\mathbf{q}^2+M^2)^2} & \frac{q_2^2+M^2}{(\mathbf{q}^2+M^2)^2} \end{bmatrix} + \text{regular}$$

- The first term becomes singular at the critical point.
- One can then see that as $M \to 0$, the most relevant contribution is given by a small neighbourhood of $\mathbf{k} = 0$, which we take $\{\mathbf{q} : |\mathbf{q}| < \Lambda\}$.
- The transformation $\mathbf{q} = A\mathbf{k}$ is linear and has a constant Jacobian, therefore, we can safely perform the integration in $\mathbf{q}$ and then go back to the original coordinates (this actually corresponds to changing the ϕ coordinates appropriately).

- The quantum metric $G_{\mu\nu} = \int_{B.Z.} g_{\mu\nu}(\mathbf{k}) d^2k/(2\pi)^2$ will assume the form

$$\widetilde{G} = [G_{\mu\nu}] = C \ln\left(\frac{M^2 + \Lambda^2}{M^2}\right) \begin{bmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{bmatrix} + \text{regular.}$$

- Through a conformal transformation we can make the regular part vanish as $M \to 0$ and

$$\tau = \frac{ab + cd + i|\det A|}{a^2 + c^2} = \frac{\omega_2}{\omega_1},$$

with $\omega_2 = a + ic$ and $\omega_1 = b + id$.

- Thus, the columns of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ determine the basis for the lattice determining a finite τ at the gapless point $M = 0$.
- If $A \in \mathrm{GL}(2; \mathbb{Z})$ then we end up with the lattice $\mathbb{Z}^2$ and τ is equivalent to $\tau = i$.

- We also remark that

$$V \sim \log(\frac{M^2 + \Lambda^2}{M^2}),$$

so that $V \to \infty$ when the system becomes gapless. Hence, the Riemannian volume is responsible for the singularity in the localization tensor when $M \to 0$.

Quantum metric vs Berry curvature

- The behaviour of the momentum space quantum metric associated to the Dirac point:

$$[g_{\mu\nu}(\mathbf{q})] \sim \begin{bmatrix} \frac{q_1^2+M^2}{(\mathbf{q}^2+M^2)^2} & -\frac{q_1 q_2}{(\mathbf{q}^2+M^2)^2} \\ -\frac{q_1 q_2}{(\mathbf{q}^2+M^2)^2} & \frac{q_2^2+M^2}{(\mathbf{q}^2+M^2)^2} \end{bmatrix}$$

  should be compared to the behaviour of the momentum space Berry curvature:

$$F(\mathbf{q}) \sim -\frac{i}{2}\frac{M}{\left(M^2+\mathbf{q}^2\right)^{3/2}} dq_1 \wedge dq_2 .$$

- A transition in which M changes sign, known as a band inversion, passing through $M = 0$, involves a change in topology of the Bloch bundle E as the change in the Chern number is naturally associated with the sign of M.
- This signature is present in the local form of F, it is odd under $M \to -M$, and it is natural to expect since $c_1 \propto \int_{B.Z.} F$.
- Since F is odd under $M \to -M$ and g is even, we understand that G as well as τ do not distinguish between topological phase transitions $\Delta c_1 = +1$ or $\Delta c_1 = -1$.
- Note however that G captures the gap closing point through the Riemannian volume and τ the information about the anisotropy of the local low-energy theory.
- This emphasizes the topological character of the flat $\mathcal{F}$ versus the geometric character of G.

Example: A modified massive Dirac model

Modified massive Dirac model

- We consider spinless fermions with a pseudo spin internal degree of freedom such that the tight binding model is

$$H(\mathbf{k}, M) = [\sin(k_1) + a\sin(k_2)]\sigma_1 + b\sin(k_2)\sigma_2 + [M - \cos(k_1) - \cos(k_2)]\sigma_3,$$

  with $a \in \mathbb{R}$, $b > 0$.
- The usual massive Dirac model is recovered for $a = 0$ and $b = 1$ and it has a low energy theory which is the familiar $2 + 1$ dimensional Dirac Hamiltonian with mass M $q_1\sigma_1 + q_2\sigma_2 + M\sigma_3$.

Modified massive Dirac model (cont.)

- The model has a topological phase diagram independent of a and b:

$$c_1 = \begin{cases} 0, & |M| > 2 \\ +1, & -2 < M < 0 \\ -1, & 0 < M < 2 \end{cases},$$

  where c_1 denotes the 1st Chern number of the bundle $E \to B.Z.$ defined previously and it generates transverse Hall conductivity plateaus. $M_c = -2, 0, +2$ are the points of phase transition.
- Using the previous results one can show that $\tau = a + ib$ for $M_c = \pm 2$ and $\tau = -a + ib$ for $M_c = 0$.
- Observe the usual isotropic massive Dirac model has $\tau = i$.
- Thus, this model allows to swipe the whole space of complex structures.

Conclusions and outlook

Conclusions

- We have shown how in two spatial dimensions the anisotropy of the localization tensor is related to a complex structure τ over the twist-angle torus.
- τ is a geometric quantity and not topological. It is sensitive to adiabatic perturbations.

Conclusions

- τ is finite, even when undergoing a phase transition where the gap closes, and thus going through a metallic state.
- τ is intimately related to the anisotropy of the low-energy Dirac theory near the critical points. Indeed, at the critical points of phase transition, τ is determined by the low energy theory.
- The complex structure τ and the Riemannian volume V are physically sensible gauge-invariant observables which completely characterize the localization tensor.

What about interactions?

- τ can be defined in the presence of interactions, provided some gap condition exists and we have a family $\{|GS(\phi)\rangle\}_{\phi\in T^2}$.
- The family $\{|GS(\phi)\rangle\}_{\phi\in T^2}$ cannot be trivial since if it were trivial, i.e., constant, the metric would be automatically degenerate.
- In the interacting case G will not, generically, be flat.

What about interactions? (cont.)

- The procedure to determine τ is to determine a flat metric in the conformal class of G, which involves solving a differential equation for the conformal factor which enforces the Ricci scalar to be zero.
- τ and V will no longer, in general, completely specify the localization tensor since it will not be flat.
- In the presence of translation invariance, this can also be seen as a measure of how interacting the system is.
- Namely, the failure of describing the localization tensor completely through τ and V measures the fluctuations from a quasi-free-fermion description.

If you want to read through details check my paper: Phys. Rev. B, 101:115128, Mar 2020.

A big thanks to João, Zé, Thomas, Nathan, Nikola, Raffaele and Roger for fruitful discussions!

Thank you, and stay safe!