The geometry and topology of free fermions

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Free fermions in $d = 0$ for physicists

If we have n degrees of freedom, we can promote them to n fermionic modes by declaring the canonical fermionic (anti-)commutation relations:

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{ij}, \tag{1}$$

$$\psi_i \psi_j + \psi_j \psi_i = 0, \tag{2}$$

$$\psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0. \tag{3}$$

Together with a vacuum state $|0\rangle$ defined by $\psi_i|0\rangle = 0$ for all i, these ingredients give rise to a Hilbert space, a basis of which is given by "creating states over the vacuum":

$$\psi_{i_1}^* \dots \psi_{i_p}^* |0\rangle, \; 1 \leq i_1 < \dots < i_p \leq n \text{ and } p = 0, \dots, n. \tag{4}$$

Free fermions in $d = 0$ for physicists (cont.)

A (charge conserving) free fermion Hamiltonian is then identified a linear operator

$$\mathcal{H} = \sum_{i,j} \psi_i^* h_{ij} \psi_j, \tag{5}$$

with $H = [h_{ij}]_{1 \le i,j \le n}$ Hermitian. It is clear that there exists $S \in \mathrm{U}(n)$ such that

$$H = SDS^*, \tag{6}$$

with $D = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ and $\{\lambda_i\}_{i=1}^n$ being the (real) eigenvalues of H. The columns of the matrix S are o.n. eigenvectors $\{v_i\}_{i=1}^n$ of H in the canonical basis of $\mathbb{C}^n$.

We can identify each of the v_i's in S as elements of the Hilbert space as follows

$$\mathbb{C}^n \ni v_i = [v_i^j]_{j=1}^n \leftrightarrow \sum_{j=1}^{n} v_i^j \psi_j^* |0\rangle \tag{7}$$

Actually, we can think of them as being operators acting on the Hilbert space, $v_i := \sum_{j=1}^n v_i^j \psi_j^*$. Together with the adjoints $v_i^* = \sum_{j=1}^n \bar{v}_i^j \psi_j$, they satisfy

$$v_i v_j^* + v_j^* v_i = \delta_{ij}. \tag{8}$$

With these identifications,

$$\mathcal{H} = \sum_i \lambda_i v_i v_i^*. \tag{9}$$

From the algebraic relations it then follows that the eigenvectors and eigenvalues of $\mathcal{H}$ are

$$v_{i_1} \ldots v_{i_p}|0\rangle \text{ with eig. } \sum_{j=1}^{p} \lambda_{i_j}, \tag{10}$$

for $1 \leq i_1 < \ldots < i_p \leq n$ and $p = 0, \ldots, n$, with $\mathcal{H}|0\rangle \equiv 0$ being the $p = 0$ state.

Free fermions in $d = 0$ for mathematicians

We are given an n-dimensional Hilbert space V and we take $\Lambda^* V = \oplus_{p=0}^{n} \Lambda^k V$ with the Hermitian inner product given by $\langle v_1 \wedge \ldots \wedge v_p, w_1 \wedge \ldots \wedge w_p \rangle = \det(\langle v_i, w_j \rangle)$, extended by linearity. If we are given an o.n. basis for V say $\{v_i\}_{i=1}^{n}$, we have a basis for $\Lambda^* V$ given by

$$v_{i_1} \wedge \ldots \wedge v_{i_p},\ 1 \leq i_1 < \ldots < i_p \leq n \text{ and } p = 0, \ldots, n. \tag{11}$$

V is realised within the endomorphisms of $\Lambda^* V$ by the exterior multiplication map $v \mapsto v \wedge \cdot$. The adjoint of v with respect to the inner product is denoted by v^* and satisfies the relation $vv^* + v^* v = \langle v, v \rangle 1$.

Free fermions in $d = 0$ for mathematicians (cont.)

If we are given a map $A \in \mathrm{End}(V)$, there is a natural way for it to act on $\Lambda^* V$, namely,

$$\Lambda(A) : v_1 \wedge \ldots \wedge v_p \mapsto (Av_1) \wedge \ldots \wedge (Av_p). \tag{12}$$

We can also extend it as a degree 0-derivation:

$$d\Lambda(A) : v_1 \wedge \ldots \wedge v_p \mapsto \sum_{j=1}^{p} v_1 \wedge \ldots \wedge (Av_j) \wedge \ldots \wedge v_p. \tag{13}$$

A (charge conserving) free fermion Hamiltonian is $d\Lambda(H)$ for some $H \in \mathrm{Herm}(V)$.

Dictionary

Take $V = \mathbb{C}^n$ and let $\{e_i\}_{i=1}^n$ be the standard basis, then

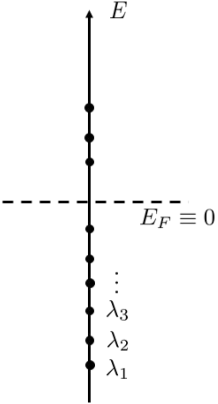
$$|0\rangle \longleftrightarrow 1, \tag{14}$$

$$\psi_{i_1}^* ... \psi_{i_p}^* |0\rangle \longleftrightarrow e_{i_1} \wedge ... \wedge e_{i_p} \tag{15}$$

$$\mathcal{H} = \sum_{i,j} h_{ij} \psi_i^* \psi_j \longleftrightarrow d\Lambda(H) = \sum_{i,j} h_{ij} e_i e_j^*. \tag{16}$$

The unique ground state for gapped H: Fermi sea

Suppose $H \in \text{Herm}(V)$ is gapped. And here we will assume that means H is invertible.



That means that we can separate its eigenvalues into positive and negative eigenvalues.

The unique ground state for gapped H: Fermi sea (cont.)

In particular we can form the negative energy vector subspace $E \subset V$ given by the direct sum of all eigenspaces with negative eigenvalue. Let $\{v_i\}_{i=1}^r$ be an o.n. basis for $E \subset V$, with $r = \dim E$. The multi-vector

$$\alpha = \bigwedge_{j=1}^{r} v_j \in \Lambda^r(V) \subset \Lambda^*(V), \tag{17}$$

is an eigenvector of $d\Lambda(H)$ for the smallest possible eigenvalue, namely $\sum_{\lambda<0} \lambda$.

Fermions and the Plücker Embedding

In quantum mechanics a (pure) state is specified by a one-dimensional subspace of a Hilbert space. This means that the ground state of a gapped free fermion Hamiltonian $d\Lambda(H)$ should be identified with the line $\det E \subset \Lambda^r(V)$. This is precisely the image of $E \in \mathrm{Gr}_r(V)$ under the *Plücker embedding* $j_r : \mathrm{Gr}_r(V) \hookrightarrow \mathbb{P}\Lambda^r(V)$.

"Classifying space"

In dimension 0, the ground states of gapped charge symmetric fermionic systems are described by prescribing E and mapping it to $\det E$. The "classifying space" for ground states of (charge conserving) gapped free fermion Hamiltonians is

$$\mathcal{M}(V) = \bigcup_{r=0}^{\dim V} \mathrm{Gr}_r(V) \tag{18}$$

We could define the ground state in this scenario to be a (continuous) map $i : \mathrm{pt} \to \mathcal{M}(V)$, i.e., a choice of a negative energy subspace of V.

Topological phases in $d = 0$

For fixed $r = \dim E$, connectedness of $\mathrm{Gr}_r(V)$ implies that any two different ground states can be "adiabatically connected", i.e., there exists a family of gapped Hamiltonians $d\Lambda(H_t)$, $t \in [0, 1]$, such that the endpoints correspond to any two ground states with fixed r. In other words, the equivalence classes $[\mathrm{pt}, \mathcal{M}(V)] \cong \{1, ..., \dim V\}$ parametrize the different ground states modulo adiabatic transformations preserving the gap.

Orbits of U(n) and coherent states

As a representation of the unitary group U(n), $\Lambda^*(\mathbb{C}^n)$ splits, with irreducible modules $\Lambda^r(\mathbb{C}^n)$. Given a choice of a "fiducial" decomposable vector

$$e_1 \wedge \ldots \wedge e_r, \tag{19}$$

we can act with U(n),

$$v_1 \wedge \ldots \wedge v_r = (Ue_1) \wedge \ldots \wedge (Ue_r), \text{ for } U \in \mathrm{U}(n), \tag{20}$$

where the action of $\mathrm{U}(r) \times \mathrm{U}(n-r)$ is through a phase and, thus, identifies the same physical state.

Orbits of U(n) and coherent states (cont.)

The set of all ground states at degree r is precisely the orbit of $e_1 \wedge \ldots \wedge e_r$,

$$\mathrm{Gr}_r(\mathbb{C}^n) = \mathrm{U}(n)/(\mathrm{U}(r) \times \mathrm{U}(n-r)). \tag{21}$$

These "states" are also known as coherent states of the group U(n) (a la Perelomov and Gilmore). The collection of the orbits label precisely all the possible gapped free fermion ground states.

Higher dimensionality: lattices

Suppose that our gapped Hamiltonian is defined on a lattice $\mathbb{Z}^d$ and $V = \ell^2(\mathbb{Z}^d, \mathbb{C}^n)$. Fourier transformation: $V \cong L^2(T^d, \mathbb{C}^n)$. Remember that plane waves satisfy

$$\exp(ik \cdot x) = \exp(i(k + 2\pi n) \cdot x) \text{ for } x, n \in \mathbb{Z}^d,$$

so,

$$k \sim k + 2\pi n, \text{ with } n \in \mathbb{Z}^d,$$

so the set of momenta is, topologically, $T^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d$.

If the Hamiltonian is local and translation invariant it provides a smooth map

$$T^d \ni k \mapsto H(k) \in \mathrm{Herm}(\mathbb{C}^n). \tag{22}$$

Then we have a smooth map $\Phi : T^d \to \mathcal{M}(\mathbb{C}^n)$. By connectedness, it is enough to think of it as a map $\Phi : T^d \to \mathrm{Gr}_r(\mathbb{C}^n)$. Because Φ is smooth, it gives rise to a smooth vector bundle $\Phi^* E$, with $E = \{(W, v) \in \mathrm{Gr}_r(\mathbb{C}^n) \times \mathbb{C}^n : v \in W\}$. Formally, the ground state is the tensor product over T^d of the determinants of all the fibers $\det E_{\Phi(k)}$, $k \in T^d$, as a 1-dimensional subspace of $\Lambda^* V$ (Infinite Grassmannians?).

(complex) K-theory

Now two ground states (specified by maps) Φ and Φ' are "adiabatically connected" if there is a path of Hamiltonians whose groundstates at the endpoints coincide. In other words, if Φ and Φ' can be joined by homotopy. If we allow the dimensionality of V to be large, we obtain a sensible geometric classification of ground states modulo adiabatic deformation: an equivalence class of ground states is an isomorphism class of a vector bundle, i.e., the space of inequivalent equivalence classes of ground states is $\bigcup_{k\in\mathbb{N}} \mathrm{Vect}_k^{\mathbb{C}}(T^d) = [T^d, \mathcal{M}(\mathbb{C}^\infty)]$. This set is a monoid (w.r.t. direct sum) and it can be made into a group through the Grothendieck construction, yielding the K-theory of T^d [A. Kitaev, 2009].

Phase transitions

- The space $\mathrm{Herm}(\mathbb{C}^n)$ is stratified according to the multiplicity of eigenvalues $\lambda = (\lambda_1, ..., \lambda_n)$.
- The subvariety of those matrices which have multiple eigenvalues has codimension 3. [V. I. Arnold, 1995] and also [von Neumann and E. Wigner, 1927]
- Indeed, a component of highest dimension is given by those Hermitian matrices having only two equal eigenvalues call it $X \subset \mathrm{Herm}(\mathbb{C}^n)$. Then

$$\begin{aligned} f\colon \mathrm{U}(n) \times \mathbb{R}^{n-1} &\longrightarrow X \\ (U, \lambda) &\longmapsto U \operatorname{diag}(\lambda_1, \lambda_1, \lambda_2, ..., \lambda_{n-1})\ U^* \end{aligned}$$

renders $X \cong \big(\mathrm{U}(n)/(\mathrm{U}(2) \times \mathrm{U}(1)^{n-2})\big) \times \mathbb{R}^{n-1}$, so

$$\begin{aligned} \dim X &= n^2 + (n-1) - \big(2^2 + (n-2)\big) \\ &= n^2 - 3. \end{aligned}$$

Phase transitions (cont.)

- If $H : M \to \mathrm{Herm}(\mathbb{C}^n)$ is a smooth family of Hermitian matrices, then transversal crossing of the image with the subvariety of multiple eigenvalues will occur in codimension 3.
- In other words, we need at least 3 real parameters to have a generic level crossing (otherwise it can be removed by some small perturbation).

Topological phase transitions

Suppose we are given a smooth one parameter family of translation invariant free fermion systems over a $d = 2$ lattice. This gives rise to a smooth family of Hermitian matrices $\{H(k, M)\}_{(k,M)\in T^2\times I}$. $T^2 \times I$ is a 3-manifold and level crossing will generically occur at isolated points $\{(k_i, M_i)\}_{i=1}^n$. W.l.o.g., suppose $n = 1$. Outside these points, the two energy levels that are going to cross define line bundles which we can follow as the parameter M is varied.

Topological phase transitions (cont.)

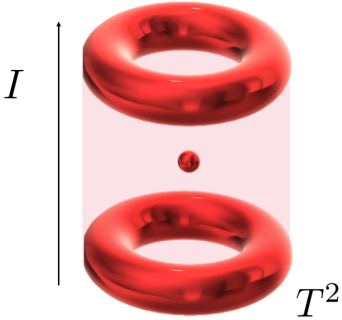


Figure: The line bundle over the torus upstairs is isomorphic to the line bundle over $T^2 \# S^2 \cong T^2$ where on the T^2 we take the vector bundle downstairs, on the S^2 we take the canonical or anti-canonical line bundle and we glue along the boundary of a disk trivially.

Topological phase transitions (cont.)

Transversality, together with the inverse function theorem, implies that there exist local coordinates (x^1, x^2, x^3) so that the two level crossing can be described by

$$H(x) = \lambda I_2 + \begin{bmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{bmatrix},$$

in which crossing corresponds to $x = 0$. The eigenvalues are $\lambda \pm |x|$. Taking x in a sphere of fixed radius, we see that the frame bundle of the eigenbundle associated with $\lambda + |x|$ corresponds to the Hopf fibration realized as

$$U = \begin{bmatrix} z_0 & -\bar{z}_1 \\ z_1 & \bar{z}_0 \end{bmatrix} \mapsto U \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} U^* = \begin{bmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{bmatrix}.$$

Topological phase transitions (cont.)

From the point of view of characteristic classes of vector bundles, and here the relevant one is $c_1(E|_{\{M\}\times T^2}) \in H^2(T^2;\mathbb{Z})$ as M varies in $[M_i, M_f]$, we can use Stokes' theorem:

$$
\begin{aligned}
0 &= \int_{[M_i,M_f]\times T^2 - B^3} dc_1(E) \\
&= \int_{T^2} c_1(E|_{\{M_f\}\times T^2}) - \int_{T^2} c_1(E|_{\{M_i\}\times T^2}) - \int_{S^2} c_1(E|_{S^2})
\end{aligned}
$$

Yielding

$$
\int_{T^2} c_1(E|_{\{M_f\}\times T^2}) = \int_{T^2} c_1(E|_{\{M_i\}\times T^2}) + \int_{S^2} c_1(E|_{S^2}).
$$

Remark: If the level crossing occurs below or above the Fermi level, the direct sum of the line bundles before and after the crossing are isomorphic, so there is no phase transition.

The Quantum Hall effect

The previous statement gives rise to the Quantum Hall effect since the transverse or Hall conductivity can be shown to be equal to, in the thermodynamic limit,

$$\sigma_{xy}(M) = \frac{q^2}{2\pi} \times \int_{T^2} c_1(E|_{\{M\}\times T^2})$$

As we vary M, the level crossing at the Fermi level induces jumps in the Hall conductivity.

Outlook: finer structures

- What we have seen in these slides refers mostly to the topology of gapped free fermions.
- The vector bundles associated to ground states of translation invariant free fermion Hamiltonians are naturally equipped with an Hermitian structure and a unitary connection, the Berry connection.
- The curvature of this connection is singular if we cross a point of phase transition.

Outlook: finer structures (cont.)

- Smooth families of ground states define maps to Grassmannians, which define, by pullback, Riemannian metrics (possibly degenerate).
- In particular, in dimension 2 they can induce complex structures on smooth surfaces.
- These Riemannian metrics and complex structures are sensitive to phase transitions. In particular, the orbifold points on the moduli space of complex structures seem to have a relevant role at the points of phase transition (at these points, it seems the induced complex structure lies exactly in this component).
- Generalizations to mixed states (Bures metric, Uhlmann connection).

Thank you!