On quantumness in multi-parameter quantum critical metrology

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Outline

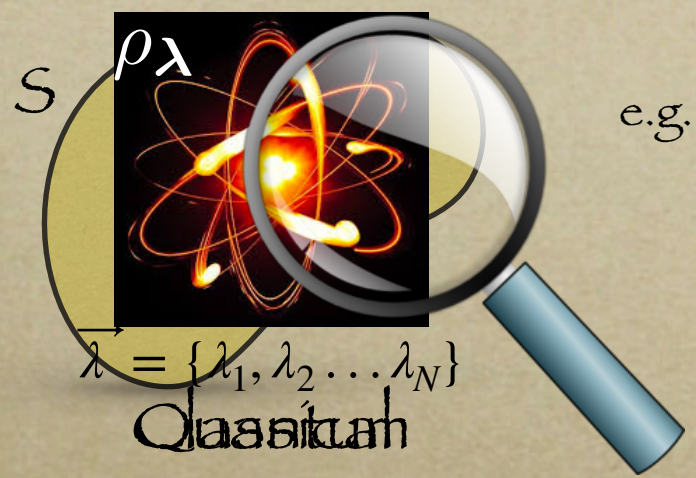
- *Introduction to Quantum Multi-Parameter Estimation*
  -Quantum Metrology & Quantum Incompatibility

- *Geometric interpretation:*
  -Geometric Information & Geometric phases
  -Uhlmann phase

- *Equilibrium Quantum Phase transition*
  -Critical Quantum Metrology & Incompatibility

- *Non-equilibrium Quantum phase transition*
  -Intro to Dissipative QPT
  -Metrology & Incompatibility in Dissipative QPT

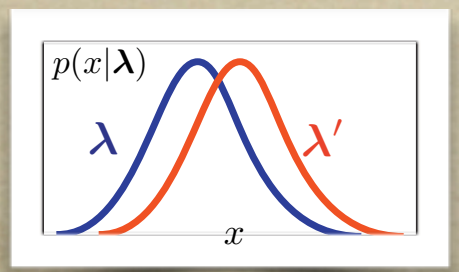
Classical&Quantum parameter estimation



Aim: Estimate some physical parameter λ

e.g. temperature, chemical potential, phase of the field…

Measurement → Estimator $\hat{\lambda}$ (unbiased)



λ_{ext}

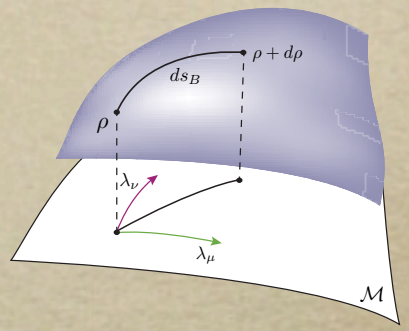
$$\min(\Delta\hat{\lambda})^2$$

Why Multiparameter Quantum Metrology?

$\vec{\lambda} = \{\lambda_1, \lambda_2 \ldots \lambda_N\}$

-Quantum Information: Information extracting capabilities of quantum measurements/Multipartite entanglement

Geometric Information: Defines a distance between quantum states

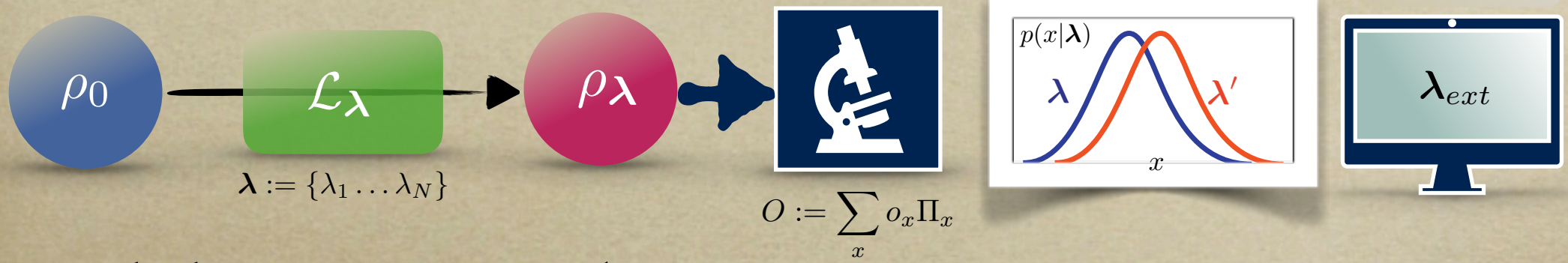


Statistical Physics: Witness of phase transition/ Understanding nature of correlations/fluctuation

Technology: Quantum meas. enhanced schemes with multi-parameters: Imaging, microscopy, spectroscopy, sensing, gravitational waves…quantum technologies.

Quantum parameter estimation:Standard Scenarios

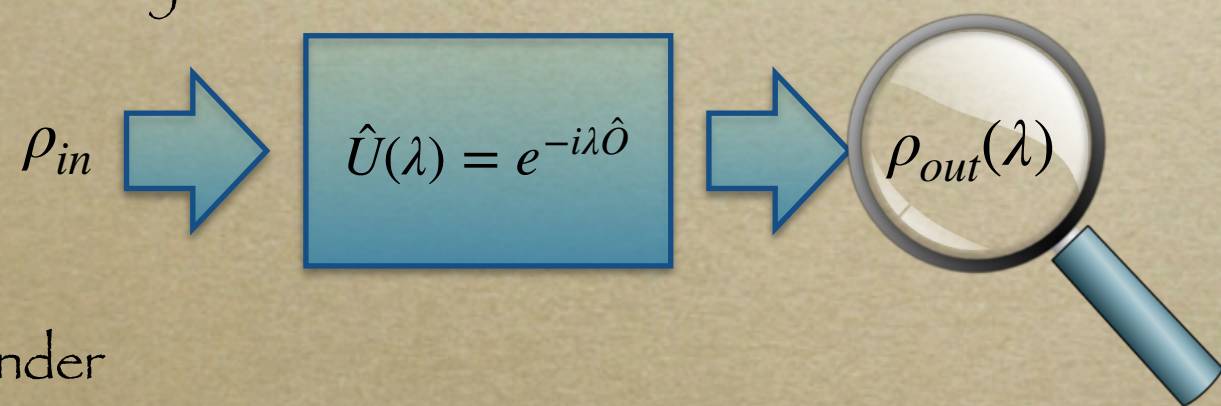
probe state → Dynamics → Final state → Measurement → Estimator $\hat{\boldsymbol{\lambda}}$ (unbiased)



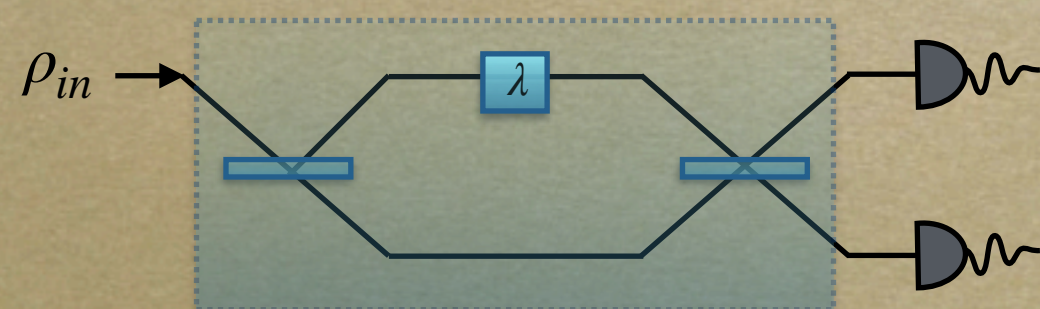
$\boldsymbol{\lambda} := \{\lambda_1 \dots \lambda_N\}$

$O := \sum_x o_x \Pi_x$

Standard Scenario: Unitary Evolution

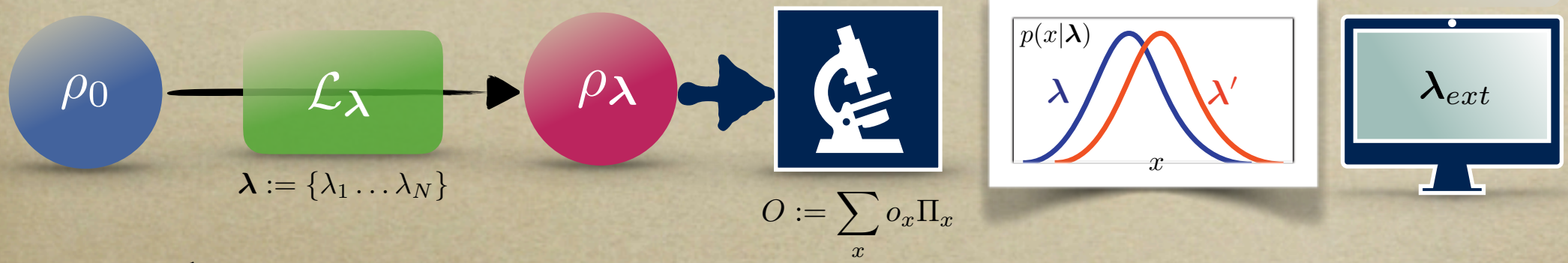


Example: Mach Zehnder

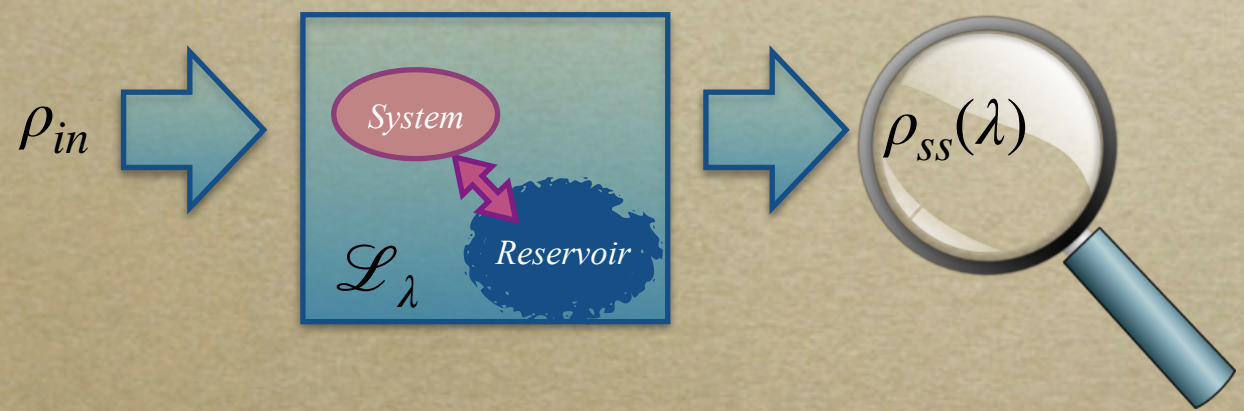


Quantum parameter estimation:Standard Scenarios

probe state → Dynamics → Final state → Measurement → Estimator $\hat{\boldsymbol{\lambda}}$ (unbiased)



Open (dissipative) Dynamics



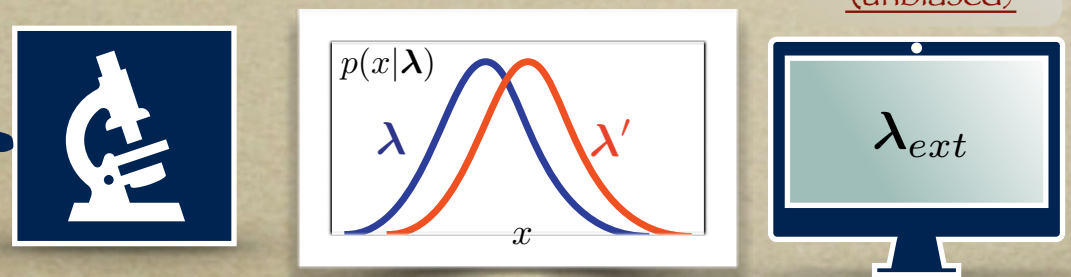
Example: System in equilibrium with a thermal Bath

Classical parameter estimation

Aim: Estimate some physical parameter λ

e.g. temperature, chemical potential, phase of the field…

Measurement → Estimator $\hat{\lambda}$ (unbiased)



$$\min(\Delta\hat{\lambda})^2$$

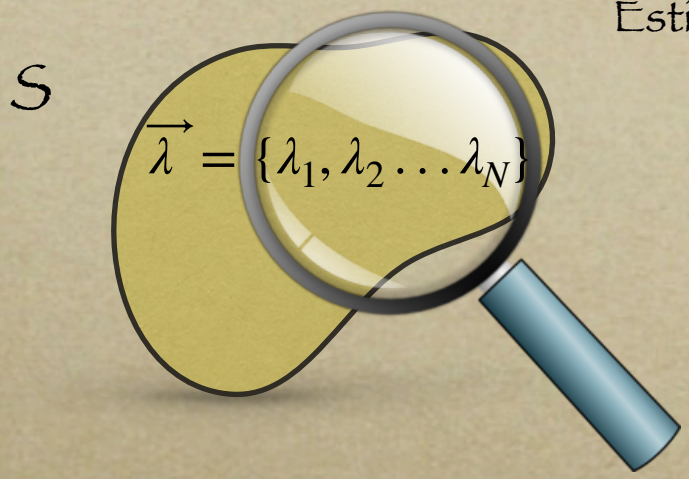
(Classical) Cramér-Rao Bound

$$(\Delta\hat{\lambda})^2 \geq \frac{1}{J} \qquad J = \sum_x \frac{1}{p(x|\lambda)} \left(\frac{\partial p(x|\lambda)}{\partial \lambda} \right)^2$$

Fisher Information

Multi-parameter estimation

Estimate more parameters simultaneously…



$$\mathrm{Cov}(\lambda) = \begin{pmatrix} \mathrm{var}(\lambda_1) & \mathrm{cov}(\lambda_1, \lambda_2) & \ldots & \mathrm{cov}(\lambda_1, \lambda_N) \\ \mathrm{cov}(\lambda_2, \lambda_1) & \mathrm{var}(\lambda_2) & \ldots & \mathrm{cov}(\lambda_2, \lambda_N) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{cov}(\lambda_N, \lambda_1) & \mathrm{cov}(\lambda_N, \lambda_2) & \ldots & \mathrm{var}(\lambda_N) \end{pmatrix}$$

$$\min(W \,\mathrm{Cov}\, \vec{\lambda})$$

(weighting matrix)

(Multiparameter) Cramér-Rao Bound

$$\mathrm{Cov}(\vec{\lambda}) \geq J^{-1}$$

$$J_{\mu\nu} = \sum_x \frac{1}{p(x|\lambda)} \left(\frac{\partial p}{\partial \lambda_\mu}\right)\left(\frac{\partial p}{\partial \lambda_\nu}\right)$$

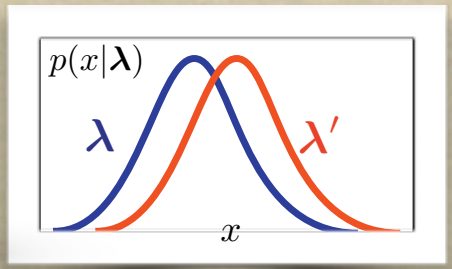
Fisher Information Matrix

Quantum parameter estimation

Quantum state → Measurement → Estimator $\hat{\lambda}$ (unbiased)

$$\min(\Delta\hat{\lambda})^2$$

$$O := \sum_x o_x \Pi_x$$



(Classical) Cramér-Rao Bound

$$(\Delta\hat{\lambda})^2 \geq 1/J$$

Fisher Information

$$J = \sum_x \frac{1}{p(x|\lambda)} \left(\frac{\partial p(x|\lambda)}{\partial \lambda} \right)^2$$

Problem: J depends on the chosen POVM!

Solution: optimise over all possible POVM!

Quantum Cramér-Rao Bound

$$(\Delta\hat{\lambda})^2 \geq 1/H$$

Always attainable!
Optimal POVM:
Eigenbasis of $L_\mu(\boldsymbol{\lambda})$

Quantum Fisher Information

$$\max_{\{\text{POVM}\}} J = H = \mathrm{Tr}\left(\rho L^2\right)$$

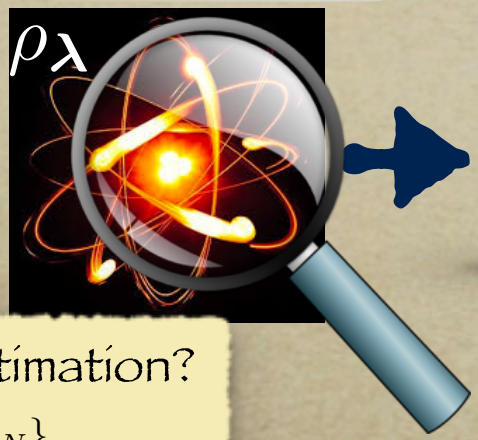
$$\frac{\partial \rho}{\partial \lambda} = \frac{1}{2}\{L, \rho\}$$

Symmetric Logarithmic Derivative

Quantum Multi-parameter estimation

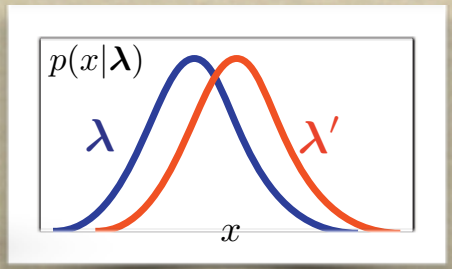
Quantum state → Measurement → Estimator $\hat{\boldsymbol{\lambda}}$ (unbiased)

$\min(W \operatorname{Cov} \vec{\lambda})$



POVM

$$O := \sum_x o_x \Pi_x$$



Simultaneous Estimation?

$\boldsymbol{\lambda} := \{\lambda_1 \dots \lambda_N\}$

SLD's are the Optimal POVM

Classical Multiparameter CRB

$$\operatorname{Cov}(\vec{\lambda}) \geq J^{-1}$$

Can always be saturated

Fisher Information Matrix

$$J_{\mu\nu} = \sum_x \frac{1}{p(x|\lambda)} \left(\frac{\partial p}{\partial \lambda_\mu}\right)\left(\frac{\partial p}{\partial \lambda_\nu}\right)$$

But if...

$$[L_\mu, L_\nu] \neq 0$$

Quantum Multiparameter CRB

$$\operatorname{Cov}(\vec{\lambda}) \geq H^{-1}$$

Cannot always be saturated!

Quantum Fisher I. Matrix

$$H_{\mu\nu} = \frac{1}{2}\langle \{L_\mu, L_\nu\} \rangle$$

Symmetric Logarithmic Derivative

$$\frac{\partial \rho}{\partial \lambda_\mu} = \frac{1}{2}\{L_\mu, \rho\}$$

Quantum Multi-parameter compatibility

When is the QCRB saturated?

Answer: iff

Multi-parameter Quantum CRB

$$\mathrm{Cov}(\hat{\boldsymbol{\lambda}}) \geq H^{-1}(\boldsymbol{\lambda})$$

(1) S. Ragy, et al., Phys. Rev. A 94, 052108 (2016).

$$[L_\mu, L_\nu] \neq 0$$

Mean Uhlmann Curvature

$$\mathcal{U}_{\mu\nu} = \frac{i}{4}\langle [L_\mu, L_\nu] \rangle = 0$$

MUC marks the ***incompatibility*** *of λ_μ and λ_ν due to* ***Heisenberg uncertainty***

We introduce:

Incompatibility Measure

$$\mathcal{R} := ||2i\mathcal{U} \cdot H^{-1}||_\infty$$

One can show that

$$0 \leq \mathcal{R} \leq 1$$

Maximal incompatibility condition

$$\mathcal{R} \simeq 1$$

A.C. et al.. On quantumness in multi-parameter quantum estimation. J. Stat. Mech. Theory Exp. 2019, 094010 (2019).

Geometric Information (Bures metrics)

How do we distinguish two pure states?

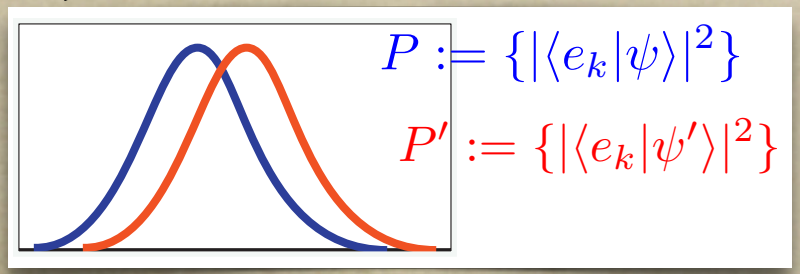
By measuring!

Observable

Outcome probability distributions

$|\psi\rangle$?

$|\psi'\rangle$



$$O = \sum_k o_k |e_k\rangle\langle e_k|$$

Distinguishing experiments

Distance in Probability Space

Distance in Projective Hilbert Space

Fisher-Rao statistical distance

$$d_{FR}(P, P') := \arccos \sum_k \sqrt{p_k p'_k}$$

Problem: it depends on the observable!

Solution: optimise over all possible observables!

Bures distance

$$d_B(\psi, \psi') := \max_O d_{FR}(P, P') = \arccos |\langle\psi|\psi'\rangle|$$

W. K. Wootters, Phys. Rev. D, 23, 357(1981)

Geometric Information (Bures metrics)

Question: What about non-pure states?

Solution:

Bures distance

$$d_B(\rho, \rho') := \arccos \mathcal{F}(\rho, \rho')$$

$$\mathcal{F}(\rho, \rho') := \sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}$$

Fidelity

S.Braunstein C.M.Caves
Phys. Rev. Lett., 72,3439 (1994).

For nearly "overlapping" states it defines a metric:

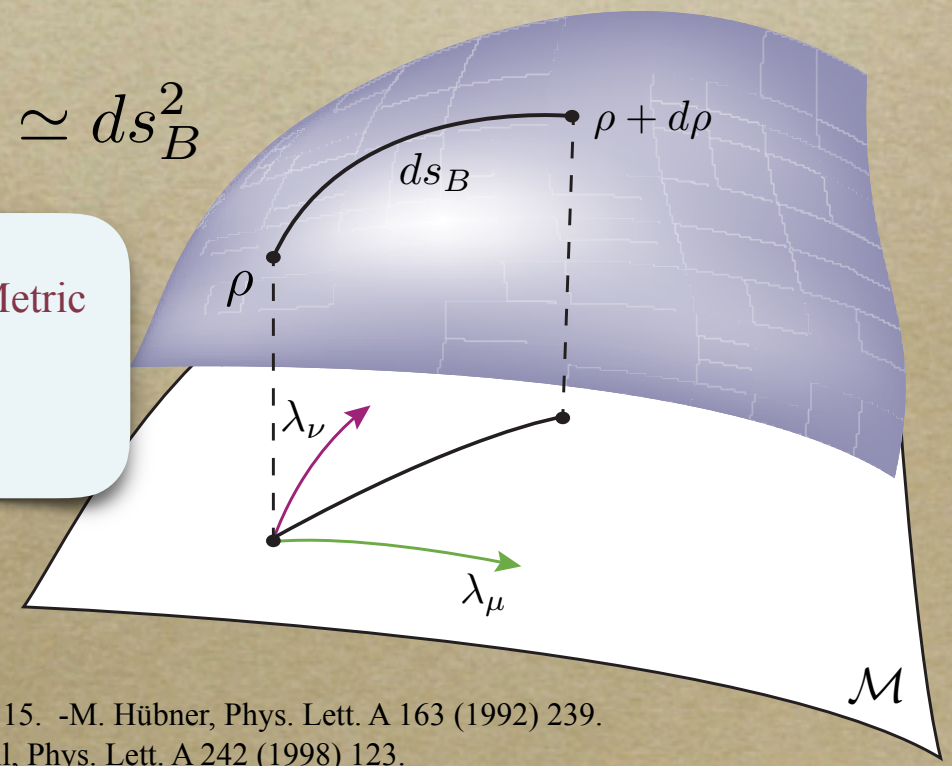
$$d[\rho, \rho + d\rho]_{\text{Bures}} \simeq ds_B^2$$

Bures Metric

$$ds_B^2 = \sum_{\mu\nu} g_{\mu\nu} d\lambda_\mu d\lambda_\nu$$

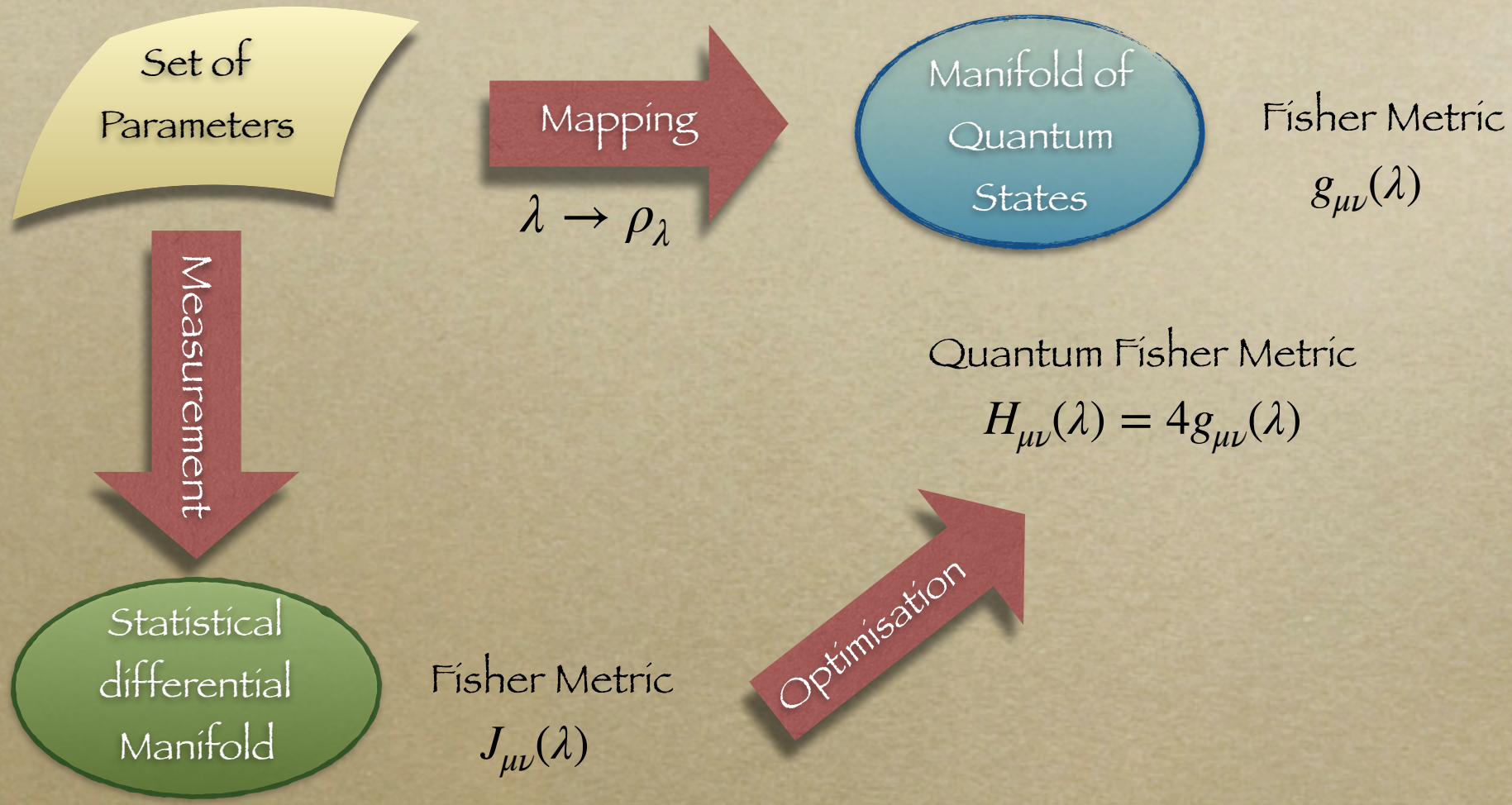
Quantum Fisher Metric

$$g_{\mu\nu} = \frac{1}{4} H_{\mu\nu}$$



-A. Uhlmann, Rep. Math. Phys. 9 (1976) 273. -R. Josza, J. Mod. Opt. 41 (1994) 2315. -M. Hübner, Phys. Lett. A 163 (1992) 239.
-P. B. Slater, J. Phys. A 29 (1996) L271; Phys. Lett. A 244 (1998) 35. - M. J. W. Hall, Phys. Lett. A 242 (1998) 123.
-J. Dittmann, J. Phys. A 32 (1999) 2663.

Geometry of quantum estimation



-A. Uhlmann, Rep. Math. Phys. 9 (1976) 273. -R. Josza, J. Mod. Opt. 41 (1994) 2315. -M. Hübner, Phys. Lett. A 163 (1992) 239.
-P. B. Slater, J. Phys. A 29 (1996) L271; Phys. Lett. A 244 (1998) 35. - M. J. W. Hall, Phys. Lett. A 242 (1998) 123.
-J. Dittmann, J. Phys. A 32 (1999) 2663.

M. Berry, Proc. Roy. Soc. A **392**, 45 (1984)

Introduction to Geometric phase

Berry phase

$$\mathscr{H}(\lambda_t)\,|n(\lambda_t)\rangle = E_n(\lambda_t)\,|n(\lambda_t)\rangle$$

$$|n(t)\rangle = e^{-i\beta_n(t)}e^{-i\gamma_n(t)}|n(0)\rangle$$

Adiabatic cyclic evolution

$$\mathscr{H}(T) = \mathscr{H}(0)$$

$$\beta_n(T) := -\int_0^T E_n(t)dt$$

Usual dynamical phase

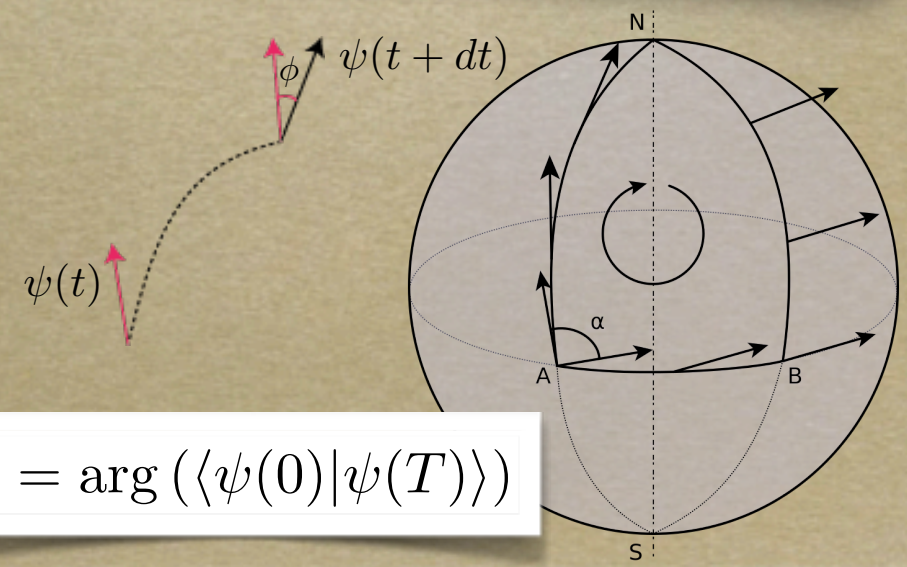
$$\gamma_n(T) = i\oint \langle n(t)|\frac{d}{dt}|n(t)\rangle dt$$

Geometric phase (GP): It depends only on the path taken in the configuration space of parameter λ.

Parallel transport:

GP is a consequence of the parallel transport condition imposed by the adiabatic evolution

$$\phi = \arg(\langle\psi(t)|\psi(t+\delta t)\rangle) = 0 + o(dt^2)$$



$$\alpha = \arg\left(\langle\psi(0)|\psi(T)\rangle\right)$$

M. Berry, Proc. Roy. Soc. A **392**, 45 (1984)

Introduction to Geometric phase

Berry phase

Adiabatic cyclic evolution

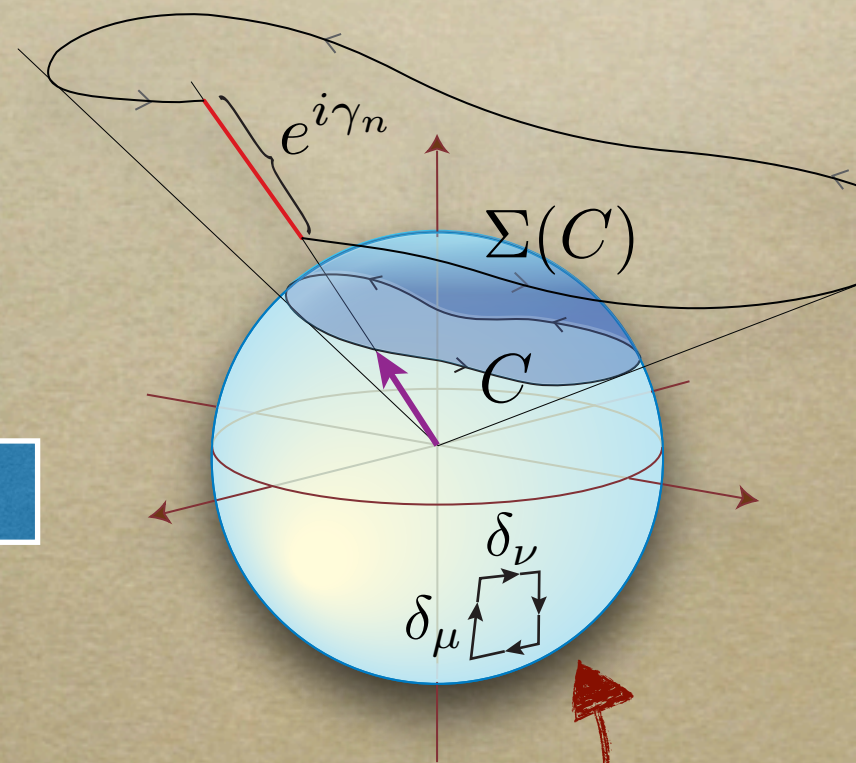
$$\mathscr{H}(T) = \mathscr{H}(0)$$

$$|\Psi_n(T)\rangle = e^{i\beta_n(T)} e^{i\gamma_n(T)} |\Psi_n(0)\rangle$$

Berry Connection
(analog to e.m. vector potential)

$$\gamma_n = \oint_C \vec{A} d\vec{\lambda}$$

$$\vec{A} := i\langle \Psi_\lambda | \vec{\nabla}_\lambda | \Psi_\lambda \rangle$$



Berry Curvature (analog to magnetic field)

$$\gamma_n = \iint_{\Sigma(C)} F_{\mu\nu} d\lambda_\mu \wedge d\lambda_\nu$$

$$\vec{F} := \vec{\nabla} \wedge \vec{A}$$

Curvature: B.P. per unit area of an infinitesimal loop

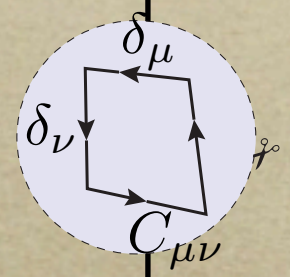
$$F_{\mu\nu} = \lim_{\delta \to 0} \frac{\gamma_{\mu\nu}}{\delta_\mu \delta_\nu}$$

Mixed state GP: Uhlmann phase[1]

Pure State $|\psi\rangle_\lambda$ → ρ_λ Mixed State

Berry Phase $e^{i\phi_B} := e^{i\oint A d\lambda}$ → $V_C := \mathcal{P} e^{i\oint A^U d\lambda}$ Uhlmann Holonomy

Berry Curvature:
BP of infinitesimal loop



Uhlmann Curvature:
Holonomy of the Infinitesimal loop

$$F_{\mu\nu} = \lim_{\delta\to 0} i \frac{1 - e^{i\phi_B}}{\delta_\mu \delta_\nu} \longrightarrow F^U_{\mu\nu} = \lim_{\delta\to 0} i \frac{1 - V_{C_{\mu\nu}}}{\delta_\mu \delta_\nu}$$

Berry Phase $\phi_B = \arg\langle\psi_0|\psi_T\rangle$ → $\phi_U = \arg(\rho V_C)$ Uhlmann GeometricPhase

Uhlmann GP (per unit area) for an infinitesimal loop

$$\mathcal{U}_{\mu\nu} := \lim_{\delta\to 0} \frac{\alpha^U[C_{\mu\nu}]}{\delta_\mu \delta_\nu} = \mathrm{Tr}\left[\rho_0 F^U_{\mu\nu}\right]$$

Mean Uhlmann Curvature

[1]Uhlmann, A. (1986). Parallel transport and "quantum holonomy" along density operators. Rep. Math. Phys., 24(2), 229–240.

Fisher Metrics & Uhlmann phase

Symmetric Log derivative

$$L_\mu \longrightarrow \partial_\mu \rho = \frac{L_\mu \rho + \rho L_\mu}{2}$$

Mean Uhlmann Curvature

$$\mathcal{U}_{\mu\nu} = \frac{i}{4}\langle [L_\mu, L_\nu] \rangle$$

Fisher Information Matrix

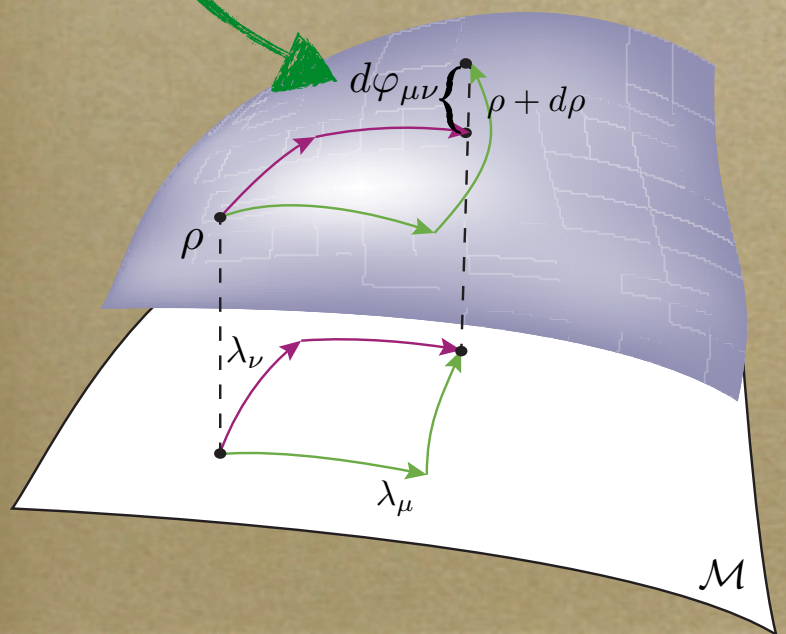
$$H_{\mu\nu} = \frac{1}{2}\langle \{L_\mu, L_\nu\} \rangle$$

Infinitesimal Uhlmann phase

$$d\varphi_{\mu\nu} = \mathcal{U}_{\mu\nu} d\lambda_\mu \wedge d\lambda_\nu$$

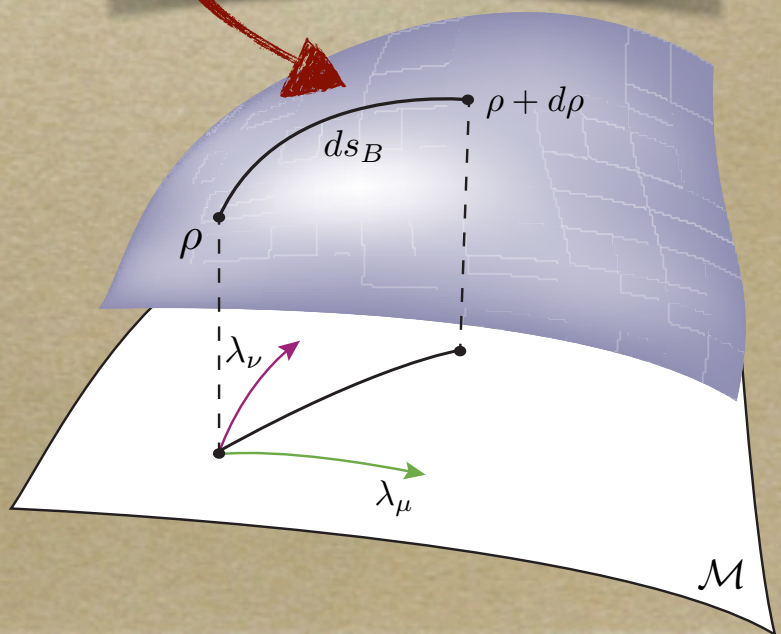
Bures Metric

$$ds_B^2 = \frac{1}{4}\sum_{\mu\nu} H_{\mu\nu} d\lambda_\mu d\lambda_\nu$$



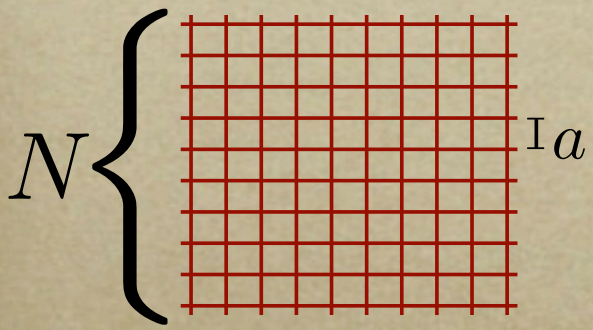
Non-commuting infinitesimal paths

$$[L_\mu, L_\nu] \neq 0$$



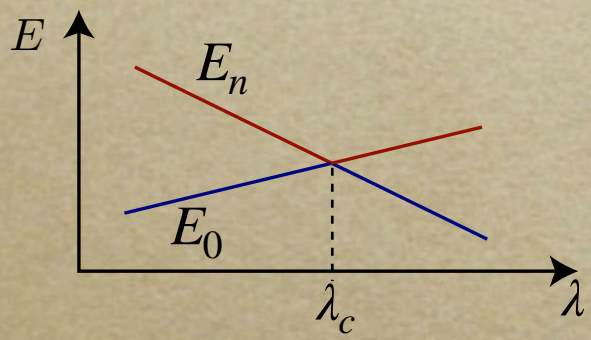
Criticality and quantum phase transition

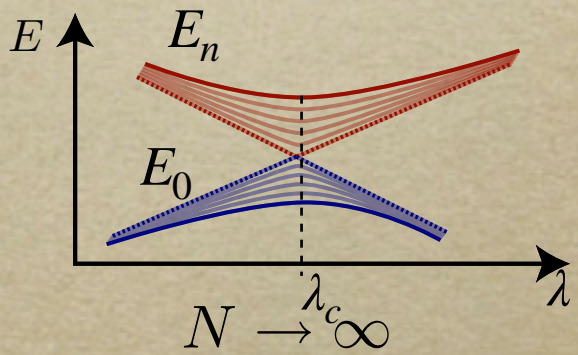
Equilibrium Quantum Phase transition



$$\mathcal{H}_\lambda = \mathcal{H}_0 + \lambda \mathcal{H}'$$

$$T = 0$$





$g = g_c$ CRITICAL POINT: Non- analyticity of the energy density

- QUALITATIVE change of the CORRELATIONS in the GS
- Point of (INFINITE) DEGENERACY of the Ground State:

Geometric Information (Bures metrics)

Anderson Orthogonality Catastrophe

Phys. Rev. Lett. 18, 1049--1051 (1967)

Ground states of a many-body Hamiltonian $H(\lambda) = H_0 + \lambda H_1$

$|\langle\psi_G(\lambda)|\psi_G(\lambda+\delta\lambda)\rangle| \propto N^{-\kappa} \quad \kappa > 0$ Size: $N \to \infty$

$$d_B(\psi_G(\lambda), \psi_G(\lambda+\delta\lambda)) \simeq 2(1 - |\langle\psi_G(\lambda)|\psi_G(\lambda+\delta\lambda)|)$$

$$ds_B^2 = \frac{1}{4}\sum_{\mu\nu} H_{\mu\nu} d\lambda_\mu d\lambda_\nu$$

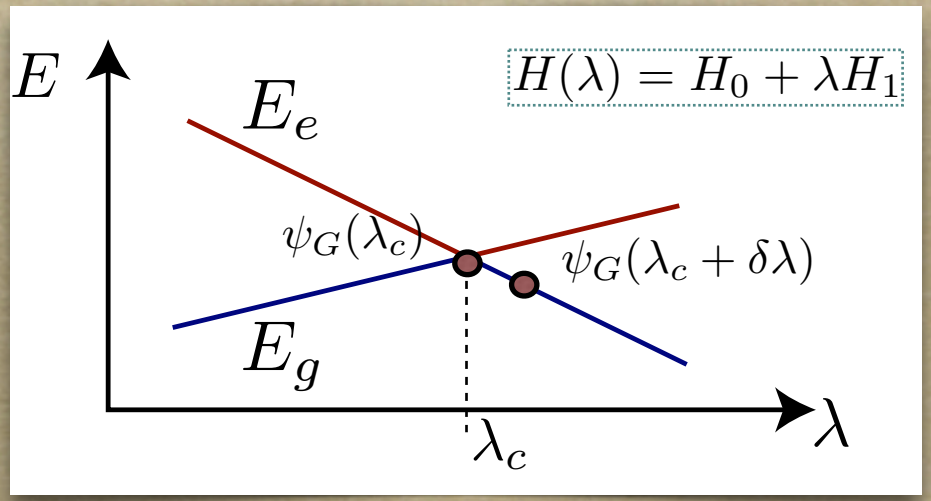
$H_{\mu\nu} \propto N^x$ *Bures metric always diverges*

In Quantum Phase Transitions:

-Maximal distinguishabilty across QPT

-Fisher information diverges at QPT ***super-extensively***

$H_{\mu\nu} \propto N^{1+x}$ Size: $N \to \infty \quad x > 0$



-P. Zanardi and N. Paunković, PRE **74**, 31123 (2006).

-P. Zanardi, P. Giorda, and M. Cozzini, Phys.Rev.Lett. 99, 100603 (2007).

S. Gu, Int. J. Mod. Phys. B, **24**, 4371 (2010)

Topological origin of Geometric phase

Conical intersections

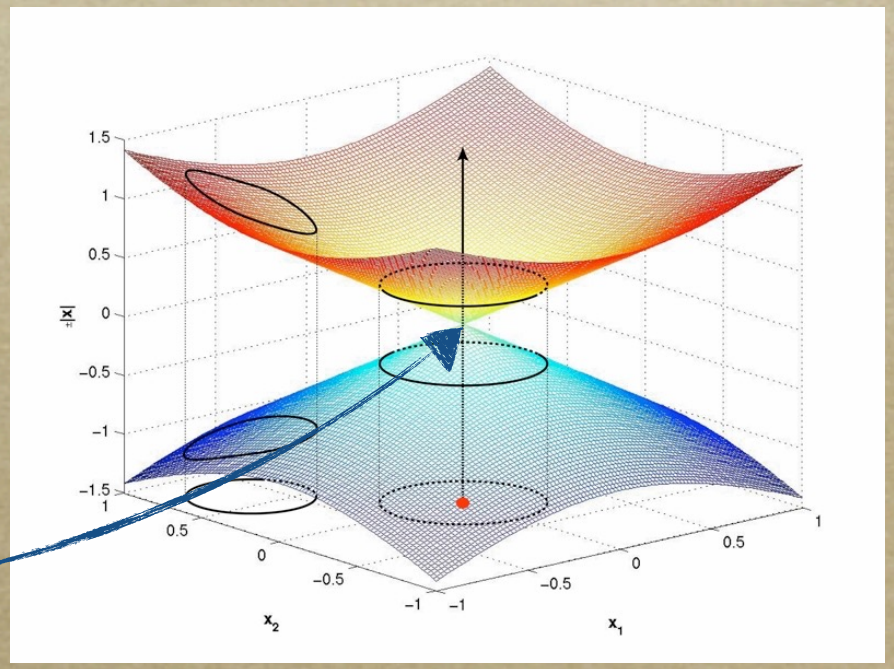
Real Hamiltonians:

$$H(r) = \begin{pmatrix} x(r) & y(r) \\ y(r) & -x(r) \end{pmatrix} = R \begin{pmatrix} \cos\theta & sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Only allowed phases: ± 1

Topological origin of -1 phase:

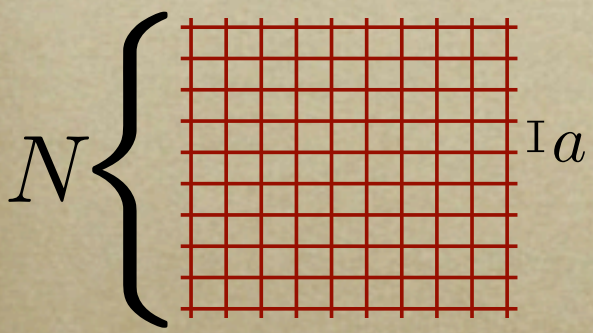
The Manifold is not Simply Connected



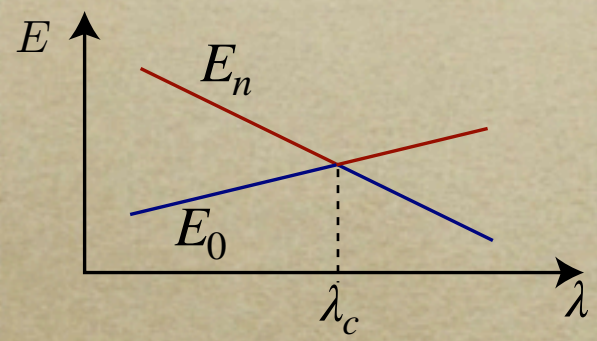
(Berry) Curvature diverges in the degeneracy point

Criticality and quantum geometric tensor

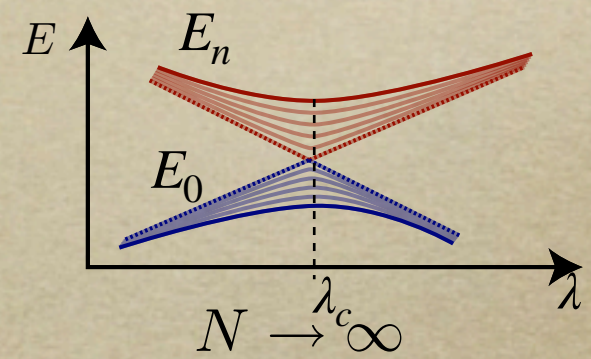
Equilibrium Quantum Phase transition



$$\mathcal{H}_\lambda = \mathcal{H}_0 + \lambda \mathcal{H}'$$



$$T = 0$$



-Point of (INFINITE) DEGENERACY of the Ground State.

- Divergence of the Quantum Geometric Tensor

$$Q_{\mu\nu} = \frac{1}{4} H_{\mu\nu} + \frac{i}{2} F_{\mu\nu} = \sum_{n \neq 0} \frac{\langle 0 | \partial_\mu \mathcal{H} | n \rangle \langle n | \partial_\nu \mathcal{H} | 0 \rangle}{(E_n - E_0)^2}$$

Fisher metric

Berry Curvature

-P. Zanardi and N. Paunković, PRE **74**, 31123 (2006).

-P. Zanardi, P. Giorda, and M. Cozzini, PRL. 99, 100603 (2007).

-L. Campos Venuti and P. Zanardi, PRL. 99, 095701 (2007).

-N. Paunković and V. R. Vieira, Phys. Rev. E **77**, 11129 (2008).

-S. Gu, Int. J. Mod. Phys. B, **24**, 4371 (2010)

Fisher Information & Finite temperature

$$\rho(\lambda,\beta) \propto e^{-\beta\mathcal{H}_\lambda}$$

$$\rho = \sum_k p_k |k\rangle\langle k|$$

$$Q_{\mu\nu} = \frac{1}{4} H_{\mu\nu} + \frac{i}{2}\mathcal{U}_{\mu\nu} = \frac{1}{4}\langle L_\mu L_\nu\rangle$$

Fisher metric

MUC

$$H_{\mu\nu} = H^c_{\mu\nu} + H^q_{\mu\nu}$$

Quantum part of FI → Due to parameter dependent eigenstates, $\partial_\mu |k\rangle\langle k| \neq 0$

Classical part of FI → Due to parameter dependent probabilities, $\partial_\mu p_k \neq 0$

$$H^c_{\mu\nu} = \sum_k \frac{\partial_\mu p_k \partial_\nu p_k}{p_k}$$

$$Q^q_{\mu\nu} = \frac{1}{4} H^q_{\mu\nu} + \frac{i}{2}\mathcal{U}_{\mu\nu} = \sum_{j\neq k}\left(\frac{p_j - p_k}{p_j + p_k}\right)^2 p_k \frac{(\partial_\mu\mathcal{H})_{kj}(\partial_\nu\mathcal{H})_{jk}}{(E_k - E_j)^2}$$

$$(O)_{kj} = \langle j|O|k\rangle$$

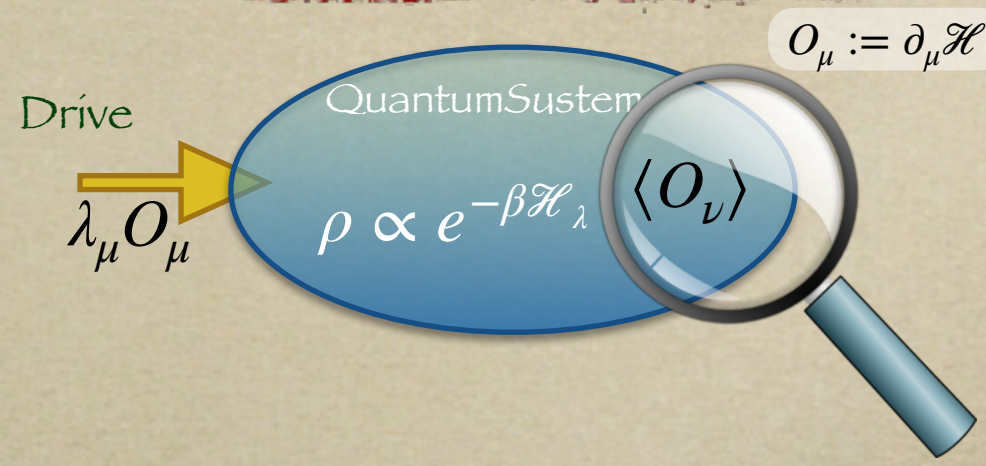
-P. Zanardi, P. Giorda, and M. Cozzini, PRL. 99, 100603 (2007).
-N. Paunković and V. R. Vieira, Phys. Rev. E **77**, 11129 (2008).
-S. Gu, Int. J. Mod. Phys. B, **24**, 4371 (2010)
-AC., Spagnolo, B., Valenti, D. Scientific Reports 8(1),9852 (2018)

Fisher Information & Linear Response

$$H^c = \beta\chi_{el}$$

Static susceptibility

$$\chi_\beta = \chi_{el} + \chi_{vV}$$



$$H^q = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{\omega^2} \tanh^2\left(\frac{\omega\beta}{2}\right) S^+(\omega)$$

$$\mathcal{U} = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{\omega^2} \tanh^2\left(\frac{\omega\beta}{2}\right) S^-(\omega)$$

Dynamical structure factor

$$S_{\mu\nu}(\omega) := \int_{-\infty}^{\infty} dt e^{i\omega t} \langle O_\mu(t) O_\nu(0) \rangle$$

$$S^{\pm}_{\mu\nu} := \frac{S_{\mu\nu} \pm S_{\nu\mu}}{2}$$

1. Leonforte, L., Valenti, D., Spagnolo, B. & AC, Sci. Rep. 9, 9106 (2019).
2. L. Campos Venuti and P. Zanardi, PRL. 99, 095701 (2007).
3. Gabbrielli, M., Smerzi, A. & Pezzè, L. Sci. Rep. 8, 1–18 (2018).
4. Hauke, P., Heyl, M., Tagliacozzo, L. & Zoller, P. Nat. Phys. 12, 778–782 (2016).
5. M. Prokopenko, J. T. Lizier, O. Obst, and X. R. Wang, Phys. Rev. E 84,041116(2011)

Quantum Critical Scaling

$$H^q = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{\omega^2} \tanh^2\left(\frac{\omega}{2T}\right) S^+(\omega)$$

$$\mathcal{U} = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{\omega^2} \tanh^2\left(\frac{\omega}{2T}\right) S^-(\omega)$$

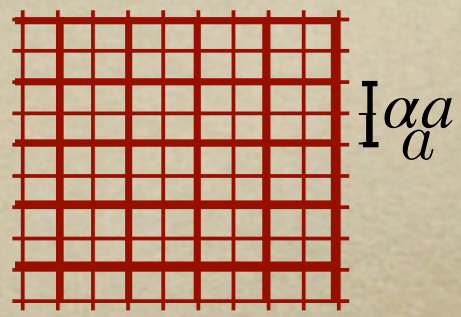
$$h^q \to H^q/\ell^d$$
$$u \to \mathcal{U}/\ell^d$$

Intensive Quantities

$$O_\mu = \sum_r O_\mu(r)$$

Local Interactions

z: dynamical critical exponent



$$x \to \alpha^{-1}x \qquad \alpha > 0$$
$$\omega \to \alpha^{-z}\omega \qquad T \to \alpha^{-z}T$$

Scaling transformation

$$O_\mu \to \alpha^{-\Delta_\mu} O_\mu$$

Scaling dimension

$$\int_{-\infty}^{\infty} d\omega S^{\pm}_{\mu\nu}(\omega) \sim \langle [O_\mu, O_\nu]_\pm \rangle \sim \alpha^{-\Delta^{\pm}_{\mu\nu}}$$

$$\Delta^+_{\mu\nu} = \Delta_\mu + \Delta_\nu$$
$$\Delta^-_{\mu\nu} \leq \Delta_\mu + \Delta_\nu$$

$$h^q_{\mu\nu} \to h^q_{\mu\nu}\alpha^{-\Delta^h_{\mu\nu}} \qquad \Delta^h_{\mu\nu} = \Delta_\mu + \Delta_\nu - d - 2z$$

$$u_{\mu\nu} \to u_{\mu\nu}\alpha^{-\Delta^u_{\mu\nu}} \qquad \Delta^u_{\mu\nu} = \Delta^-_{\mu\nu} - d - 2z$$

1. A.C., B. Spagnolo, D.Valenti, J. Stat. Mech. Theory Exp. 2019, 094010 (2019).
2. L. Campos Venuti and P. Zanardi, PRL. 99, 095701 (2007).

Quantum Critical Scaling

$$\mathscr{R} \to \mathscr{R}\alpha^{-\Delta_R} \qquad \Delta_R = \Delta_\mu + \Delta_\nu - \Delta^-_{\mu\nu} \leq 0$$

Logarithmic Corrections

$$\mathscr{R} \to \mathscr{R}\alpha^{-\Delta_R}\,(\log\alpha)^{\tilde{\Delta}_R}\,\alpha^{-\Delta_R}$$

Small temperature, non-critical (large system)
$\ell^{-1}, T^{1/z} \ll \tilde{\lambda}^{\nu}$

$$\alpha \sim \xi \sim \tilde{\lambda}^{-\nu}$$

Critical, finite temperature (large system)
$\ell^{-1}, \tilde{\lambda}^{\nu} \ll T^{1/z}$

$$\alpha \sim T^{-1/z}$$

Finite system, Critical, small temperature
$\ell^{-1}, \tilde{\lambda}^{\nu} \ll T^{1/z}$

$$\alpha \sim \ell^{-1}$$

$\tilde{\lambda} = |\lambda - \lambda_c|/\lambda_c$

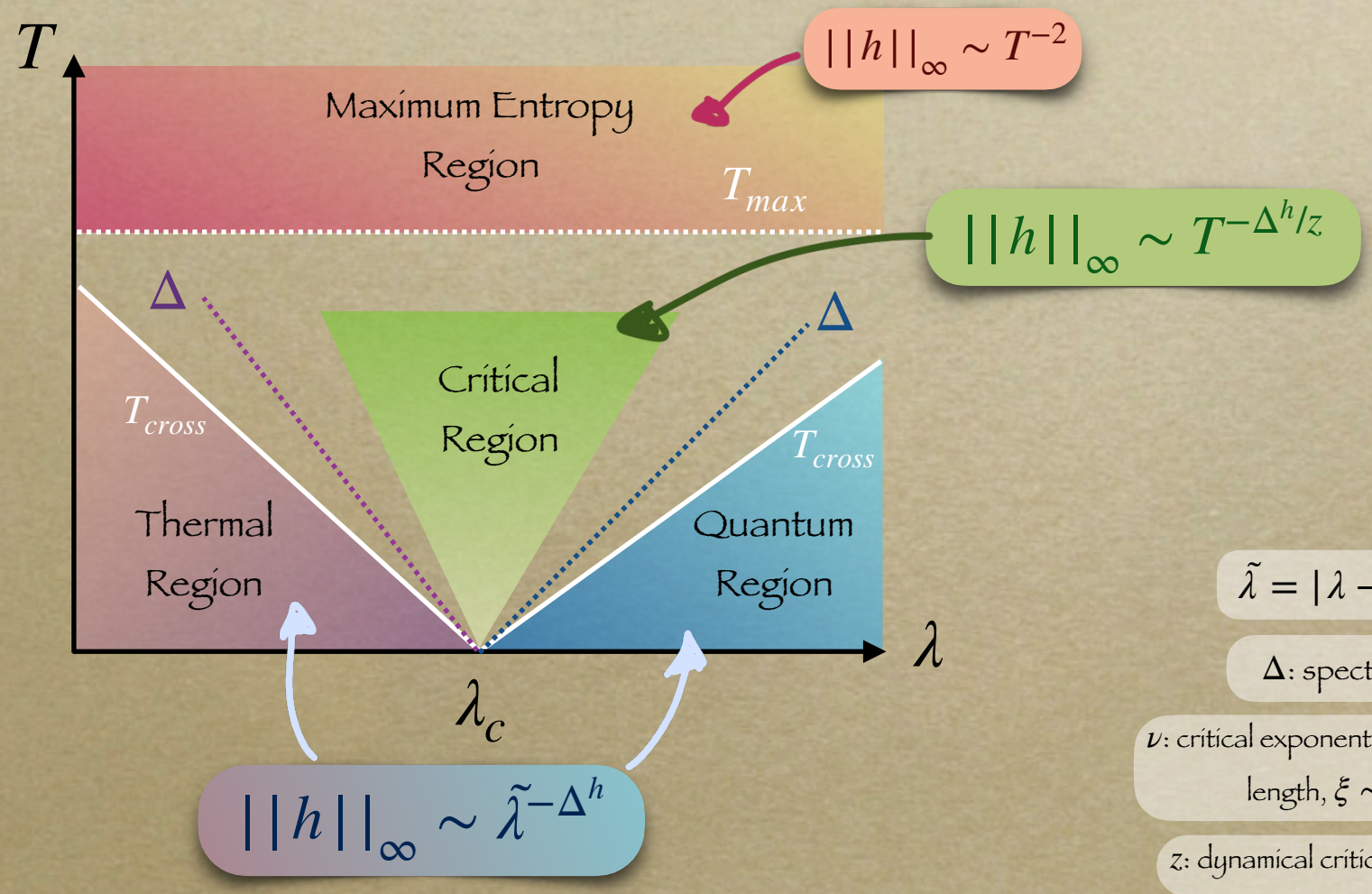
ν: critical exponent of correlation length, $\xi \sim \tilde{\lambda}^{-\nu}$

z: dynamical critical exponent

1. Leonforte, L., Valenti, D., Spagnolo, B. & AC, Sci. Rep. 9, 9106 (2019).

Fisher metric at thermal equilibrium

$$\rho_\lambda \propto e^{-\beta \mathcal{H}_\lambda}$$



$\tilde{\lambda} = |\lambda - \lambda_c|/\lambda_c$

Δ: spectral gap

ν: critical exponent of correlation length, $\xi \sim \tilde{\lambda}^{-\nu}$

z: dynamical critical exponent

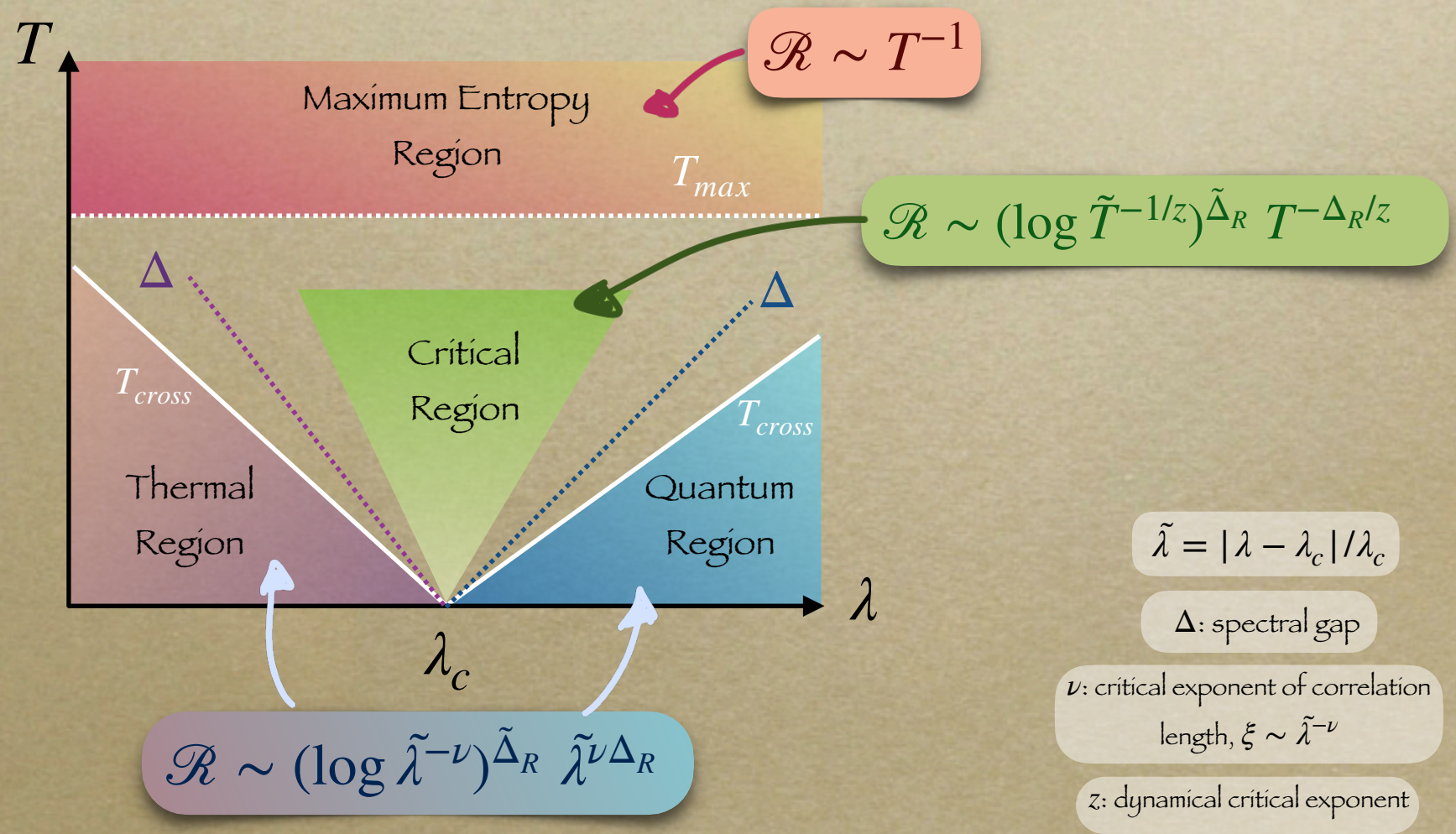
A.C. et al.. On quantumness in multi-parameter quantum estimation. J. Stat. Mech. Theory Exp. 2019, 094010 (2019).

Incompatibility Measure at thermal equilibrium

Incompatibility Measure

$$\mathscr{R} := ||2i\mathscr{U} \cdot H^{-1}||$$

$$\rho_\lambda \propto e^{-\beta H_\lambda}$$



$\tilde{\lambda} = |\lambda - \lambda_c|/\lambda_c$

Δ: spectral gap

ν: critical exponent of correlation length, $\xi \sim \tilde{\lambda}^{-\nu}$

z: dynamical critical exponent

A.C. et al.. On quantumness in multi-parameter quantum estimation. J. Stat. Mech. Theory Exp. 2019, 094010 (2019).

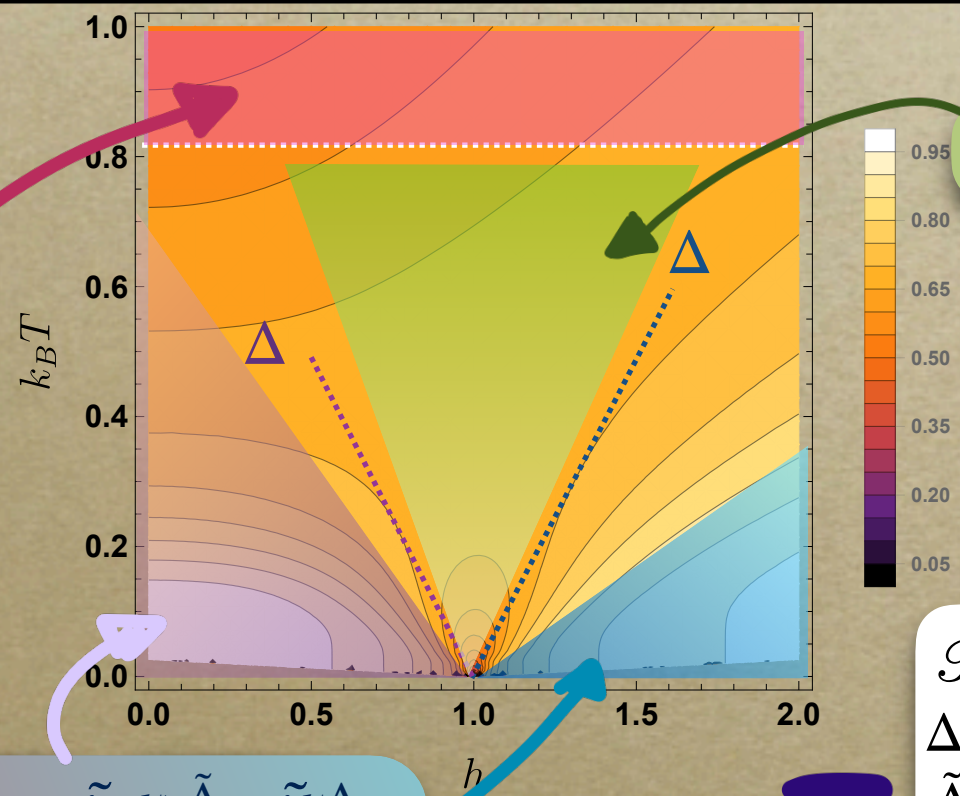
Example: (Rotated) Ising chain

$$\rho_{h\phi} \propto e^{-\beta \mathcal{H}_{h\phi}}$$

$$\mathcal{H}_{Ising} = -\frac{1}{2}\sum_{j=1}^{M}\left[\sigma_j^x \sigma_{j+1}^x + h\sigma_j^z\right] \qquad g_\phi = e^{i\phi \sum_n \sigma_n^z} \quad \rightarrow \quad H(h,\phi) = g_\phi H_{Ising} g_\phi^\dagger$$

$\nu = 1$, and $z = 1/2$

$\mathcal{R} \sim T^{-1}$



$\mathcal{R} \sim (\log \tilde{T}^{-1/z})^{\tilde{\Delta}_R}\ T^{-\Delta_R/z}$

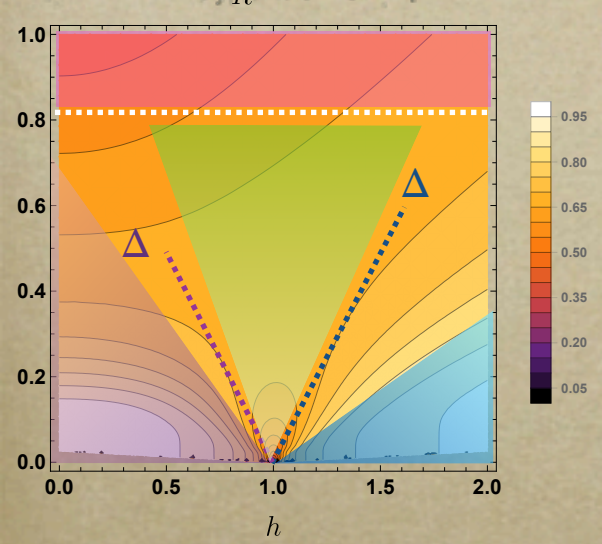
$\mathcal{R} \sim \log(T^{-1})\ T^{\frac{1}{2}}$

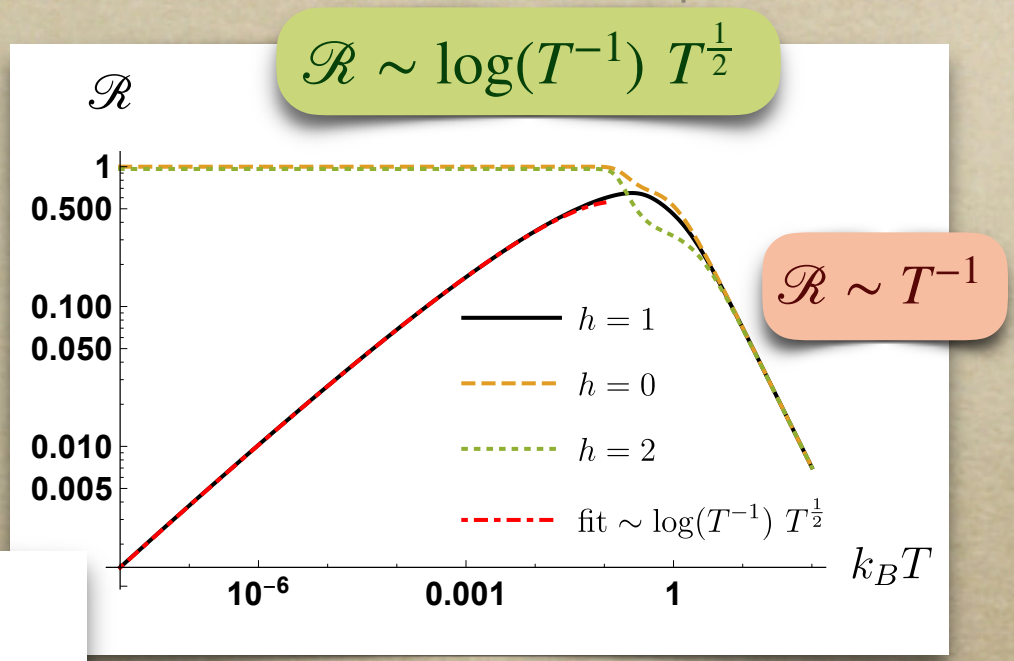
$\mathcal{R} \sim (\log \tilde{\lambda}^{-\nu})^{\tilde{\Delta}_R}\ \tilde{\lambda}^{\nu\Delta_R}$

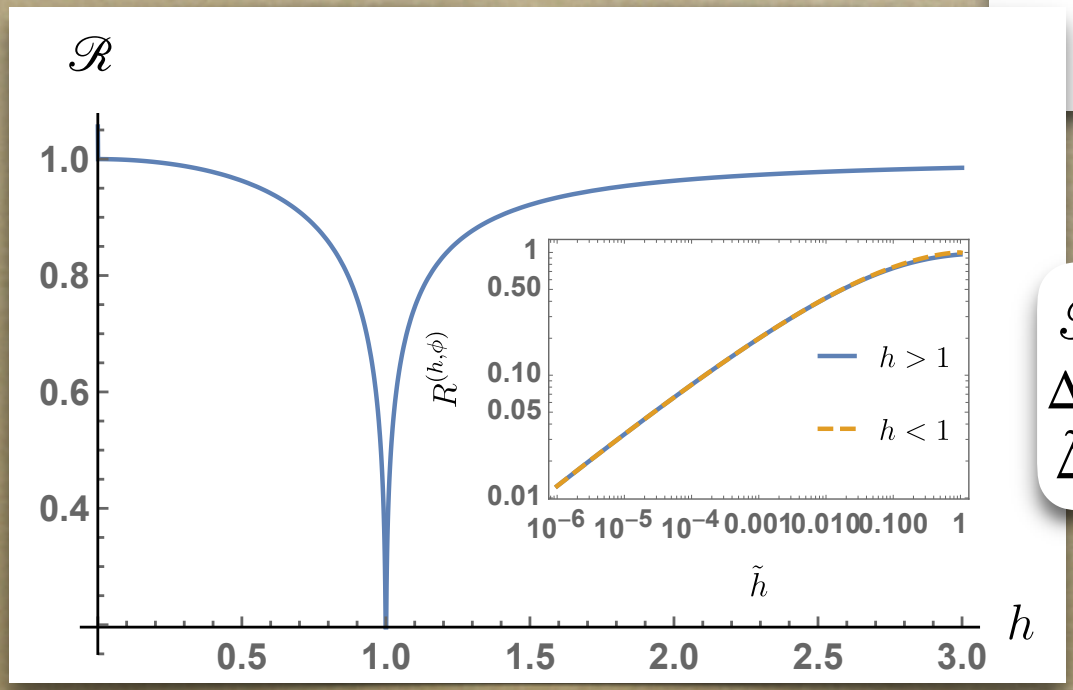
$\mathcal{R} \sim -\log(\tilde{h})\ \tilde{h}^{\frac{1}{2}}$,
$\Delta_R = -1/2$,
$\tilde{\Delta}_R = 1$

A.C. et al.. On quantumness in multi-parameter quantum estimation. J. Stat. Mech. Theory Exp. 2019, 094010 (2019).

Example: (Rotated) Ising chain







$$\mathscr{R} \sim -\log(\tilde{h})\ \tilde{h}^{\frac{1}{2}},$$
$$\Delta_R = -1/2,$$
$$\tilde{\Delta}_R = 1$$

A.C. et al.. On quantumness in multi-parameter quantum estimation. J. Stat. Mech. Theory Exp. 2019, 094010 (2019).

Equilibrium vs Non-equilibrium Many-body

Equilibrium Quantum phase Transition:

$$\mathcal{H}_\lambda = \mathcal{H}_0 + \lambda \mathcal{H}'$$

Competing dynamics

Standard scenario in Condensed Matter (& Quantum Optics)

$$\mathcal{H}(\lambda)|0(\lambda)\rangle = E_0(\lambda)|0(\lambda)\rangle$$

Many Body Hamiltonian

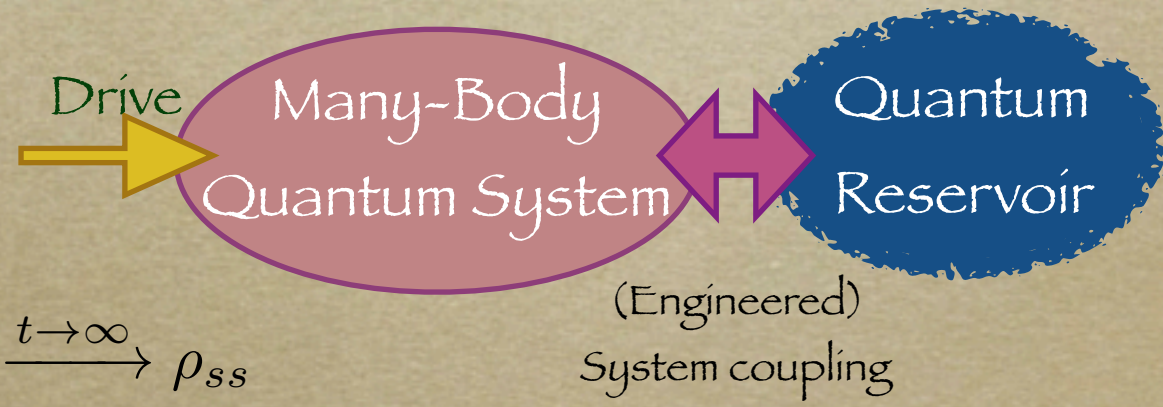
$$\rho \propto e^{-H/k_B T} \xrightarrow{T\to 0} |E_q\rangle\langle E_q|$$

Cooling to T=0

- *Interesting ground state properties*
- *Quantum Phases*

Non-equilibrium

Quantum Optics



Competing dynamics

$$\frac{d\rho}{dt} = -i[H_{sys}, \rho] + \mathcal{L}(\rho)$$

Master Equation

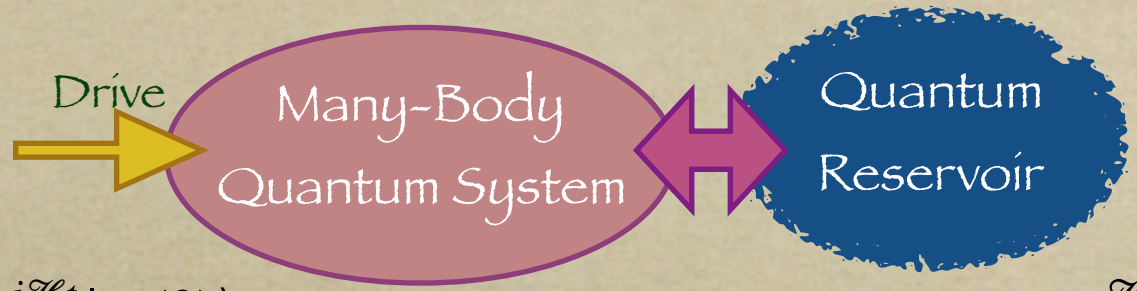
$$\rho(t) \xrightarrow{t\to\infty} \rho_{ss}$$

Steady state (Possibly pure)

Liouvillian (Engineered) Dynamics:

✓many body (pure states) / driven quantum phases
✓mixed states ~ "finite temperature"
✓useful and interesting fermion states

Non-equilibrium Dissipative Dynamics



$$|\psi(t)\rangle = e^{-i\mathscr{H}t}|\psi(0)\rangle$$

$$\mathscr{H} = \mathscr{H}_S + \mathscr{H}_{SR} + \mathscr{H}_R$$

Approximations:

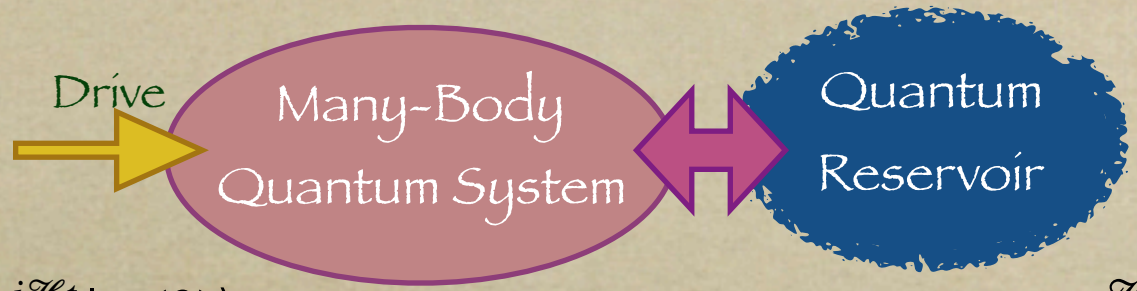
1) $\rho(0) = \rho_S \otimes \rho_R$ *(System-Reservoir initially un-correlated)*
2) $||H_{SR}|| << ||H_R||$ *(weak coupling)*
3) *The environment <u>equilibrates fast</u> (Markov approximation)*
4) *Fast rotating terms are averaged-out (secular approximation)*

Lindblad Master Equation

$$\partial_t \rho = \mathscr{L}[\rho] = -i[\mathscr{H}_S, \rho] + \sum_{i=1}^{n} \left(L_i \rho L_i^\dagger - \frac{1}{2}\left\{ L_i^\dagger L_i, \rho \right\} \right)$$

Jump operators L_i

Non-equilibrium Dissipative Dynamics



$$|\psi(t)\rangle = e^{-i\mathscr{H}t}|\psi(0)\rangle$$

$$\mathscr{H} = \mathscr{H}_S + \mathscr{H}_{SR} + \mathscr{H}_R$$

The most general evolution:

1) $\mathrm{Tr}(\rho(t)) = 1$ *(trace preserving)*
2) *Completely Positive*
3) $\mathscr{E}(t+s) = \mathscr{E}(t) \circ \mathscr{E}(t)$
4) $\mathscr{E}(0) = \mathscr{I}$

Lindblad Master Equation

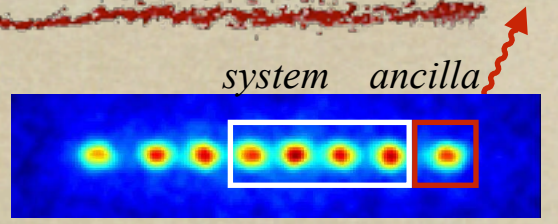
$$\partial_t \rho = \mathscr{L}[\rho] = -i[\mathscr{H}_S, \rho] + \sum_{i=1}^{n} \left(L_i \rho L_i^\dagger - \frac{1}{2}\left\{ L_i^\dagger L_i, \rho \right\} \right)$$

Jump operators L_i

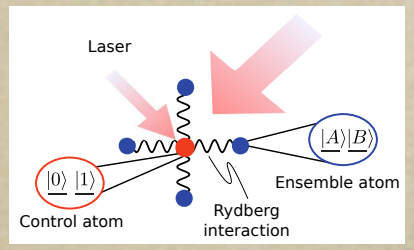
Experimental platforms

- *Ion trap experiments [1]*
- *Rydberg atoms [2]*
- *Jaynes-cummings lattice in CQED [3]*
- *Ultracold atoms[4]…*

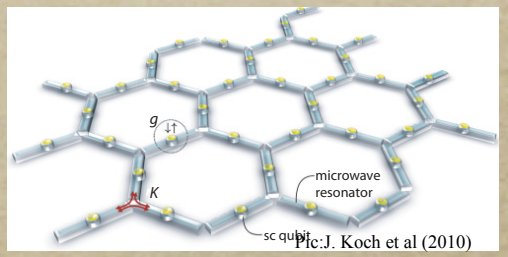
[1]



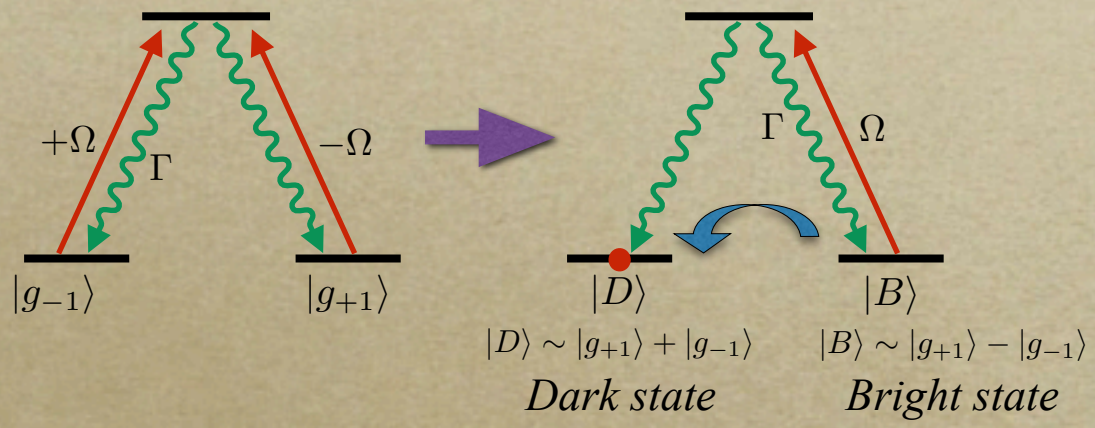
[2]



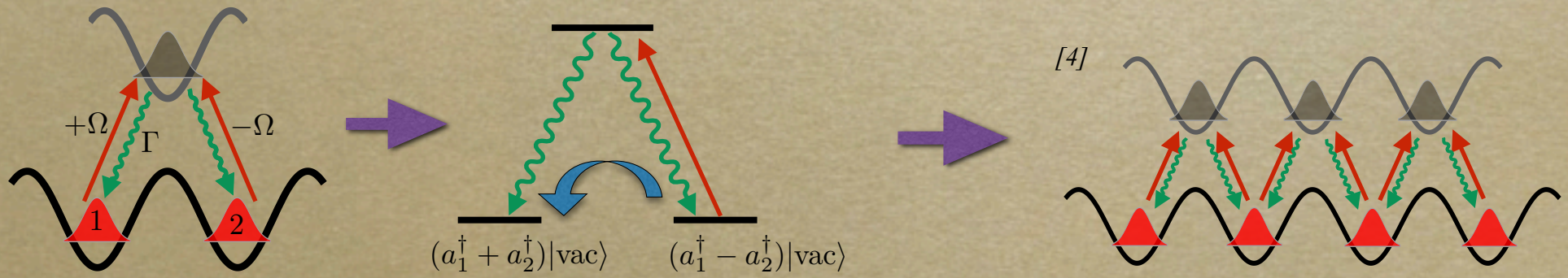
[3]



Pic:J. Koch et al (2010)



[4]



Example: State symmetrization (dissipative BEC condensation) via coherent driving of atoms in optical lattice

[1] J. T. Barreiro et al, Nature **470**, 486 (2011).
P. Schindler et al. Nat. Phys. **9**, 361 (2012).

[2] H. Weimer et al., Nat. Phys. **6**, 382 (2009).

[4] S. Diehl et al., Nat. Phys. **4**, 878 (2008).

[3] M. J. Hartmann et al.Nat. Phys. 2, 849 (2006)
A.D. Greentree et al, Nat. Phys. 2, 856 (2006)
D.G. Angelakis et al.Phys. Rev. A (RC) 76, 031805 (2007)

Criticality driven by Dissipative Dynamics

QPT vs Dissipative-QPT

$$\partial_t \rho = -i[H, \rho]$$

Dynamics: Hamiltonian

$$\partial_t \rho = \mathcal{L}[\rho] = -i[H, \rho] + \sum_{\mu=1}^{M} \left(L_\mu \rho L_\mu^\dagger - \frac{1}{2} \{ L_\mu^\dagger L_\mu, \rho \} \right)$$

Dynamics: Linbladian

GS of Hamiltonian (Pure)

$$\rho_0 = |\psi_{gs}\rangle\langle\psi_{gs}|$$

Steady state (generally Mixed)

$$\mathcal{L}[\rho_0] = 0$$

Criticality: Non analiticity of GS wave func.
Hamiltonian Gap/Infinite c. length

$$\Delta_H \to 0 \quad \longleftrightarrow \quad \xi \to \infty$$

Criticality: Non analyticity of Steady state.
(Dissipative Gap)/Infinite c. length

$$\Delta_R \to 0 \quad \longleftarrow \quad \xi \to \infty$$

?? !!

Pure state Metrics: Fubini-Study

Mixed state Metrics: Bures/Fisher Information

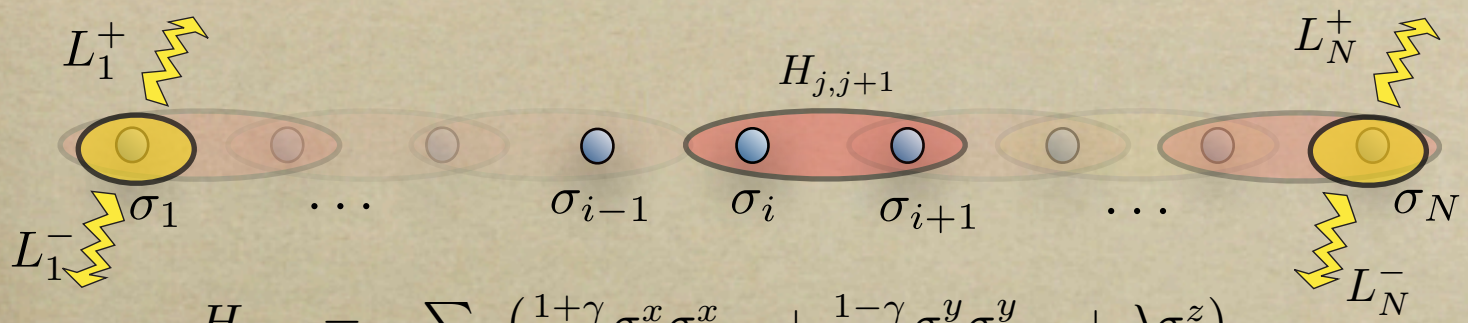
Pure state Geometric Phase: Berry phase

Mixed state Geometric Phase: Uhlmann phase

T. Prosen and I. Pizorn, Phys. Rev. Lett. 101, 105701 (2008).
S. Diehl, A. Micheli, A. Kantian, B. Kraus, H. Buchler, andP. Zoller, Nat. Phys. 4, 878 (2008).

S. Diehl, A. Tomadin, A. Micheli, R. Fazio, and P. Zoller, Phys. Rev. Lett. 105, 015702 (2010).
E.G. Dalla Torre, E. Demler, T. Giamarchi, and E.Altman, Nat. Phys. 6, 806 (2010).

Example: Boundary driven XY



Boundary Reservoirs

$$L^-_{1,N} \propto \sigma^-_{1,N} \quad L^+_{1,N} \propto \sigma^+_{1,N}$$

$$H_{XY} = -\sum_j \left(\tfrac{1+\gamma}{2}\sigma^x_j\sigma^x_{j+1} + \tfrac{1-\gamma}{2}\sigma^y_j\sigma^y_{j+1} + \lambda\sigma^z_j\right)$$

T. Prosen and I. Pizorn, Phys. Rev. Lett. 101, 105701 (2008)

$\lambda > \lambda_c$ Short range phase

$$\langle\sigma^z_j\sigma^z_k\rangle \simeq e^{-|j-k|/\xi} \qquad \xi \simeq \sqrt{2\lambda_c/(\lambda-\lambda_c)}/8$$

$\lambda \simeq \lambda_c$ Critical

$$\langle\sigma^z_j\sigma^z_k\rangle \simeq |j-k|^{-4}$$

$\lambda < \lambda_c$ Long range phase

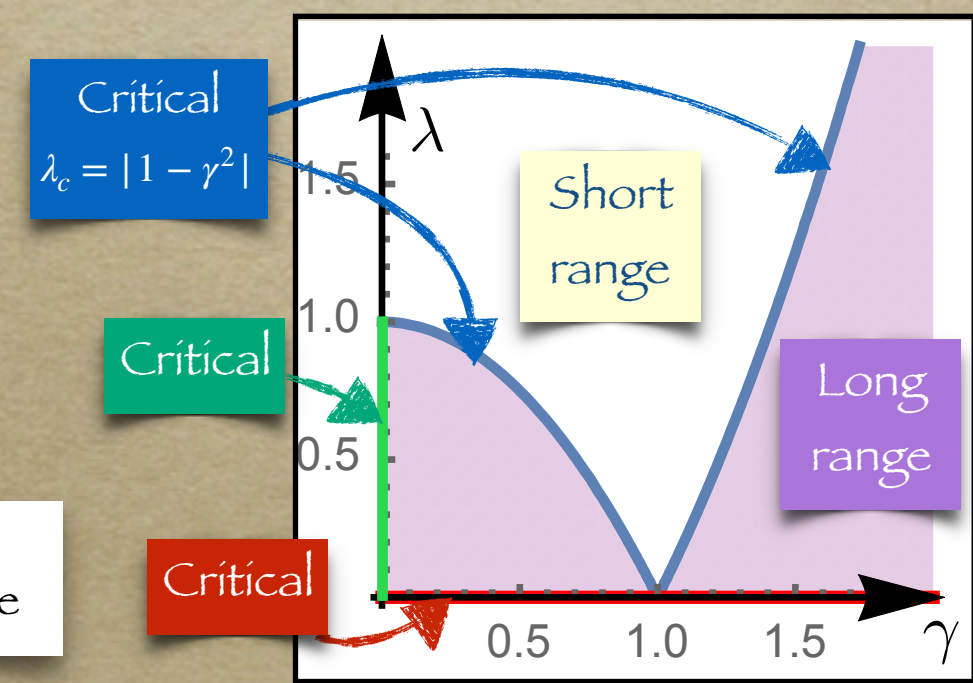
$\langle\sigma^z_j\sigma^z_k\rangle \rightarrow$ Non decaying structures

$\gamma = 0$ Critical

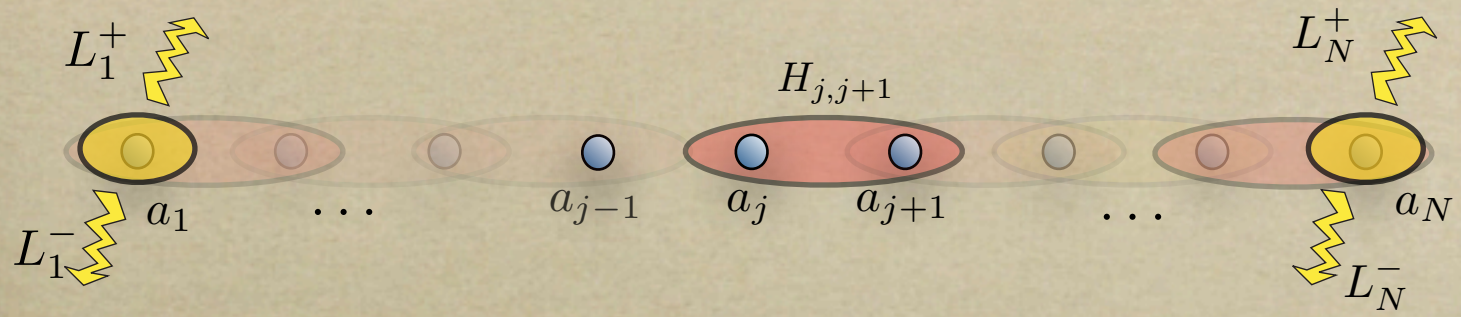
$\lambda = 0$ Critical

Short range phases embedded in a long range phase

Phase diagram



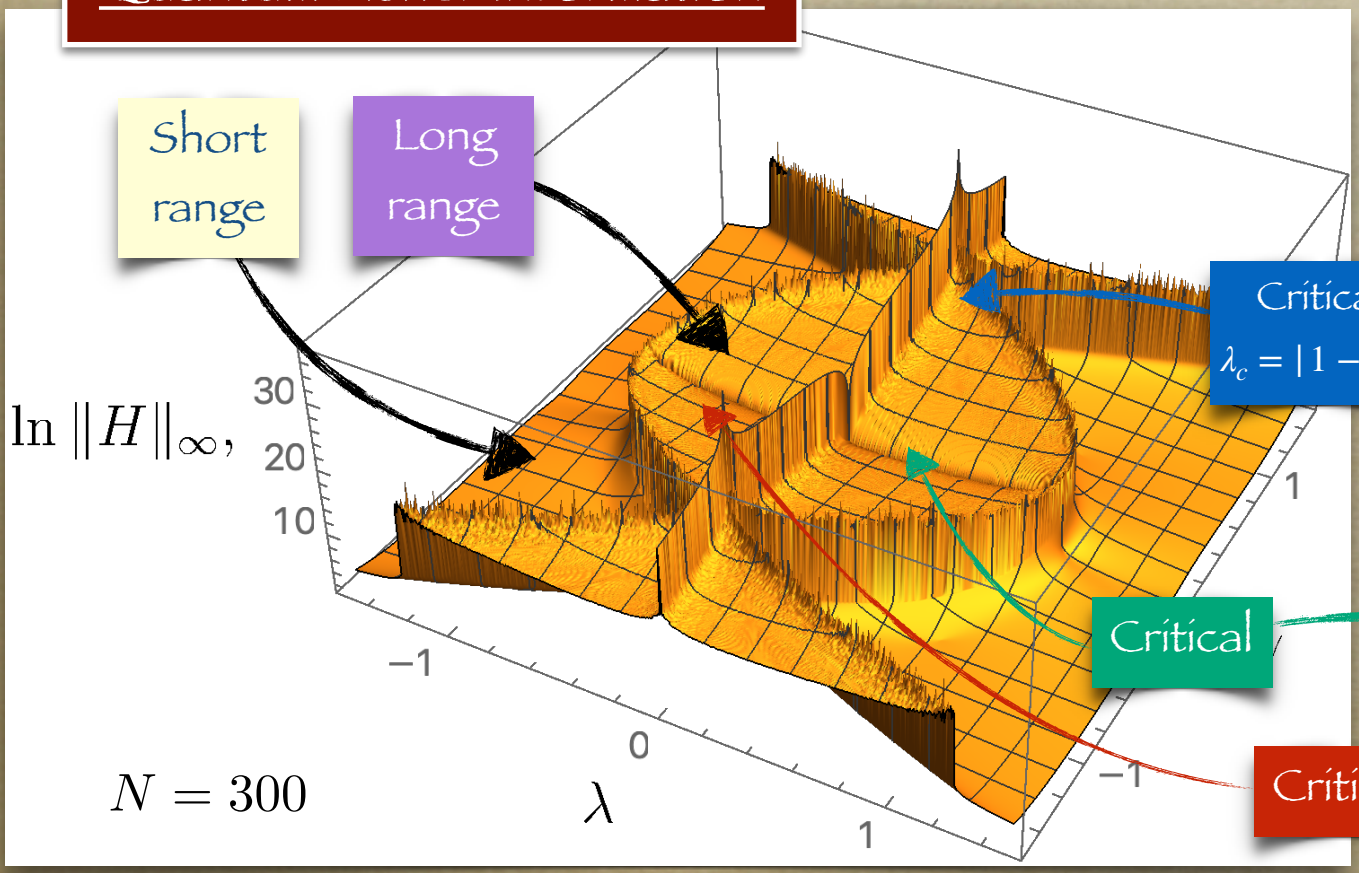
Example: Boundary driven XY



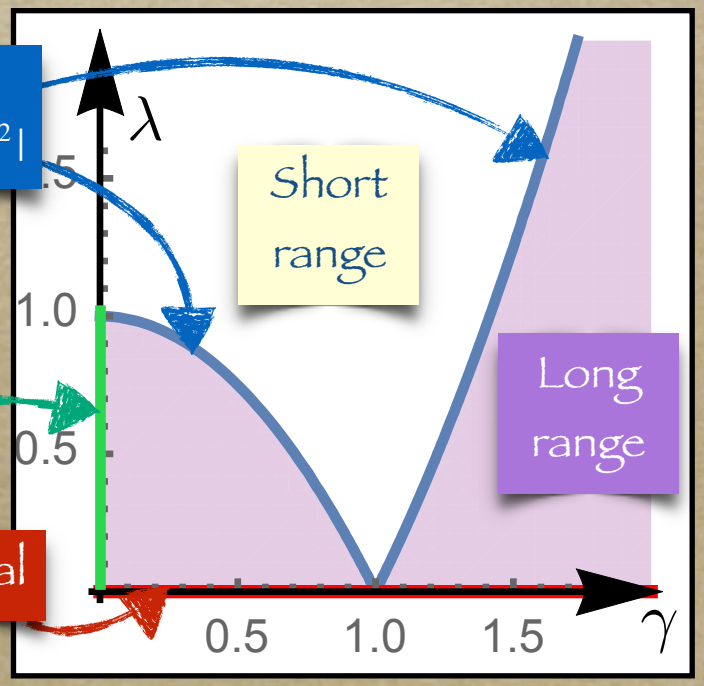
Boundary Reservoirs

$$L^-_{1,N} \propto a_{1,N} \quad L^+_{1,N} \propto a^\dagger_{1,N}$$

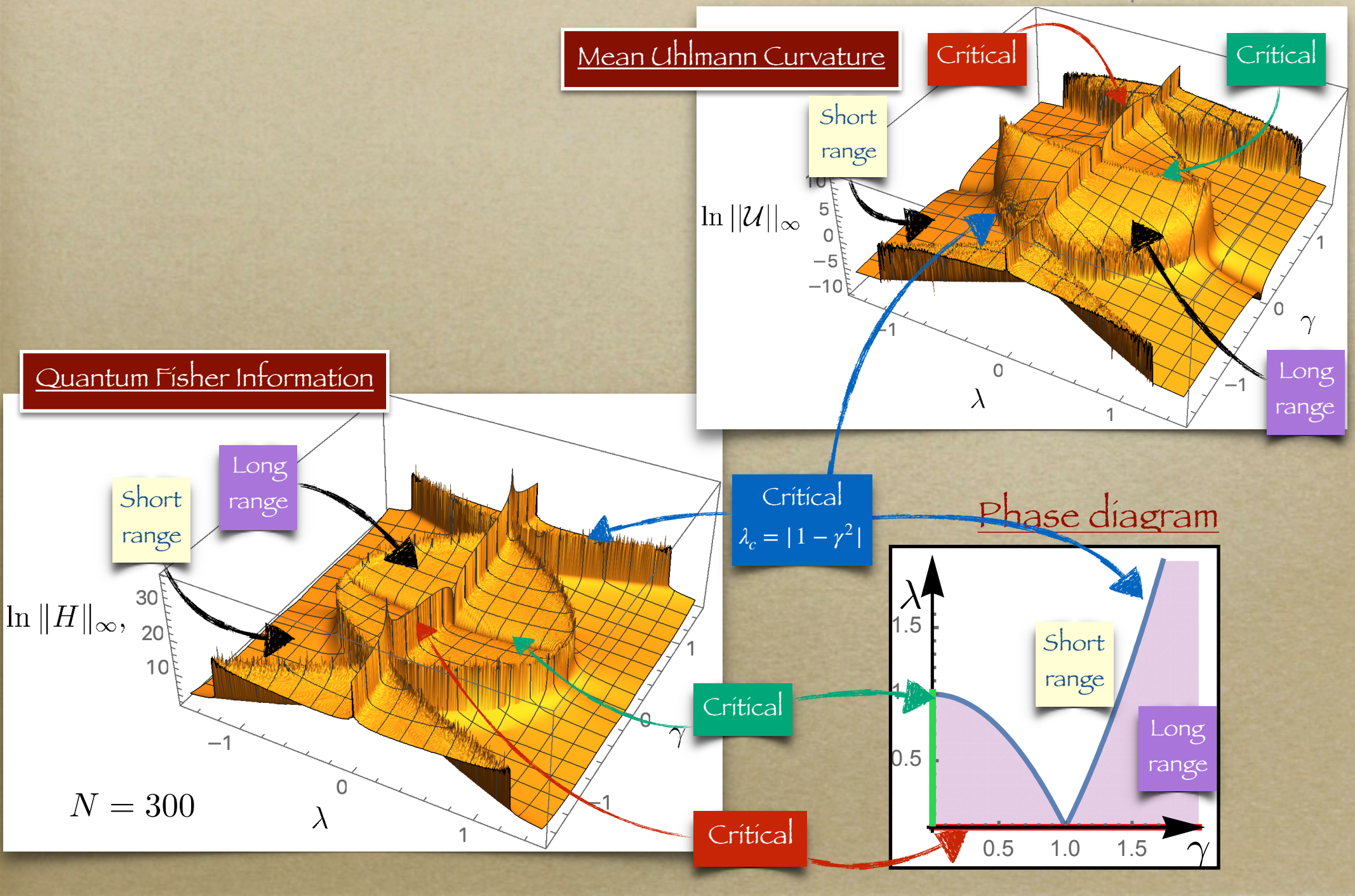
Quantum Fisher Information



Phase diagram



Example: Boundary driven XY



Boundary driven XY: Scaling laws

Phase	Parameters	Δ	$\|\|H\|\|_\infty$	$\|\|\mathcal{U}\|\|_\infty$	$\mathcal{R}$
Critical	$\lambda = 0$	N^{-3}	N^6	N^3	$N^{-1/2}$
Long range	$0 < \|\lambda\| < \lambda_c$	N^{-3}	N^3	N^2	const.
Critical	$\lambda \simeq \lambda_c$	N^{-5}	N^6	const.	$N^{-7/2}$
Short range	$\lambda > \lambda_c$	N^{-3}	N	const.	N^{-1}
Critical	$\gamma = 0, \|\lambda\| < \lambda_c$	N^{-3}	N^2	N^3	N^{-1}

(Quantum) critical

Maximal incompatibility

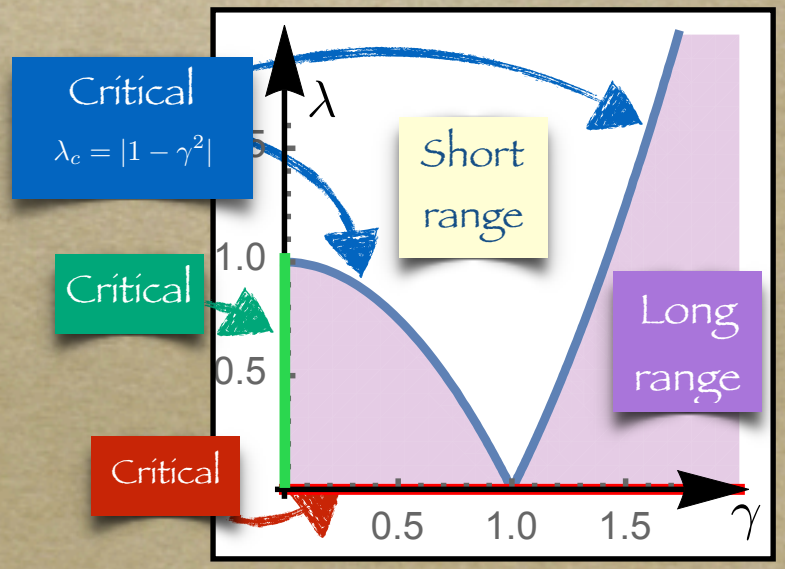
$$\lim_{N\to\infty} \mathscr{R} = 1$$

Asymptotic incompatibility

No criticality

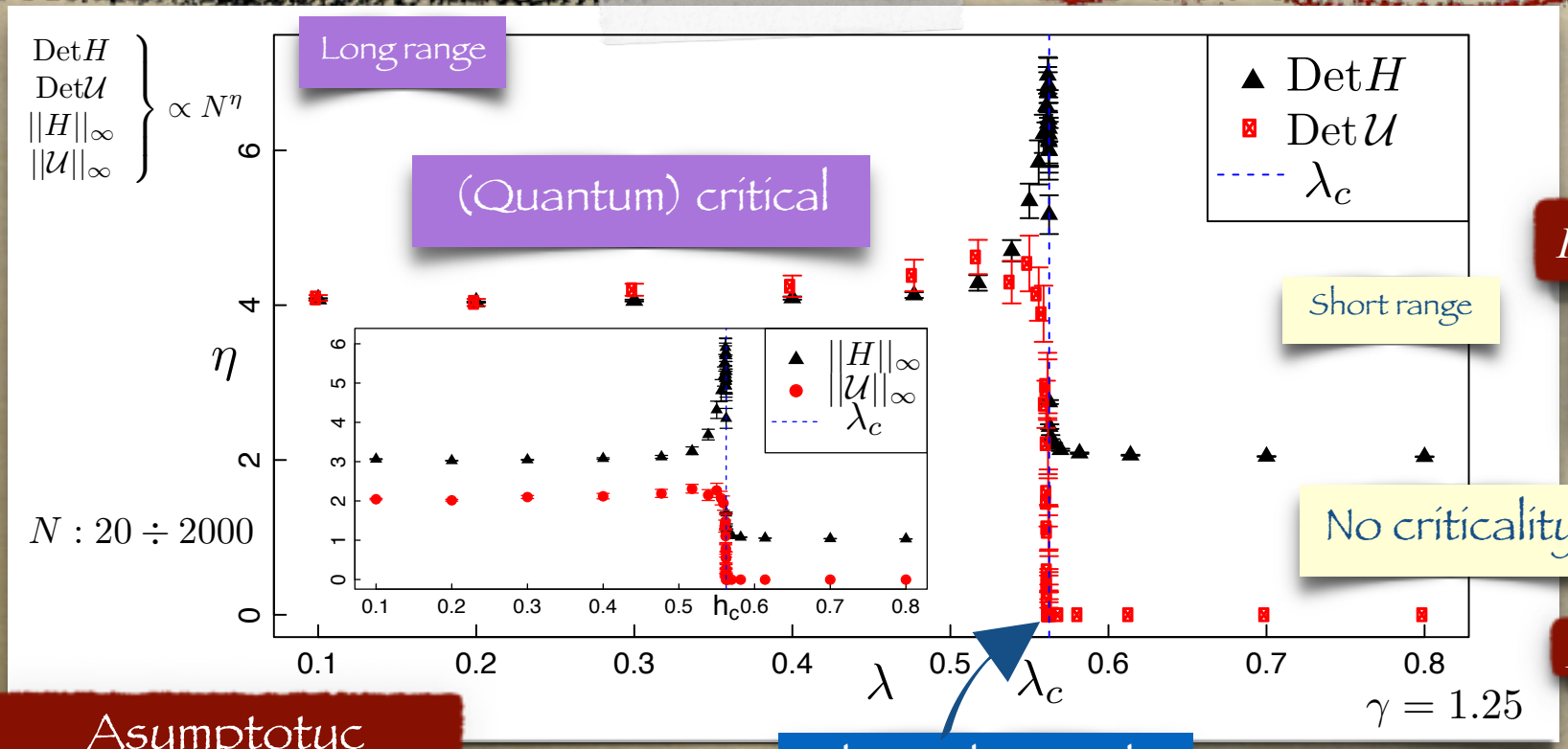
(Classical) critical

Phase diagram



(1) A.C.,, B. Spagnolo, and D. Valenti, Sci. Rep. **8**, 9852 (2018).
A.C., Spagnolo, B., Dubkov, A. A. & Valenti, *J. Stat* **2019**, 094010 (2019).
L. Banchi, P. Giorda, and P. Zanardi, Phys. Rev. E 89, 22102 (2014).

Boundary driven XY: Scaling laws



Incompatibility

$$\mathscr{R} = \sqrt{\frac{\det 2\mathcal{U}}{\det H}}$$

Bound on $\mathscr{R}$

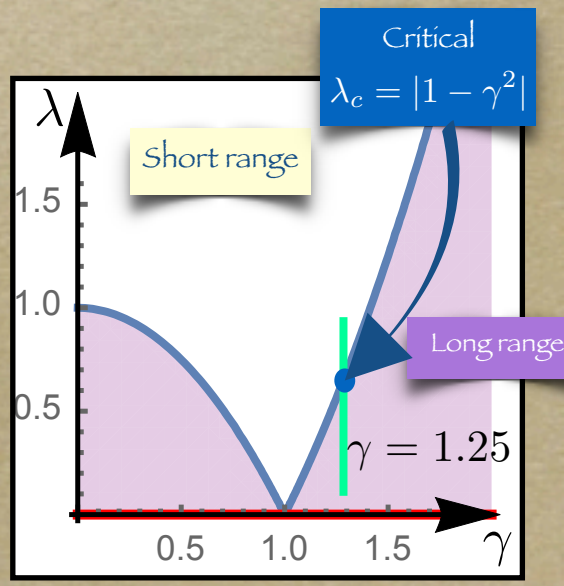
$$\det H \geq \det 2\mathcal{U}$$

Asymptotyc incompatibility

Discriminate the Quantum vs Classical nature of the parameter estimation problem, at the thermodynamical limit

Maximal incompatibility

$$\det H \sim \det 2\mathcal{U}$$
$$\mathscr{R} \sim 1$$



(1) A.C.,, B. Spagnolo, and D. Valenti, Sci. Rep. **8**, 9852 (2018)..

Conclusions

-We introduced a measure of incompatibility in multiparameter-quantum metrology

-We applied this idea to many-body models in equilibrium and out-of equilibrium open systems.

-In equilibrium models we demonstrated a general scaling behaviour, in different regions of the Phase diagram

-Close to criticality, multi-parameter incompatibility scales to zero.

-Non equilibrium phase transition show a richer structure, with different scaling as criticality is approached.

Thanks for the attention!