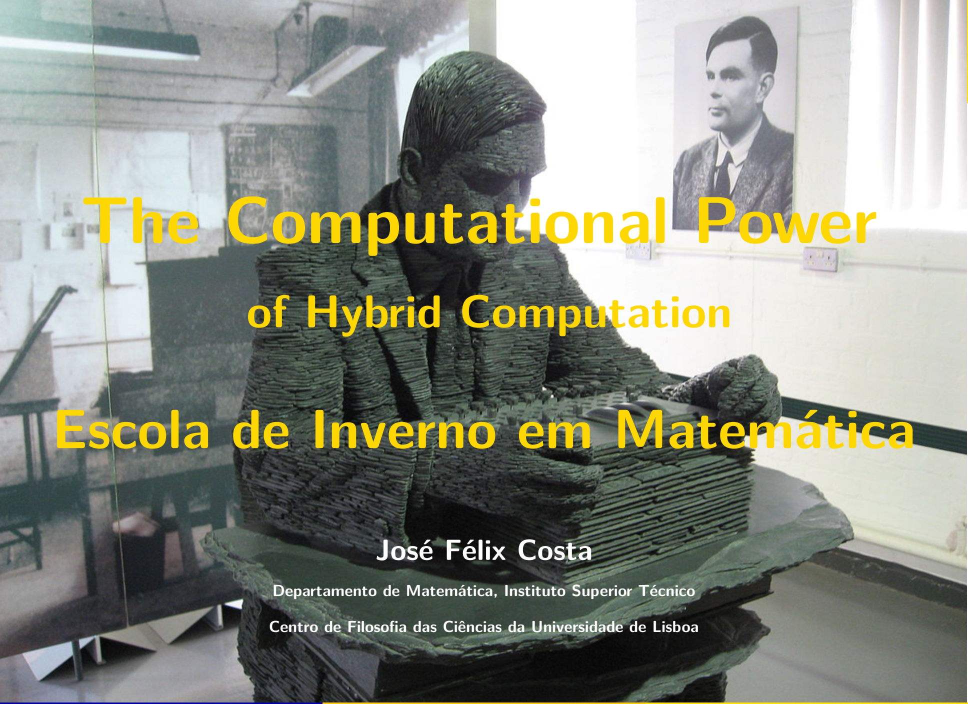


The Computational Power of Hybrid Computation

Escola de Inverno em Matemática

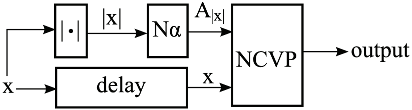
José Félix Costa

Departamento de Matemática, Instituto Superior Técnico

Centro de Filosofia das Ciências da Universidade de Lisboa

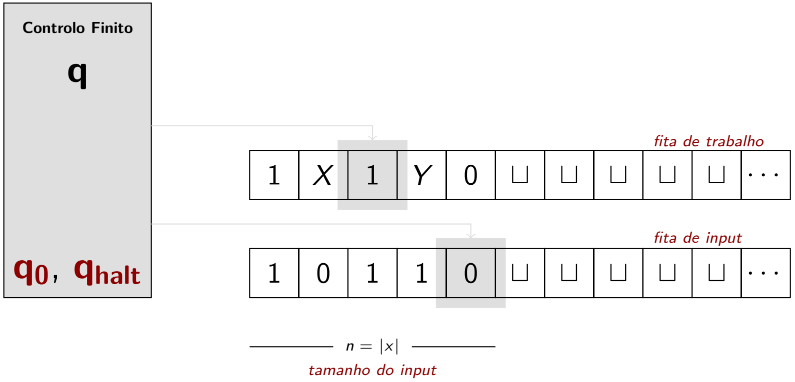
Contents

1. The *ARNN* model
2. Measurement theory
3. Timed measurement systems
4. The three types of measurements
5. Resume to type I: The scatter machine model I
6. Resume to type I: The scatter machine model II
7. Concept of a measurable quantity
8. Open problems



The *ARNN* model

The Turing machine



Sending π at once!

"

```
#! / usr / bin / env python
import random
import math
countinside = 0
for count in range(0, 10000):
    d = math.hypot(random.random(),
                   random.random())
    if d < 1: countinside += 1
count += 1
print 4.0 * countinside / count
```

"

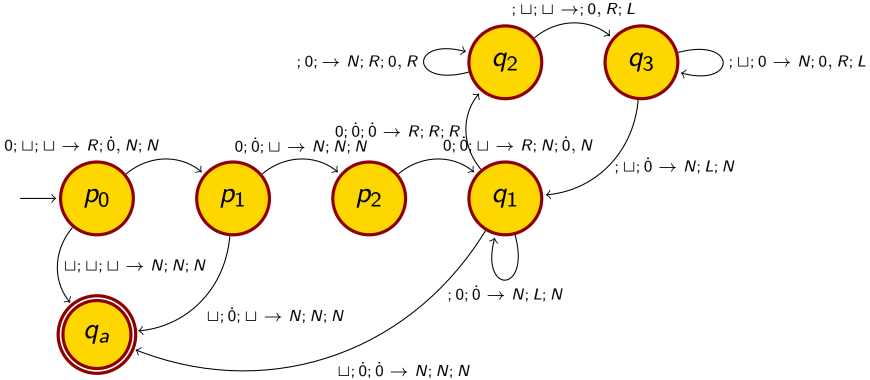
Poe, E., Near a Raven

Midnights so dreary, tired and weary.
    Silently pondering volumes extolling all by-now obsolete lore.
During my rather long nap – the weirdest tap!
    An ominous vibrating sound disturbing my chamber's antedoor.
        "This", I whispered quietly, "I ignore".

Perfectly, the intellect remembers: the ghostly fires, a glittering ember.
    Inflamed by lightning's outbursts, windows cast penumbras upon this floor.
Sorrowful, as one mistreated, unhappy thoughts I heeded:
    That inimitable lesson in elegance – Lenore –
        Is delighting, exciting... nevermore.

(Mike Keith, 1995)

Turing machine as an alarm clock



Collatz function

Iterating Collatz function

input n;

while $n \neq 1$ do if $even(n)$ then $n := n/2$ else $n = 3n + 1$

Collatz function

Iterating Collatz function

input n;

while $n \neq 1$ do if $even(n)$ then $n := n/2$ else $n = 3n + 1$

Sequences generated by different *inputs*

Collatz function

Iterating Collatz function

input n;

while $n \neq 1$ do if $even(n)$ then $n := n/2$ else $n = 3n + 1$

Sequences generated by different *inputs*

4, 2, 1 HALT

Collatz function

Iterating Collatz function

input n;

while $n \neq 1$ do if $even(n)$ then $n := n/2$ else $n = 3n + 1$

Sequences generated by different *inputs*

4, 2, 1 HALT

5, 16, 8, 4, 2, 1 HALT

Collatz function

Iterating Collatz function

input n;

while $n \neq 1$ do if $even(n)$ then $n := n/2$ else $n = 3n + 1$

Sequences generated by different *inputs*

4, 2, 1 HALT

5, 16, 8, 4, 2, 1 HALT

7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1 HALT

Collatz function

Open problem

Collatz function

Open problem

1. It is an open problem to know if the Collatz function takes value 1 after finitely many iterations.

Collatz function

Open problem

1. It is an open problem to know if the Collatz function takes value 1 after finitely many iterations.
2. If a suitable version of the halting problem were decidable, then it would easy to solve Collatz open problem.

The *ARNN* model

Development of Physical Super-Turing Analog Hardware

A. Steven Younger[1], Emmett Redd[1], Hava Siegelmann[2]

[1] Missouri State University, Springfield, MO, USA
{steveyounger,emmettredd}@missouristate.edu
[2] University of Massachusetts-Amherst, Amherst, MA, USA hava@cs.umass.edu

Abstract. In the 1930s, mathematician Alan Turing proposed a mathematical model of computation now called a Turing Machine to describe how people follow repetitive procedures given to them in order to come up with final calculation result. This extraordinary computational model has been the foundation of all modern digital computers since the World War II. Turing also speculated that this model had some limits and that more powerful computing machines should exist. In 1993, Siegelmann and colleagues introduced a Super-Turing Computational Model that may be an answer to Turing's call. Super-Turing computation models have no inherent problem to be realizable physically and biologically. This is unlike the general class of hyper-computer as introduced in 1999 to include the Super-Turing model and some others. This report is on research to design, develop and physically realize two prototypes of analog recurrent neural networks that are capable of solving problems in the Super-Turing complexity hierarchy, similar to the class BPP/log*. We present plans to test and characterize these prototypes on problems that demonstrate anticipated Super-Turing capabilities in modeling Chaotic Systems.

Analogue Recurrent Neural Net [SS94, SS95, Sie99]

System equation

$$x(t+1) \quad = \quad \sigma(Ax(t) + Bu(t) + c)\ .$$

Analogue Recurrent Neural Net [SS94, SS95, Sie99]

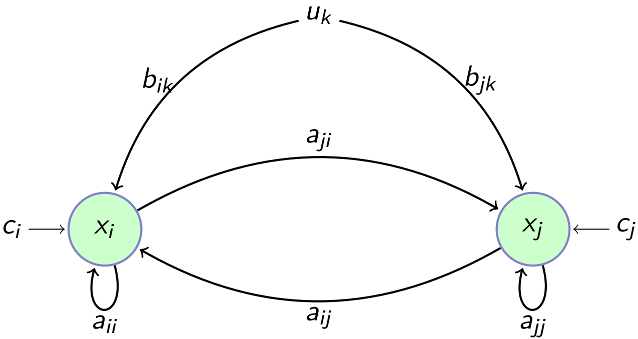


Figure: $x_i[t+1] = \sigma\left(\sum_{j=1}^{N} a_{ij}x_j[t] + \sum_{j=1}^{M} b_{ij}u_j[t] + c_i\right)$

Common sigmoids

Sigmoids [MP43], [SS94, SS95] and [Hay94]

(a) The McCulloch-Pitts sigmoid,

$$\sigma_d(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

(b) The saturated sigmoid,

$$\sigma(x) = \begin{cases} 1 & \text{if } x > 1 \\ x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x < 0 \end{cases}$$

(c) The analytic sigmoid of parameter k,

$$\sigma_k(x) = \frac{1}{1 + e^{-kx}}$$

Computing successor in unary

Example (Successor in unary)

$$y_1^+ = \sigma(a)$$
$$y_a^+ = \sigma(a + y_1)$$

Computing successor in unary

Example (Successor in unary)

$$\begin{aligned} y_1^+ &= \sigma(a) \\ y_a^+ &= \sigma(a + y_1) \end{aligned}$$

Example (Successor in unary)

t	a	y_1	y_a
0	0	0	0
1	1	0	0
2	1	1	1
3	0	1	1
4	0	0	1
5	0	0	0

Computing addition in binary

Example (Adition in binary)

$$
\begin{aligned}
y_1^+ &= \sigma(a + b + v + y_1 - 2) \\
y_2^+ &= \sigma(a + b + v + y_1 - 3) \\
y_3^+ &= \sigma(a - 2b + v - 2y_1 - 1) \\
y_4^+ &= \sigma(-2a + b + v - 2y_1 - 1) \\
y_5^+ &= \sigma(-2a - 2b + v + y_1 - 1) \\
y_6^+ &= \sigma(-a - b - v + y_1) \\
y_7^+ &= \sigma(a + b + v - 3y_1) \\
y_8^+ &= \sigma(a - 3b + v + y_1) \\
y_9^+ &= \sigma(-3a + b + v + y_1) \\
y_{10}^+ &= \sigma(-a - b + v + y_1) \\
y_{a+b}^+ &= \sigma(y_2 + y_3 + y_4 + y_5 + y_6) \\
y_v^+ &= \sigma(y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10})
\end{aligned}
$$

Computing addition in binary

Example (Addition in binary)

t	a	b	v	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{a+b}	y_v
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	1	0	0	1	0	0	0	1	1	0	0	0	0
3	0	0	0	1	0	0	0	0	0	1	0	0	0	1	1
4	0	0	0	0	0	0	0	0	1	0	1	1	0	0	1
5	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Decidability

System equation

$$x(t+1) \quad = \quad \sigma(Ax(t) + Bu(t) + c)\ .$$

Decidability

System equation

$$x(t+1) \quad = \quad \sigma(Ax(t) + Bu(t) + c)\ .$$

Definition

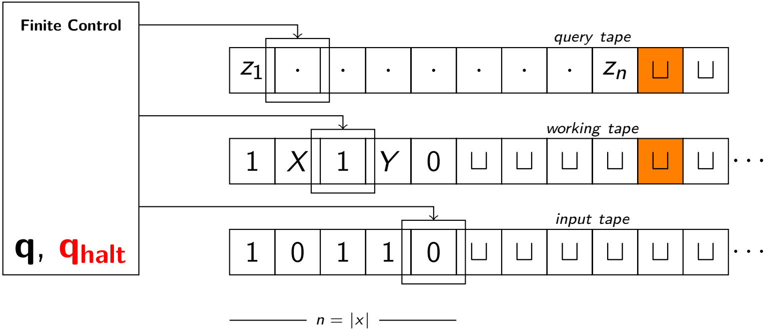
A word $w \in \{0,1\}^+$ is said to be classified in time τ by a system $\mathcal{N}$ if the input streams are (u_1, u_2), with $u_1 = 0w0^\omega$ and $u_2 = 01^{|w|}0^\omega$, and the output streams are (v_1, v_2) with $v_2(t) \equiv (t = \tau)$. If $v_1(\tau) = 1$, then the word is said to be accepted, otherwise (if $v_1(\tau) = 0$) rejected.

Advice function vs oracle

Definition

Let $\mathcal{B}$ be a class of sets and $\mathcal{F}$ a class of total functions of signature $\mathbb{N} \to \Sigma^\star$. The non-uniform class $\mathcal{B}/\mathcal{F}$ is the class of sets A for which some $B \in \mathcal{B}$ and some $f \in \mathcal{F}$ are such that, for every w, $w \in A$ if and only if $\langle w, f(|w|)\rangle \in B$. If we take $\mathcal{B}$ as P and $\mathcal{F}$ as poly, then we get class $P/$poly.

Advice function vs oracle



Lower and upper bounds in polynomial time

Proposition

The output of an ARNN after t steps is affected only by the first $O(t)$ digits in the expansion of the weights.

Lower and upper bounds in polynomial time

Proposition

The output of an ARNN after t steps is affected only by the first $O(t)$ digits in the expansion of the weights.

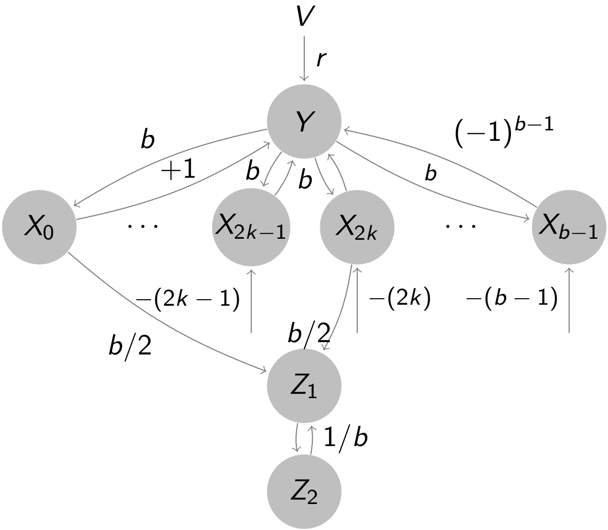
Proposition

$ARNN[\mathbb{R}]P = P/poly$.

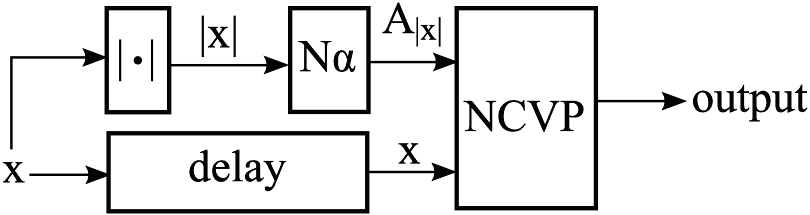
Computational power of *ARNN* under various restrictions

Weights	**Time restriction**	**Computational power**
$\mathbb{Z}$	none	Regular sets
$\mathbb{Q}$	none	Recursively enumerable sets
$\mathbb{R}$	polynomial	P/poly
$\mathbb{R}$	none	All sets

The BAM



The standard sigmoid

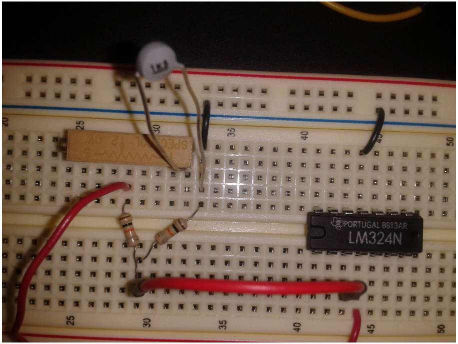


$P = NP$ relativise [CL13]

Proposition

The following propositions are equivalent for the standard analytic activation function on the real weight:

1. $P = NP$
2. $ARNN[\mathbb{Q}]P = ARNN[\mathbb{Q}]NP$
3. $ARNN[\mathbb{R}]P = ARNN[\mathbb{R}]NP$



Measurement theory

Measurement according to Hempel [Hem52, KSLT09]

Definition

Given two binary relations $\mathcal{E}$ and $\mathcal{L}$ in $\mathcal{O}$, $\mathcal{L}$ is *$\mathcal{E}$-irrefexive* if, for all objects a and b in a set $\mathcal{O}$, if $a\mathcal{E}b$ is the case, then $a\mathcal{L}b$ does not hold.

Measurement according to Hempel [Hem52, KSLT09]

Definition

Given two binary relations $\mathcal{E}$ and $\mathcal{L}$ in $\mathcal{O}$, $\mathcal{L}$ is *$\mathcal{E}$-irrefexive* if, for all objects a and b in a set $\mathcal{O}$, if $a\mathcal{E}b$ is the case, then $a\mathcal{L}b$ does not hold.

Definition

Given two binary relations $\mathcal{E}$ and $\mathcal{L}$ in a set $\mathcal{O}$, $\mathcal{L}$ is *$\mathcal{E}$-connected* if, for all objects a and b in $\mathcal{O}$, if $a\mathcal{E}b$ is not the case, then either $a\mathcal{L}b$ or $b\mathcal{L}a$ holds.

Measurement according to Hempel [Hem52, KSLT09]

Definition

Given two binary relations $\mathcal{E}$ and $\mathcal{L}$ in $\mathcal{O}$, $\mathcal{L}$ is *$\mathcal{E}$-irrefexive* if, for all objects a and b in a set $\mathcal{O}$, if $a\mathcal{E}b$ is the case, then $a\mathcal{L}b$ does not hold.

Definition

Given two binary relations $\mathcal{E}$ and $\mathcal{L}$ in a set $\mathcal{O}$, $\mathcal{L}$ is *$\mathcal{E}$-connected* if, for all objects a and b in $\mathcal{O}$, if $a\mathcal{E}b$ is not the case, then either $a\mathcal{L}b$ or $b\mathcal{L}a$ holds.

Definition

Two binary relations $\mathcal{E}$ and $\mathcal{L}$ determine a *comparative concept*, or a *quasi-series*, for the elements of $\mathcal{O}$, if $\mathcal{E}$ is an equivalence relation and $\mathcal{L}$ is transitive, $\mathcal{E}$-irreflexive, and $\mathcal{E}$-connected.

Hempel: Measurement map [Hem52, KSLT09]

Definition

The map $M : \mathcal{O} \to \mathbb{R}$ is said to be a *measurement map* if

Hempel: Measurement map [Hem52, KSLT09]

Definition

The map $M : \mathcal{O} \rightarrow \mathbb{R}$ is said to be a *measurement map* if

Axiom 1 If $a \mathcal{E} b$, then $M(a) = M(b)$.

Hempel: Measurement map [Hem52, KSLT09]

Definition

The map $M : \mathcal{O} \rightarrow \mathbb{R}$ is said to be a *measurement map* if

Axiom 1 If $a\mathcal{E}b$, then $M(a) = M(b)$.

Axiom 2 If $a\mathcal{L}b$, then $M(a) < M(b)$.

Hempel: Propositional

Proposition

For all a, b in $\mathcal{O}$, one, and only one, of the following statements holds: (a) $a\mathcal{E}b$, (b) $a\mathcal{L}b$, or (c) $b\mathcal{L}a$.

Hempel: Propositional

Proposition

For all a, b in $\mathcal{O}$, one, and only one, of the following statements holds: (a) $a\mathcal{E}b$, (b) $a\mathcal{L}b$, or (c) $b\mathcal{L}a$.

Proposition

For all a, b in $\mathcal{O}$:

$$\begin{aligned} \text{If } M(a) &= M(b), \text{ then } a\mathcal{E}b \\ \text{If } M(a) &< M(b), \text{ then } a\mathcal{L}b \end{aligned}$$

Timed measurement systems

Bachelard, Eddington

Gaston Bachelard

Let us briefly note that the behaviour of the precision balance, though it is faithful to the mass, is not always clear: many students are surprised and disturbed by the slowness of the measurement process. We can not say that, for everyone, there is a precise idea of measurement of mass.[a]

[a] Gaston Bachelard, *The Philosophy of No: A Philosophy of the New Scientific Mind*, Viking Press, 1968 (1940).

Bachelard, Eddington

Gaston Bachelard

Let us briefly note that the behaviour of the precision balance, though it is faithful to the mass, is not always clear: many students are surprised and disturbed by the slowness of the measurement process. We can not say that, for everyone, there is a precise idea of measurement of mass.[a]

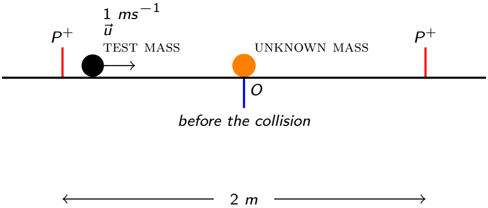
[a] Gaston Bachelard, *The Philosophy of No: A Philosophy of the New Scientific Mind*, Viking Press, 1968 (1940).

Arthur Eddington

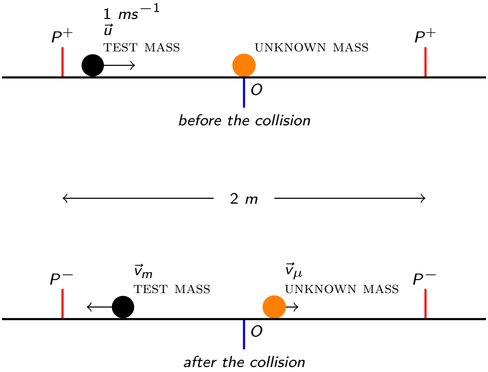
Yet space is a prominent feature of the physical world; and measurement of space — lengths, distances, volumes — is part of the normal occupation of a physicist. Indeed it is rare to find any quantitative physical observation which does not ultimately reduce to measuring distances.[a]

[a] Arthur Eddington, *The Expanding Universe*, Cambridge University Press, First published in 1933.

Collider experiment



Collider experiment



Collider experiment

Implementing a comparative concept

1. Test particle m is detected backward, in time t: $m\mathcal{L}_t\mu$;

Collider experiment

Implementing a comparative concept

1. Test particle m is detected backward, in time t: $m\mathcal{L}_t\mu$;
2. Test particle m is detected forward, in time t: $\mu\mathcal{L}_t m$;

Collider experiment

Implementing a comparative concept

1. Test particle m is detected backward, in time t: $m\mathcal{L}_t\mu$;
2. Test particle m is detected forward, in time t: $\mu\mathcal{L}_t m$;
3. Test particle m not seen within time t: $m\mathcal{E}_t\mu$.

Timed relation [BCT10a]

Definition

A relation $\mathcal{E}_t$ in $\mathcal{O} \times \mathcal{O}$, for the time bound $t > 0$, is said to be a *timed equivalence relation* if there is a $\kappa \geq 1$ so that

Timed relation [BCT10a]

Definition

A relation $\mathcal{E}_t$ in $\mathcal{O} \times \mathcal{O}$, for the time bound $t > 0$, is said to be a *timed equivalence relation* if there is a $\kappa \geq 1$ so that

1. $\mathcal{E}_t$ is *reflexive*;

Timed relation [BCT10a]

Definition

A relation $\mathcal{E}_t$ in $\mathcal{O} \times \mathcal{O}$, for the time bound $t > 0$, is said to be a *timed equivalence relation* if there is a $\kappa \geq 1$ so that

1. $\mathcal{E}_t$ is *reflexive*;
2. $\mathcal{E}_t$ is *timed symmetric*: for every a, b in $\mathcal{O}$, if $a\mathcal{E}_t b$, then $b\mathcal{E}_{t/\kappa}a$;

Timed relation [BCT10a]

Definition

A relation $\mathcal{E}_t$ in $\mathcal{O} \times \mathcal{O}$, for the time bound $t > 0$, is said to be a *timed equivalence relation* if there is a $\kappa \geq 1$ so that

1. $\mathcal{E}_t$ is *reflexive*;
2. $\mathcal{E}_t$ is *timed symmetric*: for every a, b in $\mathcal{O}$, if $a\mathcal{E}_t b$, then $b\mathcal{E}_{t/\kappa}a$;
3. $\mathcal{E}_t$ is *timed transitive*: for every a, b, and c in $\mathcal{O}$, if $a\mathcal{E}_t b$ and $b\mathcal{E}_t c$, then $a\mathcal{E}_{t/\kappa}c$;

Timed relation [BCT10a]

Definition

A relation $\mathcal{E}_t$ in $\mathcal{O} \times \mathcal{O}$, for the time bound $t > 0$, is said to be a *timed equivalence relation* if there is a $\kappa \geq 1$ so that

1. $\mathcal{E}_t$ is *reflexive*;
2. $\mathcal{E}_t$ is *timed symmetric*: for every a, b in $\mathcal{O}$, if $a\mathcal{E}_t b$, then $b\mathcal{E}_{t/\kappa}a$;
3. $\mathcal{E}_t$ is *timed transitive*: for every a, b, and c in $\mathcal{O}$, if $a\mathcal{E}_t b$ and $b\mathcal{E}_t c$, then $a\mathcal{E}_{t/\kappa}c$;
4. if $t < t'$ and $a\mathcal{E}_{t'} b$, then $a\mathcal{E}_t b$.

Timed relation [BCT10a]

Definition

A relation $\mathcal{E}_t$ in $\mathcal{O} \times \mathcal{O}$, for the time bound $t > 0$, is said to be a *timed equivalence relation* if there is a $\kappa \geq 1$ so that

1. $\mathcal{E}_t$ is *reflexive*;
2. $\mathcal{E}_t$ is *timed symmetric*: for every a, b in $\mathcal{O}$, if $a\mathcal{E}_t b$, then $b\mathcal{E}_{t/\kappa} a$;
3. $\mathcal{E}_t$ is *timed transitive*: for every a, b, and c in $\mathcal{O}$, if $a\mathcal{E}_t b$ and $b\mathcal{E}_t c$, then $a\mathcal{E}_{t/\kappa} c$;
4. if $t < t'$ and $a\mathcal{E}_{t'} b$, then $a\mathcal{E}_t b$.

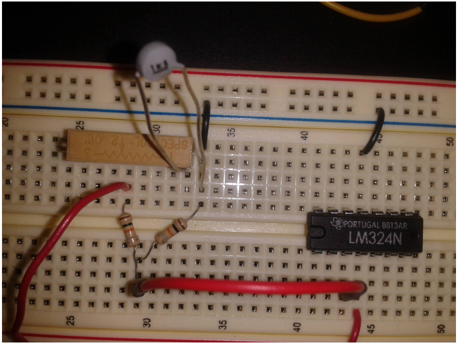
Axiom

The apparatus satisfies the *separation property* for the measurement map $M : \mathcal{O} \to \mathbb{R}$ if, for every objects a and b in $\mathcal{O}$, if $M(a) < M(b)$, then there exists a time bound t such that $a\mathcal{L}_t b$.

BCT Conjecture

Conjecture

No reasonable physical measurement has an associated measurement map with polynomial time complexity.



The three types of measurements

Three cases of measurability [BCT10c, BCT14]

The vertical axis measures the outcome of the experiment; we have to find the first zero x by trial and error on the value a:

Three cases of measurability [BCT10c, BCT14]

The vertical axis measures the outcome of the experiment; we have to find the first zero x by trial and error on the value a:

Type I

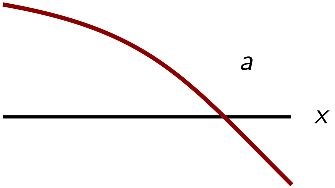


Figure: Measure both $a < x$ and $x < a$.

Three cases of measurability [BCT10c, BCT14]

Type I

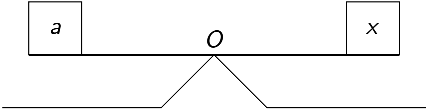


Figure: Balance.

Three cases of measurability [BCT10c, BCT14]

The vertical axis measures the outcome of the experiment; we have to find the first zero x by trial and error on the value a:

Three cases of measurability [BCT10c, BCT14]

The vertical axis measures the outcome of the experiment; we have to find the first zero x by trial and error on the value a:

Type II

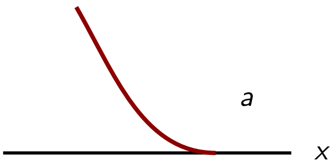


Figure: Can only measure $a < x$.

Three cases of measurability [BCT10c, BCT14]

Type II

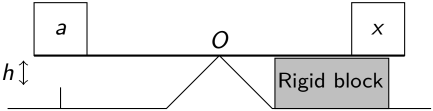


Figure: Broken balance.

Three cases of measurrement [BCT10c, BCT14]

The vertical axis measures the outcome of the experiment; we have to find the first zero x by trial and error on the value a:

Three cases of measurrement [BCT10c, BCT14]

The vertical axis measures the outcome of the experiment; we have to find the first zero x by trial and error on the value a:

Type III

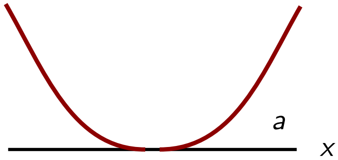


Figure: Can only measure ($a < x$ or $x < a$).

Balance scale in vanishing mode

Type III

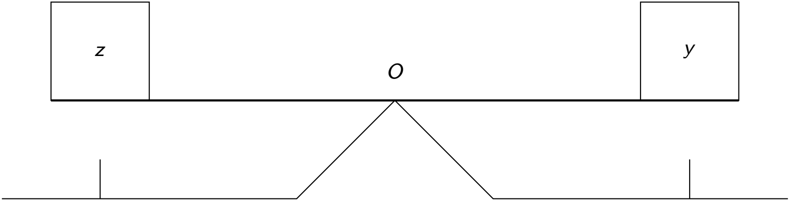
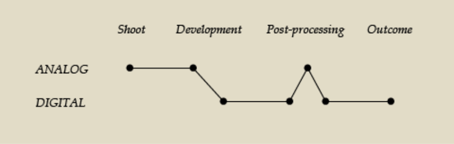
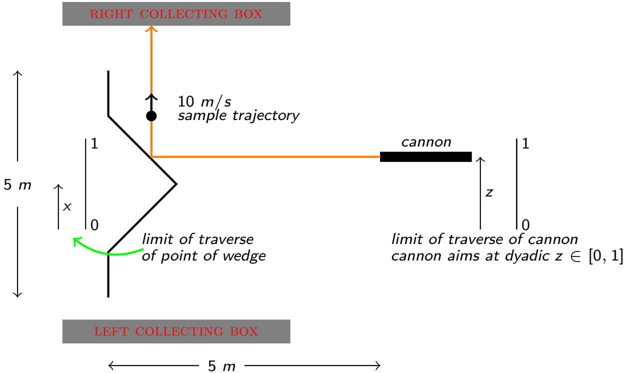


Figure: Schematic depiction of the *vanishing balance experiment*.



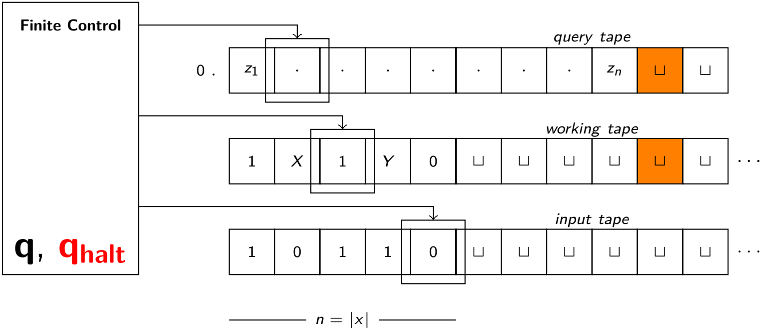
Resume to type I: The scatter machine model I

The scatter machine [BT07]

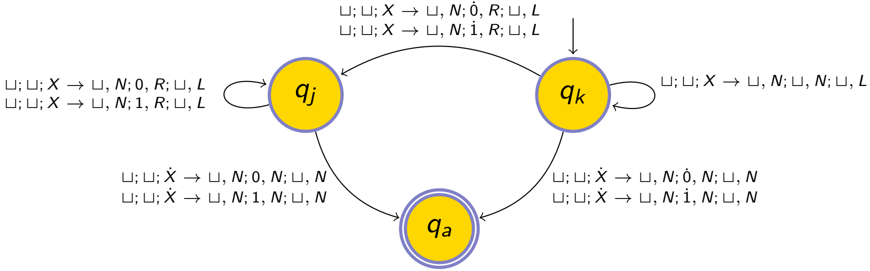


Query tape [BCLT08b, BCLT08a, BCLT09]

Query tape



Non-deterministic and probabilistic machines



Analog-digital scatter machine: decidability [BCLT08b, BCLT08a, BCLT09]

Error-free analog-digital scatter machine

Let $A \subseteq \Sigma^\star$ be a set of words over Σ. We say that an error-free analog-digital scatter machine $\mathcal{M}$ decides A if, for every input $w \in \Sigma^\star$, w is accepted if $w \in A$ and rejected if $w \notin A$.

Analog-digital scatter machine: decidability [BCLT08b, BCLT08a, BCLT09]

Error-free analog-digital scatter machine

Let $A \subseteq \Sigma^\star$ be a set of words over Σ. We say that an error-free analog-digital scatter machine $\mathcal{M}$ decides A if, for every input $w \in \Sigma^\star$, w is accepted if $w \in A$ and rejected if $w \notin A$.

Error-prone analog-digital scatter machine

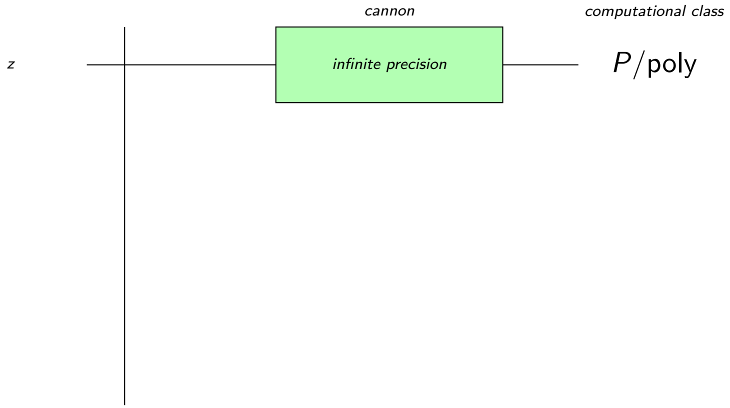
Let $A \subseteq \Sigma^\star$ be a set of words over Σ. We say that an error-prone analog-digital scatter machine $\mathcal{M}$ decides A if there is a number $\gamma < \frac{1}{2}$, such that the error probability of $\mathcal{M}$ for any input w is smaller than γ.

$BPP//\log\star$

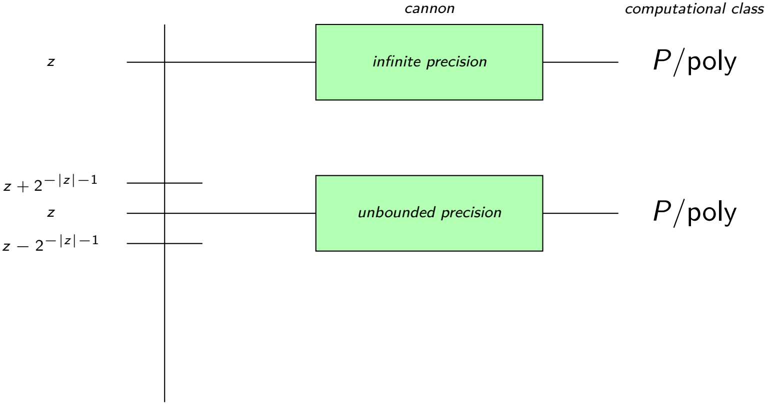
Definition

$BPP//\log\star$ is the class of sets $A \subseteq \Sigma^\star$ for which a probabilistic Turing machine $\mathcal{M}$, clocked in polynomial time, a prefix function $f \in \log$, and a constant $\gamma < \frac{1}{2}$ exist such that, for every length n and input w with $|w| \leq n$, $\mathcal{M}$ rejects $\langle w, f(n)\rangle$ with probability at most γ if $w \in A$ and accepts $\langle w, f(n)\rangle$ with probability at most γ if $w \notin A$.

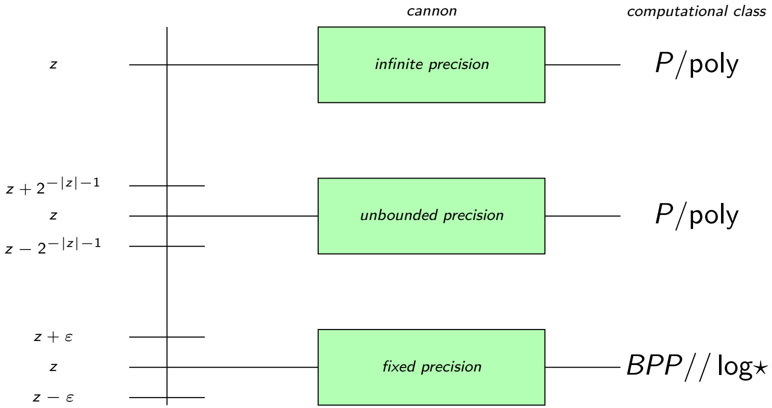
ARNN case and the sharp scatter machine

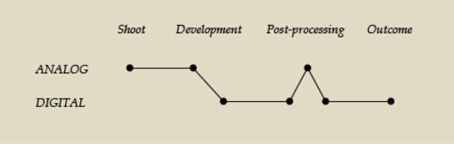


ARNN case and the sharp scatter machine



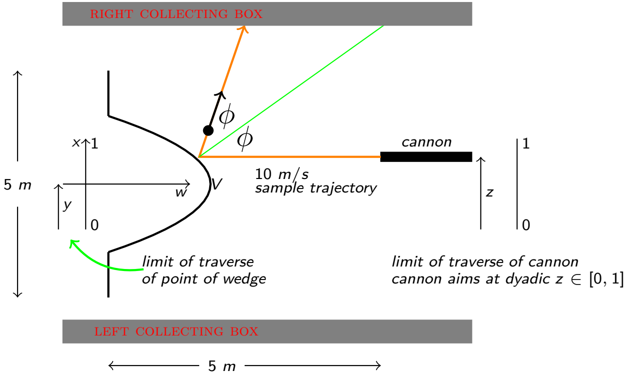
ARNN case and the sharp scatter machine





Resume to type I: The scatter machine model II

Smooth scatter machine [BCT12]



Complexity of the vertex position [BCT12]

Proposition

Consider that $g(x)$ is the function describing the shape of the wedge of a SmSE. Suppose that $g(x)$ is n times continuously differentiable near $x = 0$, all its derivatives up to $(n-1)$-th vanish at $x = 0$, and the n-th derivative is nonzero. Then, when the SmSE, with vertex position y, fires the cannon at position z, the time needed to detect the particle in one of the boxes is $t(z)$, where:

$$\frac{A}{|y-z|^{n-1}} \leq t(z) \leq \frac{B}{|y-z|^{n-1}} \ , \qquad (1)$$

for some $A, B > 0$ and for $|y-z|$ sufficiently small.

Complexity of the vertex position [BCT12]

Proposition

Any particle hitting horizontally, sufficiently closer to the vertex V, will bounce back covering an horizontal distance before detection that goes to infinity as $O(\frac{1}{|z-y|})$.

Complexity of the vertex position [BCT12]

Proposition

Any particle hitting horizontally, sufficiently closer to the vertex V, will bounce back covering an horizontal distance before detection that goes to infinity as $O(\frac{1}{|z-y|})$.

Proposition

The protocol that processes queries between a Turing machine and the generalised scatter machine takes a time that is at least exponential in the size of the dyadic rational specified by the query during the binary search procedure.

Protocol [ABC$^+$16]

The cannon can be placed at the dyadic rational z — infinite precision

Algorithm 1: Measurement algorithm for infinite precision.

Data: Positive integer ℓ representing the desired precision

$x_0 = 0$;
$x_1 = 1$;
$z = 0$;
while $x_1 - x_0 > 2^{-\ell}$ **do**
    $z = (x_0 + x_1)/2$;
    $s = Prot\_IP(z\downharpoonright_\ell)$;
    if $s ==$ "q_r" **then**
        $x_1 = z$;
    if $s ==$ "q_l" **then**
        $x_0 = z$;
    else
        $x_0 = z$;
        $x_1 = z$;
return *Dyadic rational denoted by* x_0

Protocol [ABC+16]

The cannon can be placed at the dyadic rational z, but only with unbounded but finite precision, say $2^{-|z|-1}$, i.e., the cannon can be set at position $z \pm 2^{-|z|-1}$

Algorithm 5: Measurement algorithm for unbounded precision.

Data: Positive integer ℓ representing the precision

$x_0 = 0$;
$x_1 = 1$;
$z = 0$;
while $x_1 - x_0 > 2^{-\ell}$ **do**
    $z = (x_0 + x_1)/2$;
    $s = Prot\_UP(z\downharpoonright_\ell)$;
    if $s ==$ "q_r" **then**
        $x_1 = z$;
    if $s ==$ "q_l" **then**
        $x_0 = z$;
    else
        $x_0 = z$;
        $x_1 = z$;
return *Dyadic rational denoted by* x_0

Protocol [ABC+16]

The cannon can be placed at the dyadic rational z, but only with fixed a priori precision ε (dyadic rational), i.e., the cannon can be set at position $z \pm \varepsilon$

Algorithm 9: Measurement algorithm for fixed precision.

Data: Integer ℓ representing the precision

$c = 0$;
$i = 0$;
$\xi = 2^{2\ell+h}$;
while $i < \xi$ **do**
    $s = Prot\_FP(1\downharpoonright_\ell)$;
    if $s ==$ "q_l" **then**
        $c = c + 2$;
    if $s ==$ "q_t" **then**
        $c = c + 1$;
    i++ ;
return $c/(2\xi)$

The digital-analog device as a biased coin

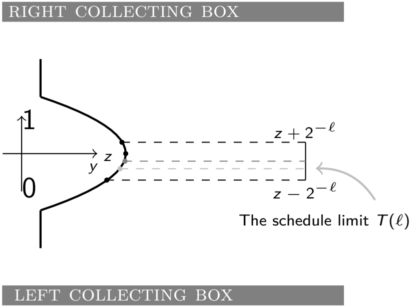


Figure: The *SmSE* with unbounded precision as a coin.

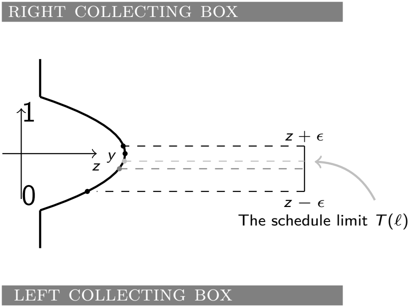
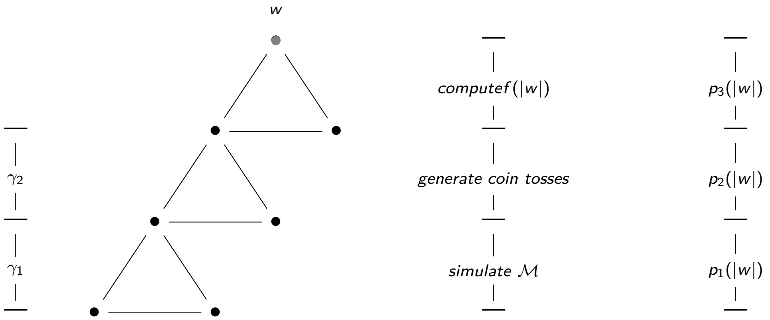


Figure: The *SmSE* with fixed precision as a coin.

Lower bounds

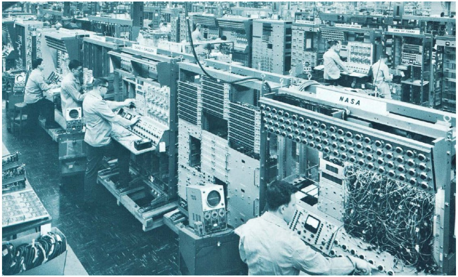


Computational power ([BCPT13, ABC$^+$16, BCCT18])

	Infinite	**Unbounded**	**Fixed**
Lower Bound	$P/\log\star$	$BPP//\log\star$	$BPP//\log\star$
Upper Bound Exponential schedule	$P/\log\star$	$BPP//\log^2\star$	$BPP//\log^2\star$
Upper Bound Explicit Time	—	$BPP//\log\star$ Exponential schedule	$BPP//\log\star$ Exponential schedule

Results for different types, [BCT14, BCPT17]

Type of Oracle		Infinite	Unbounded	Finite
Two-sided	lower bound	$P/\log\star$	$BPP//\log\star$	$BPP//\log\star$
	upper bound	P/poly	P/poly	P/poly
	upper bound (w/ exponential T)	$P/\log\star$	$BPP//\log\star$	$BPP//\log\star$
Threshold	lower bound	$P/\log\star$	$BPP//\log\star$	$BPP//\log\star$
	upper bound	——	——	——
	upper bound (w/ exponential T)	$P/\log\star$	$BPP//\log\star$	$BPP//\log\star$
Vanishing Type 1 (Parallel)	lower bound	P/poly	P/poly	$BPP//\log\star$
	upper bound	P/poly	P/poly	$BPP//\log\star$
	upper bound (w/ exponential T)	——	——	——
Vanishing Type 2 (Clock)	lower bound	$P/\log\star$	$BPP//\log\star$	$BPP//\log\star$
	upper bound	P/poly	P/poly	$BPP//\log\star$
	upper bound (w/ exponential T)	——	$BPP//\log\star$	——



Concept of a measurable quantity

Geroch and Hartle [GH86]

Geroch and Hartle [GH86]

Every computable number is measurable. This is easy to see: Let the instructions direct that the raw materials be assembled into a computer, and that a certain [...] program — one specified in the instructions — be run on that computer. That is, every digital computer is at heart an analog computer. [a]

[a] Robert Geroch and James B. Hartle, *Computability and Physical Theories*, Foundations of Physics, 16(6), 1986.

Geroch and Hartle [GH86]

Geroch and Hartle [GH86]

Every computable number is measurable. This is easy to see: Let the instructions direct that the raw materials be assembled into a computer, and that a certain [...] program — one specified in the instructions — be run on that computer. That is, every digital computer is at heart an analog computer. [a]

[a]Robert Geroch and James B. Hartle, *Computability and Physical Theories*, Foundations of Physics, 16(6), 1986.

Geroch and Hartle [GH86]

We now ask whether, conversely, every measurable number is computable — or, in more detail, whether current physical theories are such that their measurable numbers are computable. This question must asked with care. [a]

[a]Robert Geroch and James B. Hartle, *Computability and Physical Theories*, Foundations of Physics, 16(6), 1986.

Concept of measurable [BCT10b, ABC^{+}16]

Definition

A distance y is said to be measurable if there exists a Turing machine, equipped with a physical oracle with a computable schedule T, such that it prints the first n bits of y on the output tape in less than $T(n)$ time steps without timing out in any query.

Concept of measurable [BCT10b, ABC+16]

Definition

A distance y is said to be measurable if there exists a Turing machine, equipped with a physical oracle with a computable schedule T, such that it prints the first n bits of y on the output tape in less than $T(n)$ time steps without timing out in any query.

Proposition

There are uncountable many $y \in [0,1]$ so that, for any program P with specified waiting times, there is a n so that P can not determine the first n binary places of y.

Measurable distances [BCT10b, ABC$^+$16]

Proposition

For the SmSM with vertice at y (not a dyadic rational), written according to the pattern:

$$y = 0{\cdot}\underbrace{1\ldots1}_{u_1}\underbrace{0\ldots0}_{u_2}\underbrace{1\ldots1}_{u_3}\underbrace{0\ldots0}_{u_4}\underbrace{1\ldots1}_{u_5}\underbrace{0\ldots0}_{u_6}\ldots$$

where $u_1 \geq 0$, $u_i \geq 1$ ($i \geq 2$).

Measurable distances [BCT10b, ABC+16]

Proposition

For the SmSM with vertice at y (not a dyadic rational), written according to the pattern:

$$y = 0 \cdot \underbrace{1 \ldots 1}_{u_1} \underbrace{0 \ldots 0}_{u_2} \underbrace{1 \ldots 1}_{u_3} \underbrace{0 \ldots 0}_{u_4} \underbrace{1 \ldots 1}_{u_5} \underbrace{0 \ldots 0}_{u_6} \ldots$$

where $u_1 \geq 0$, $u_i \geq 1$ $(i \geq 2)$.

1. *If y is measurable by any program, then the sequence u_k is bounded by a computable function.*

Measurable distances [BCT10b, ABC+16]

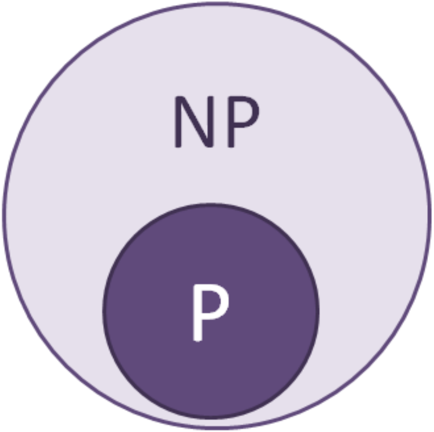
Proposition

For the SmSM with vertice at y (not a dyadic rational), written according to the pattern:

$$y = 0{\cdot}\underbrace{1\ldots1}_{u_1}\underbrace{0\ldots0}_{u_2}\underbrace{1\ldots1}_{u_3}\underbrace{0\ldots0}_{u_4}\underbrace{1\ldots1}_{u_5}\underbrace{0\ldots0}_{u_6}\ldots$$

where $u_1 \geq 0$, $u_i \geq 1$ $(i \geq 2)$.

1. *If y is measurable by any program, then the sequence u_k is bounded by a computable function.*
2. *If the sequence u_k is bounded by a computable function, then y is measurable by the linear search method.*



Open problems

Bibliography I

[ABC+16] Tânia Ambaram, Edwin Beggs, José Félix Costa, Diogo Poças, and John V. Tucker. An analogue-digital model of computation: Turing machines with physical oracles. In Andrew Adamatzky, editor, *Advances in Unconventional Computing, Volume 1 (Theory)*, volume 22 of *Emergence, Complexity and Computation*, pages 73–115. Springer, 2016.

[BCCT18] Edwin Beggs, Pedro Cortez, José Félix Costa, and John V. Tucker. Classifying the computational power of stochastic physical oracles. *International Journal of Unconventional Computing 14(1): 68–90*, 2018.

[BCLT08a] Edwin Beggs, José Félix Costa, Bruno Loff, and John V. Tucker. Computational complexity with experiments as oracles. *Proceedings of the Royal Society, Series A (Mathematical, Physical and Engineering Sciences)*, 464(2098):2777–2801, 2008.

[BCLT08b] Edwin Beggs, José Félix Costa, Bruno Loff, and John V. Tucker. On the complexity of measurement in classical physics. In Manindra Agrawal, Dingzhu Du, Zhenhua Duan, and Angsheng Li, editors, *Theory and Applications of Models of Computation (TAMC 2008)*, volume 4978 of *Lecture Notes in Computer Science*, pages 20–30. Springer, 2008.

[BCLT09] Edwin Beggs, José Félix Costa, Bruno Loff, and John V. Tucker. Computational complexity with experiments as oracles II. Upper bounds. *Proceedings of the Royal Society, Series A (Mathematical, Physical and Engineering Sciences)*, 465(2105):1453–1465, 2009.

[BCPT13] Edwin Beggs, José Félix Costa, Diogo Poças, and John V. Tucker. Oracles that measure thresholds: The Turing machine and the broken balance. *Journal of Logic and Computation*, 23(6):1155–1181, 2013.

[BCPT17] Edwin Beggs, José Félix Costa, Diogo Poças, and John V. Tucker. Computations with oracles that measure vanishing quantities. *Mathematical Structures in Computer Science*, 23(6):1155 – 1181, 2017.

[BCT10a] Edwin Beggs, José Félix Costa, and John V. Tucker. Computational Models of Measurement and Hempel's Axiomatization. In Arturo Carsetti, editor, *Causality, Meaningful Complexity and Knowledge Construction*, volume 46 of *Theory and Decision Library A*, pages 155–184. Springer, 2010.

[BCT10b] Edwin Beggs, José Félix Costa, and John V. Tucker. Limits to measurements in experiments governed by algorithms. *Mathematical Structures in Computer Science*, 20(06):1019–1050, 2010. Special issue on Quantum Algorithms, Editor Salvador Elías Venegas-Andraca.

Bibliography II

[BCT10c] Edwin Beggs, José Félix Costa, and John V. Tucker. The Turing machine and the uncertainty in the measurement process. In Hélia Guerra, editor, *Physics and Computation, P&C 2010*, pages 62–72. CMATI – Centre for Applied Mathematics and Information Technology, University of Azores, 2010.

[BCT12] Edwin Beggs, José Félix Costa, and John V. Tucker. The impact of models of a physical oracle on computational power. *Mathematical Structures in Computer Science*, 22(5):853–879, 2012. Special issue on Computability of the Physical, Editors Cristian S. Calude and S. Barry Cooper.

[BCT14] Edwin Beggs, José Félix Costa, and John V. Tucker. Three forms of physical measurement and their computability. *The Review of Symbolic Logic*, 7(4):618–646, 2014.

[BT07] Edwin Beggs and John V. Tucker. Experimental computation of real numbers by Newtonian machines. *Proceedings of the Royal Society, Series A (Mathematical, Physical and Engineering Sciences)*, 463(2082):1541–1561, 2007.

[CL13] José Félix Costa and Raimundo Leong. The arnn model relativizes P=NP and P≠NP. *Theoretical Computer Science*, 499(1):2–22, 2013.

[GH86] Robert Geroch and James B. Hartle. Computability and physical theories. *Foundations of Physics*, 16(6):533–550, 1986.

[Hay94] S. Haykin. *Neural Networks: A Comprehensive Foundation*. MacMillan College Publishing, 1994.

[Hem52] Carl G. Hempel. Fundamentals of concept formation in empirical science. *International Encyclopedia of Unified Science*, 2(7), 1952.

[KSLT09] David H. Krantz, Patrick Suppes, R. Duncan Luce, and Amos Tversky. *Foundations of Measurement*. Dover, 2009.

[MP43] Warren McCulloch and Walter Pitts. A logical calculus of ideas immanent in nervous activity. *Journal of Mathematical Analysis and Applications*, 5:115–133, 1943.

[Sie99] Hava T. Siegelmann. *Neural Networks and Analog Computation: Beyond the Turing Limit*. Birkhäuser, 1999.

[SS94] Hava T. Siegelmann and Eduardo D. Sontag. Analog computation via neural networks. *Theoretical Computer Science*, 131(2):331–360, 1994.

[SS95] Hava T. Siegelmann and Eduardo D. Sontag. On the computational power of neural networks. *J. Comp. Syst. Sciences*, 50(1):132–150, 1995.