# RESURGENCE, SUPERCONDUCTORS AND RENORMALONS

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# Introduction

# Overview - What?

**Resurgence** A framework to study perturbation theory and its relation to non-perturbative phenomena.

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**Superconductors** A non-perturbative phenomenon! And a very physical one. We took the first steps into fitting superconductivity into the language of resurgence.

**Renormalons** Looking at superconductivity from the perspective of perturbation theory, we find it to be a renormalon effect.

(The papers: arXiv:1905.09575 and arXiv:1905.09569)

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# Overview - The menu

- Resurgence in a nutshell
- The Gaudin-Yang model and its solutions
- Results, resummation and renormalons
- Beyond Gaudin-Yang, a conjecture and the conclusion

# Resurgence in a nutshell

# The divergence menace

Often in QM/QFT, observables $f(g)$ computed pertubatively in some variable $g$ result in asymptotic series.

A series expansion of a function is **asymptotic** if for any sufficiently large truncation $N$ the remainder is smaller than $g^N$ when $g \to 0$.

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Since the series doesn't converge, we can show that the best approximation we can get is

$$F(g) - F_{N_{\text{optimal}}}(g) \sim \mathrm{e}^{-|A/g|}. \tag{2.2}$$

# Borel resummation - An old hope

Many series, including some conventionally divergent series, can be resummed through **Borel (re)summation** (1899).

The Borel transform of a series is given by

$$\varphi(z) \approx \sum_{k\geq 0} b_k z^k \to \widehat{\varphi}(\zeta) = \sum_{k\geq 0} \frac{b_k}{k!}\zeta^k \tag{2.3}$$

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If its Laplace transform converges, $\varphi$ is *Borel summable* with Borel sum

$$s(\varphi)(z) = \int_0^\infty \mathrm{e}^{-\zeta}\widehat{\varphi}(z\zeta)\,d\zeta \tag{2.4}$$

which recovers the “true” function $\varphi(z)$.

# Borel resummation - The ambiguity strikes back

Let us look at the Borel transform of our example from before

$$F_p(g) \sim \sum_{k\geq 0}^{\infty} (A^{-k} k!) g^k \Rightarrow \widehat{F}(\zeta) = \frac{1}{1-\zeta/A} \tag{2.5}$$

Which has a pole on $\mathbb{R}^+$ if $A > 0$ !

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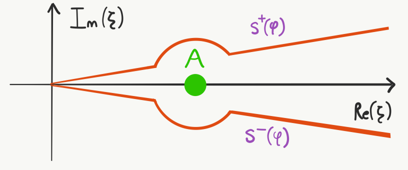

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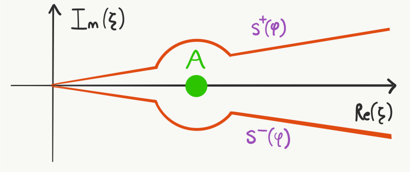


But there is an ambiguity

$$s_+(F)(g) - s_-(F)(g) = 2\pi \mathrm{i} e^{-A/g} \tag{2.6}$$

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$$\varphi(z) = \sum_{k\geq 0} c_k z^k + \sum_{l\geq 1,i} C_{l,i} \mathrm{e}^{-lA_i/z} \sum_{k\geq 0} c_k^{(l,i)} z^k \qquad (2.7)$$

we can further augment with the monomials $\log(z)$, $\exp(-\exp(A_i/z))$, $\log(\log(z))$, $\exp(-\exp(\exp(A_i/z)))$, ...

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Road to a better understanding of QFT and non-perturbative physics.

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Note that the weakly coupled gas in one dimension is dense!

# Perturbation theory

We want to find ground-state energy as a function of $\gamma$

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We can start with free Fermi gas, then Hartree-Fock, then bubble diagrams...

$$e(\gamma) = \frac{\pi^2}{12} - \frac{1}{2}\gamma - \frac{1}{12}\gamma^2 - \frac{\zeta(3)}{\pi^4}\gamma^3 + \cdots \tag{3.10}$$

$e(\gamma) - e_0 =$

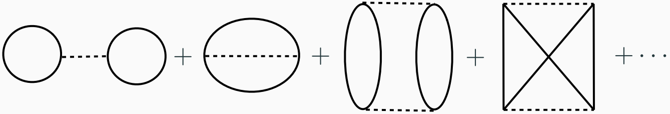

Figure 1: Feynman diagrams.

# BCS approximation

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Solving the gap equation, we find the BCS gap (binding energy of a Cooper pair)

$$\Delta_{BCS} \approx \left(\frac{32n^2}{\pi^2}\right) \mathrm{e}^{-\frac{\pi^2}{2\gamma}} \tag{3.11}$$

and

$$e_{BCS}(\gamma) \approx \frac{\pi^2}{12} - \frac{\gamma}{2} - 2\pi^2 \mathrm{e}^{-\frac{\pi^2}{\gamma}} \tag{3.12}$$

The GY model is integrable, so we can instead describe its ground state by a **Bethe Ansatz** integral equation

$$\frac{f(x)}{2} + \frac{1}{2\pi} \int_{-B}^{B} \frac{f(y)\mathrm{d}y}{(x-y)^2+1} = 1, \qquad -B < x < B. \qquad (3.13)$$

where $f(x)$ is the density of Bethe roots and $B \sim n/g$.

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We can relate it to our observables. The dimensionless coupling is given by

$$\frac{1}{\gamma} = \frac{n}{g} = \frac{1}{\pi}\int_{-B}^{B} f(x)\mathrm{d}x, \tag{3.14}$$

and the ground state energy is

$$e(\gamma) = -\frac{\gamma^2}{4} + \pi^2\frac{\int_{-B}^{B} f(x)x^2\mathrm{d}x}{\left(\int_{-B}^{B} f(x)\mathrm{d}x\right)^3}. \tag{3.15}$$

# Bethe Ansatz

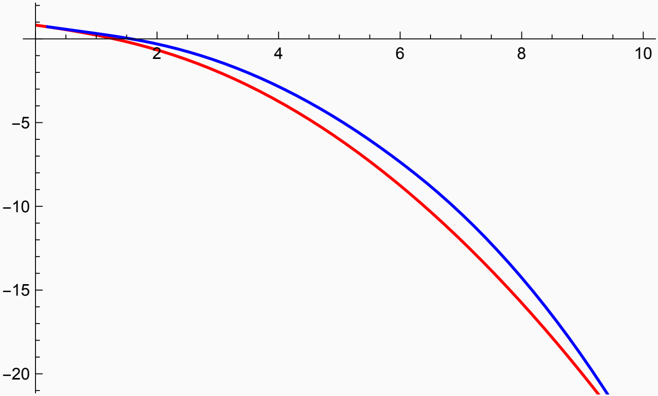


Figure 2: BCS approximation for the ground state energy (blue) vs. exact numeric solution of the ground state energy (red).

# Bethe Ansatz - Exact perturbative solution

The exact perturbative method finds a solution in for $f(x)$ as a series in $1/B$, from which we can obtain $e(\gamma)$.

It was developed by **D. Volin** (2009) in the context of AdS/CFT, though key steps had been assembled before (Hutson 1963, Popov 1977, Iida-Wadati 2005, Tracy-Widom 2016,...).

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We study the integral equation in two regimes and then match the two to fix unknown coefficients.

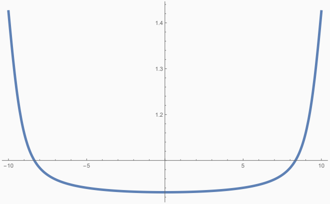


**Figure 3:** Numerical solution of $f(x)$ when $B = 10$.

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Other models described by similar integral equations

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- The GY with a repulsive interaction
- The Lieb-Liniger model (1963), a 1D gas of bosons

# Results, resummation and renormalons

# Large order behaviour - The energy

$$
\begin{aligned}
e(\gamma) &= e_0 + \sum_{k\geq 0} c_k \gamma^k \\
&= \frac{\pi^2}{12} - \frac{\gamma}{2} + \frac{\gamma^2}{6} - \frac{\zeta(3)}{\pi^4}\gamma^3 - \frac{3\zeta(3)}{2\pi^6}\gamma^4 - \frac{3\zeta(3)}{\pi^8}\gamma^5 \\
&- \frac{5(5\zeta(3)+3\zeta(5))}{4\pi^{10}}\gamma^6 - \frac{3\left(12\zeta(3)^2+35\zeta(3)+75\zeta(5)\right)}{8\pi^{12}}\gamma^7 \\
&- \frac{63\left(12\zeta(3)^2+7\zeta(3)+35\zeta(5)+12\zeta(7)\right)}{16\pi^{14}}\gamma^8 \\
&- \frac{3\left(404\zeta(3)^2+240\zeta(5)\zeta(3)+77\zeta(3)+735\zeta(5)+882\zeta(7)\right)}{4\pi^{16}}\gamma^9 \\
&+ \mathcal{O}\left(\gamma^{10}\right)
\end{aligned}
\tag{4.16}
$$

Agrees with numerical predictions (Prolhac 2017).

# Large order behaviour - Results

$$c_k \sim A^{-b-k}\Gamma(k+b) \Rightarrow s_k = \frac{kc_k}{c_{k+1}} \sim A + \mathcal{O}\left(\frac{1}{k}\right),\ k \gg 1 \quad (4.17)$$

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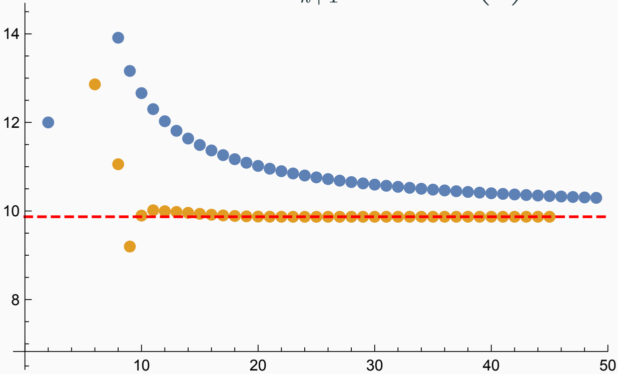


Figure 4: $s_k$ (in blue) and its Richardson transform (in orange), which should converge to $A$ faster.

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We find numerically

$$c_k \sim -\frac{1}{\pi}(\pi^2)^{-k+1}\Gamma(k-1), \qquad k \gg 1. \tag{4.18}$$

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$$-2\gamma \mathrm{e}^{-\pi^2/\gamma} \tag{4.19}$$

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$$-2\gamma \mathrm{e}^{-\pi^2/\gamma} \tag{4.19}$$

Same non-perturbative scale as the $\Delta^2_{BCS}$! We thus expect a trans-series

$$e(\gamma) \sim \sum_{n\geq 0} c_n \gamma^n + \sum_{\ell \geq 1} C_\ell \, \mathrm{e}^{-\ell\pi^2/\gamma} \gamma^{b_\ell} \sum_{n \geq 0} c_n^{(\ell)} \gamma^n. \tag{4.20}$$

What type of non-pertubative effect is it?

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They are associated to non-trivial saddle points.

# Renormalons - Less is more

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In particle physics renormalon diagram typically come from $\log$ divergences at $k \to 0, \infty$. In condensed matter, at $q \to 0, 2k_F$.

They were first found in (renormalizable) asymptotically free theories by 't Hooft, where they dominate over instantons. Hence the misnomer.

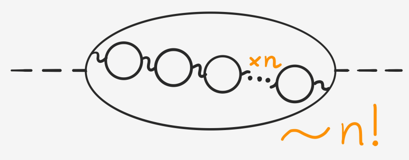


**Figure 5:** A typical renormalon diagram in particle physics.

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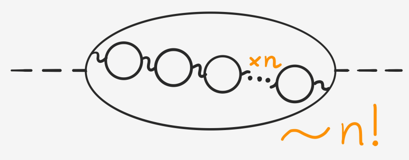


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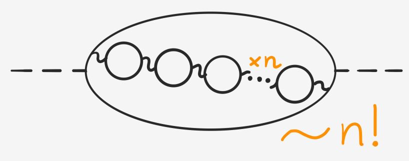


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When there is an OPE picture, we can associate renormalons with non-zero condensates in the vacuum.

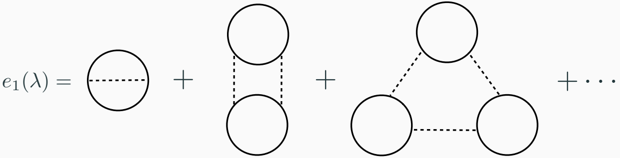
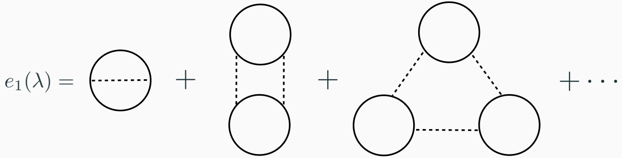



Figure 6: Ring diagrams.

# Diagrammatics - One diagram to rule them all...

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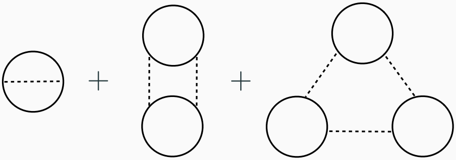
$e_1(\lambda) =$  $+ \cdots$

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Ring diagrams...

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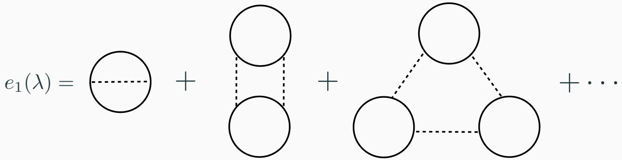


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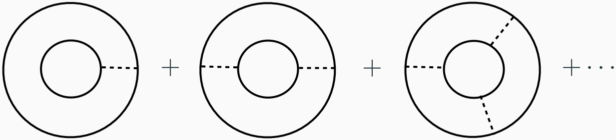
Ring diagrams...

- are factorially divergent in one dimension
- dominate at large $N_s$
- provide the correct weight

# Diagrammatics - A sequence of Escher drawings

We can also have ladder diagrams



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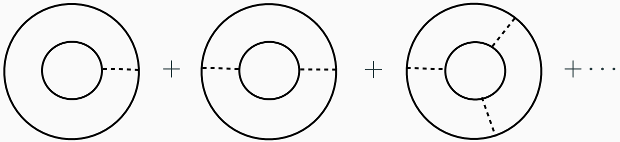

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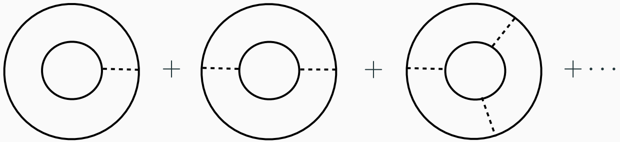

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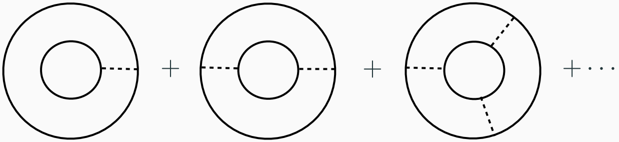

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They feature in traditional calculations of superconductivity.

But they underestimate the divergence. They are, however, more relevant in higher dimensions (Baker 1971). And there's more than one family of them.

# Discussion and conclusion

# What do we learn from GY

- Lack of resummability and bad vacua, an old tale.
- A mathematical view into the Cooper instability
- A physical picture, but no semiclassical picture.
- Renormalons in a new context.

# A conjecture

We thus conjecture

- The perturbative expansion of the ground state of a superconductor is non-Borel resummable.
- The divergence is caused by renormalon effects.
- The weight of the non-perturbative effects is given by the square of the BCS gap.

# Beyond Gaudin-Yang

We what can we say about...

- **repulsive gas?** The repulsive GY turns out to be Borel-*summable*, as expected.

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- **relativistic models?** In asymptotically free integrable QFTs we also have renormalons. (arXiv:1909.12134)

# Conclusion

- Resurgence can help understand many-body theory.
- The superconductor gap is a robust energy scale in attractive fermion systems, so it should dictate non-perturbative effects.
- The Cooper instability can be phrased more precisely using Borel summability.
- Renormalons can appear in condensed matter systems.

# The end.

Thank you!