Defects in shein theory & TQFT

[arXiv:2606.07432]

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(jt. w/ Ingo Runkel, UHH)

TQFT Club

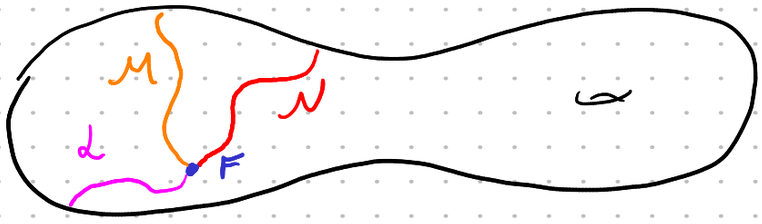
28 October 2026

Overview

Skein theory $\longleftrightarrow$ Topological field theories

$Sk_e(M)$ $\quad \delta : \Sigma \xrightarrow{\sim} \partial M$ $\quad Z_e^{RT}(\Sigma_g)$

TFTs can have defects - implement symmetries



$\mathcal{E}$

Defect skein module $Sk(M)$

Defects in RT theory - implemented by "state sums"!

$Z_e^d(\mathcal{E})$

Isomorphism $\quad Sk(M) \xrightarrow{\sim} Z_e^d(\mathcal{E})$

$\delta : \mathcal{E} \xrightarrow{\sim} \partial M$

Shein theory & Topological Field Theory

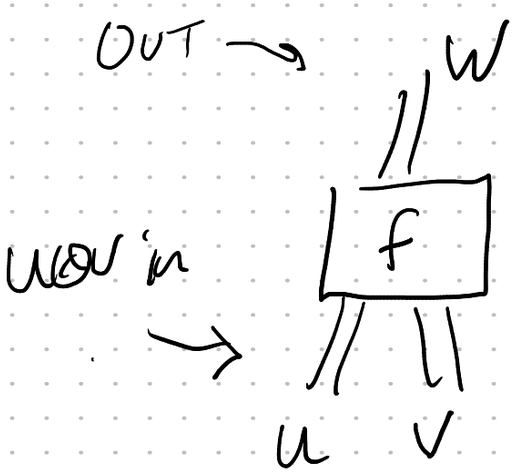
[Reshetikhin–Turaev] construction:

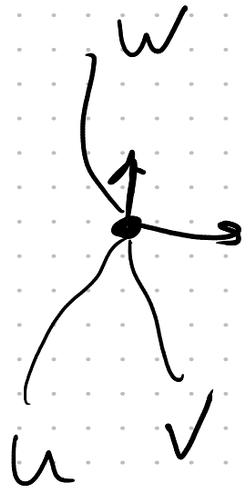
Input: $\mathcal{C}$ a modular fusion category

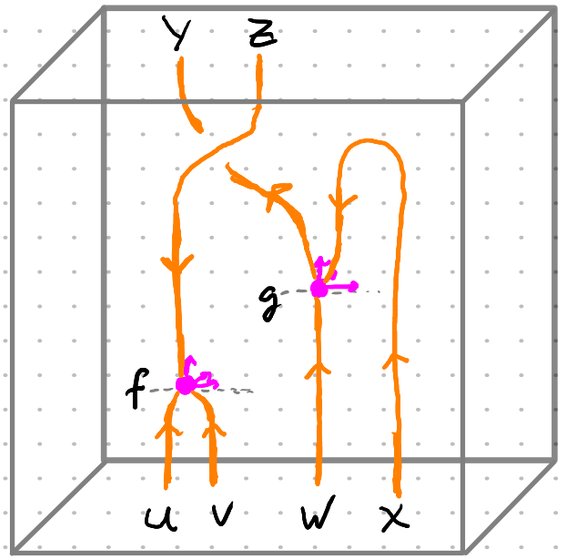
3d graphical calculus:

$\mathcal{C}$-colored graph ($\mathcal{C}$-graph)
- obc ⇝ edges — oriented ($\mathcal{C}$ rigid) / framed
- mor $\mathcal{C}$ ⇝ vertices — 3-framed (input/output order)

Ribbon graph: underlying topology (no $\mathcal{C}$-colours)







Here, $f: U \otimes V \to Z^{\vee}$ etc.

Shein theory & Topological Field Theory

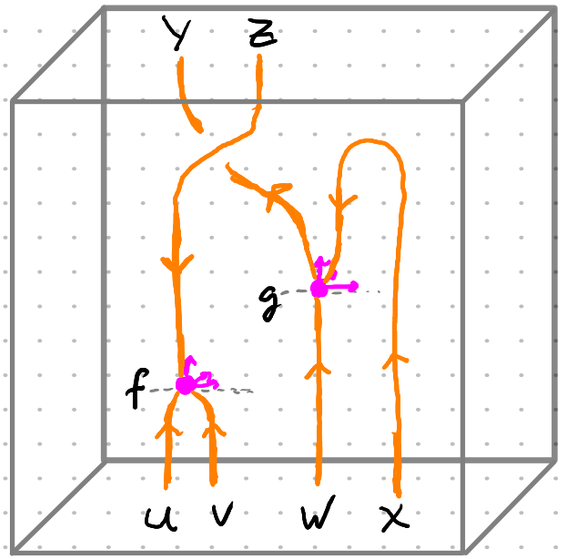
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Here, $f: u \otimes v \to z^{v}$ etc.

Output: a scalar invariant of

(M: closed 3-manifold, Γ: closed $\mathcal{C}$-graph).

Construction: • write M as surgery on L

$$\cdot\ \tau(M,\Gamma) := \alpha \sum_{v \in \mathrm{Irr}(\mathcal{C})} \dim_{\mathcal{C}}(V)\, \langle L_V \cup \Gamma \rangle \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1},\mathbb{1}) = \cdot\, k$$

- α: scalar we don't care about today!
- L coloured by V
- $\langle L_V \cup \Gamma \rangle$: evaluation.

State spaces, bordisms

[Turaev, Blanchet-Habegger-Masbaum-Vogel]

Need state spaces for marked surfaces $(\Sigma, \{P_i, V_i, v_i\})$



Idea: choose $\delta : \Sigma_g \xrightarrow{\sim} \partial H_g$.

Let $Z_e^{RT}(\Sigma, \delta) = k\langle \text{c-graphs } \Gamma \hookrightarrow H_g \rangle / \sim$

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Given $\overline{(\Sigma_1, \delta_1)} \xrightarrow{(M,\Gamma)} (\Sigma_2, \delta_2)$, have

$$Z_c^{RT}(\Sigma_1, \delta_1) \otimes Z_c^{RT}(\overline{\Sigma_2, \delta_2}) \longrightarrow k$$
$$[\Gamma_1] \otimes [\Gamma_2] \longmapsto \tau\left(H_1 \underset{\delta_1}{\cup} M \underset{\delta_2}{\cup} H_2, \Gamma_1 \cup \Gamma \cup \Gamma_2\right)$$

Equip (M,Γ) with $m \in \mathbb{Z}$, to absorb gluing anomaly.

TQFT: $Z_c^{RT} : \widehat{Bord}_3 \longrightarrow Vect$

Skein relations

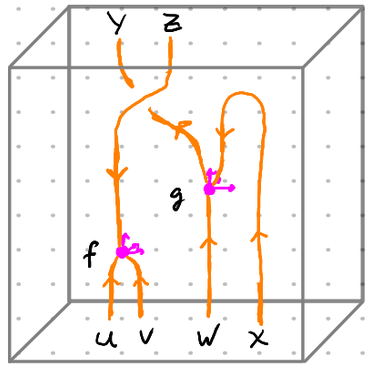
$\Gamma \hookrightarrow H$ coloured graph

$C : [0,1] \times [-\tfrac{1}{2}, \tfrac{1}{2}] \times [0,1] \hookrightarrow H$ cube s.t. $\partial C \cap \Gamma \subseteq \{y = \pm \tfrac{1}{2}\}$

- transverse
- distinct x-coordinates.

Project $\Gamma \cap C$ to xy-plane $\rightsquigarrow$ morphism in $\mathcal{C}$

$\langle \Gamma, C \rangle =$



$\in \mathrm{Hom}_{\mathcal{C}}(u \otimes v \otimes w \otimes x, \; y \otimes z^{\vee})$

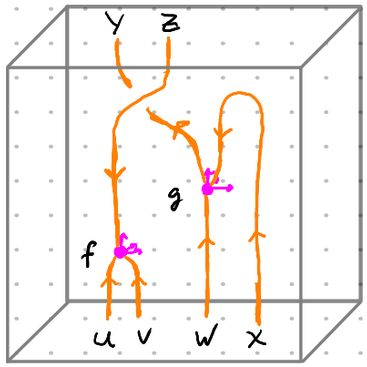
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If $\{\Gamma_i \hookrightarrow H_g\}_{i \in I}$, and $\exists C, \{\lambda_i\}$ s.t. (*) $\begin{cases} \Gamma_i \setminus C = \Gamma_j \setminus C \quad \forall i,j \in I \\ \sum_{i \in I} \lambda_i \langle \Gamma_i, C \rangle = 0 \end{cases}$

Then $\sum_{i \in I} \lambda_i \, \tau(H_g \cup M, \Gamma_i \cup \Gamma) = 0.$

So let $\sum_i \lambda_i [\Gamma_i] \sim 0$ if (*) holds : the <u>skein relations</u>.

Then $Z_{\mathcal{C}}^{RT}(\Sigma, \delta) \equiv Sk_{\mathcal{C}}(H_g)$

General 3-manifolds

The definition

$$Sk_c(M) = k\langle \substack{c\text{-graphs} \\ \Gamma \hookrightarrow M} \rangle / \text{skein relations}$$

makes sense for any 3-manifold M (with ∂-markings!)

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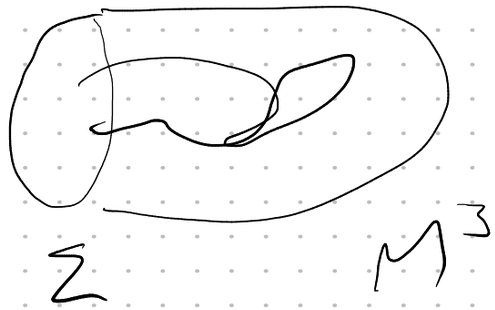
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Fact: For any M with $s_M: \Sigma \xrightarrow{\sim} \partial M$, the map

$$\varphi_M: Sk_c(M) \longrightarrow Z_c^{RT}(\Sigma, \delta_H)$$

$$[\Gamma] \longmapsto Z_c^{RT}\left(\emptyset \xrightarrow{(M,\Gamma)} \Sigma\right)(1)$$

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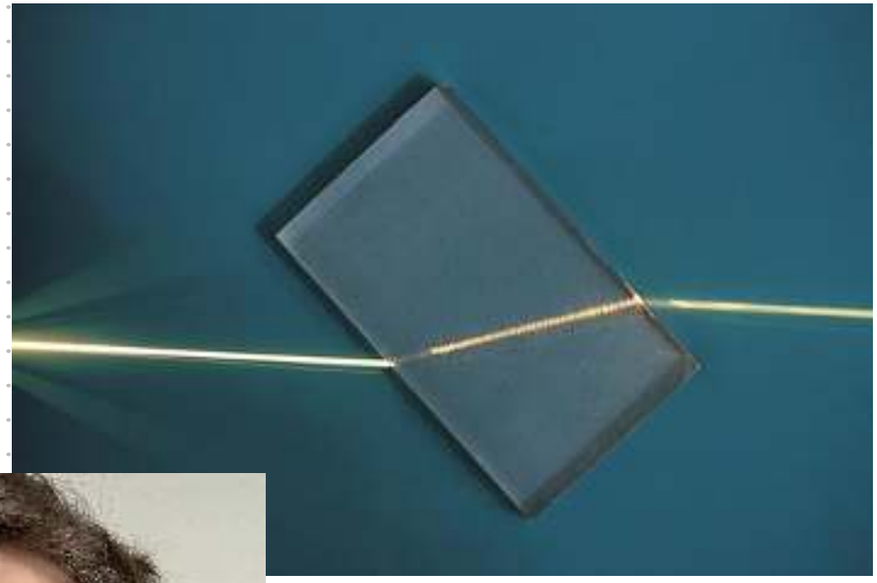
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Convention: suppress δ_H, i.e. write $\varphi_M : Sk_c(M) \xrightarrow{\sim} Z_c^{RT}(\Sigma)$

Defects: change of regime

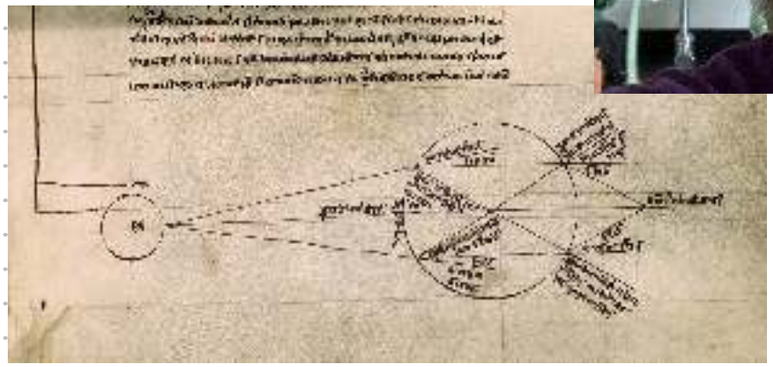




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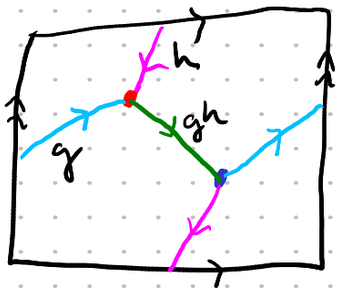
Roger Bacon: Grosseteste's theory of refraction in a spherical lens
"De Multiplicatione Specierum", 13CE. Public Domain.

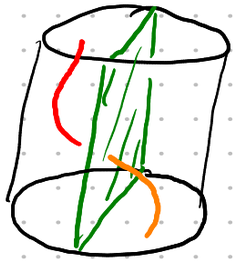
Defects: carriers of symmetry

Algebras carry symmetries of vector spaces ... $k[G] \subset H$.

TFTs carry symmetry defects of QFTs.

$Z: \mathrm{Bord}(\mathbb{D}) \longrightarrow \mathcal{V}$





E.g. $\mathcal{C}_a^\times = \bigoplus_{g \in G} \mathcal{C}_g$ a G-crossed braided fusion category, $\mathcal{C}_1 = \mathcal{C}$,

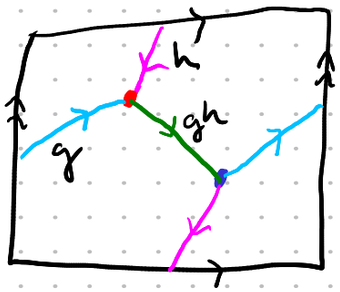
$G \to \mathrm{Aut}_\otimes(\mathcal{C}_a^\times)$, compatible braiding.

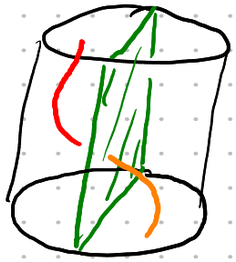
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Sometimes want to pass to symmetry-invariant setting. "Invariants" "Gauging" "Orbifolding"

$$H \rightsquigarrow eH, \quad e = \frac{1}{|G|} \sum_{g \in G} g.$$

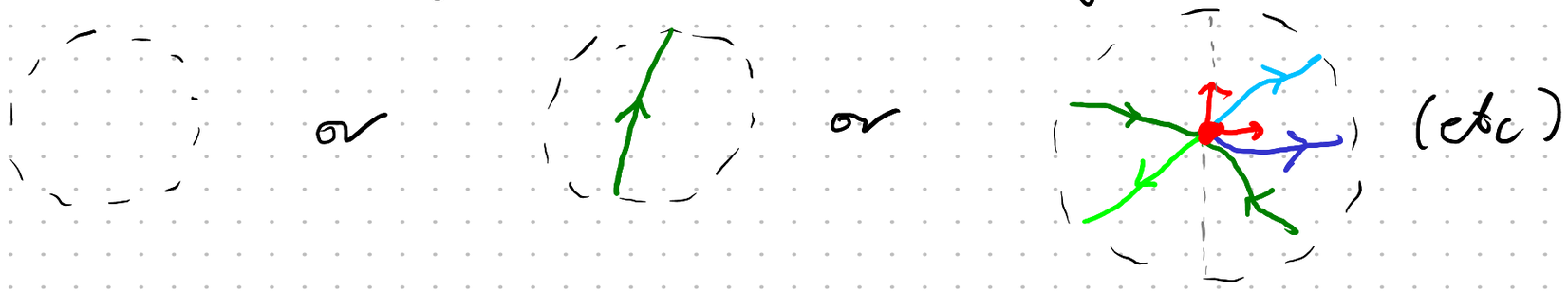
Programme of [Carqueville-Runkel-Schaumann] to add defects and orbifold them.

$G \subset Z^e$ via $\mathcal{C}_G^\times$, $G \subset Z^X$ via $G \subset X \rightsquigarrow Z^{X/\!/G}$

Stratified Surfaces

A **stratified surface** is

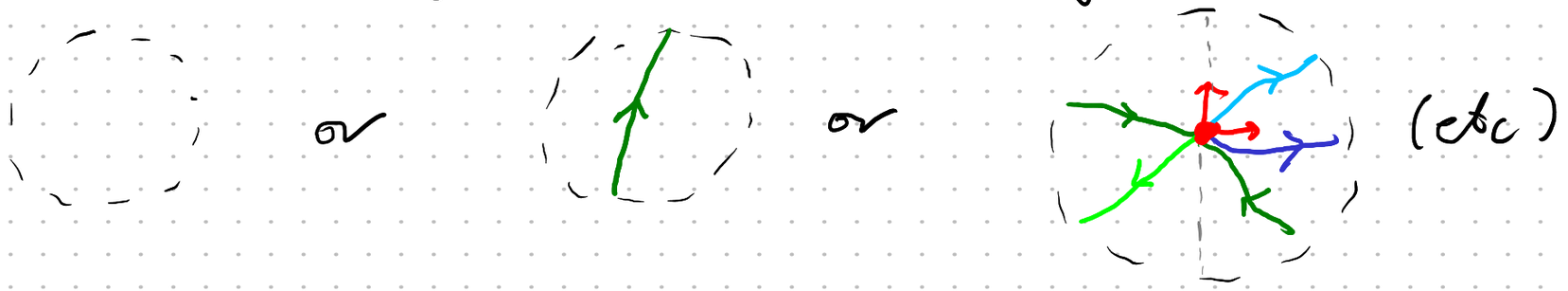
- a closed, compact, oriented surface Σ
- a finite set of embedded 0- and 1-manifolds (connected components: strata).
- such that every point has a neighbourhood like:



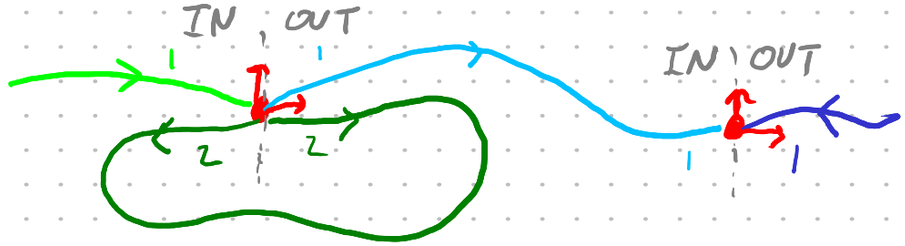
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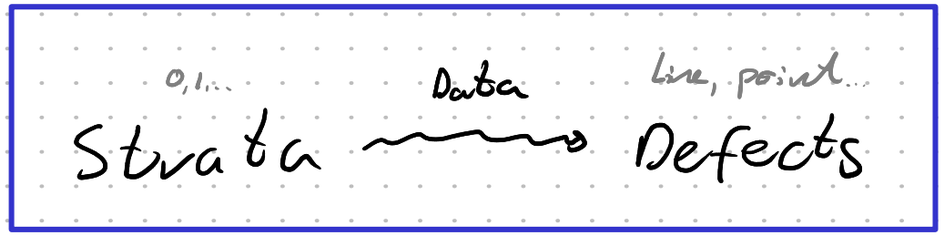
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I.e. 1-strata oriented, 0-strata framed, framing specifies in/out + ordering unambiguously.



Defect shein modules



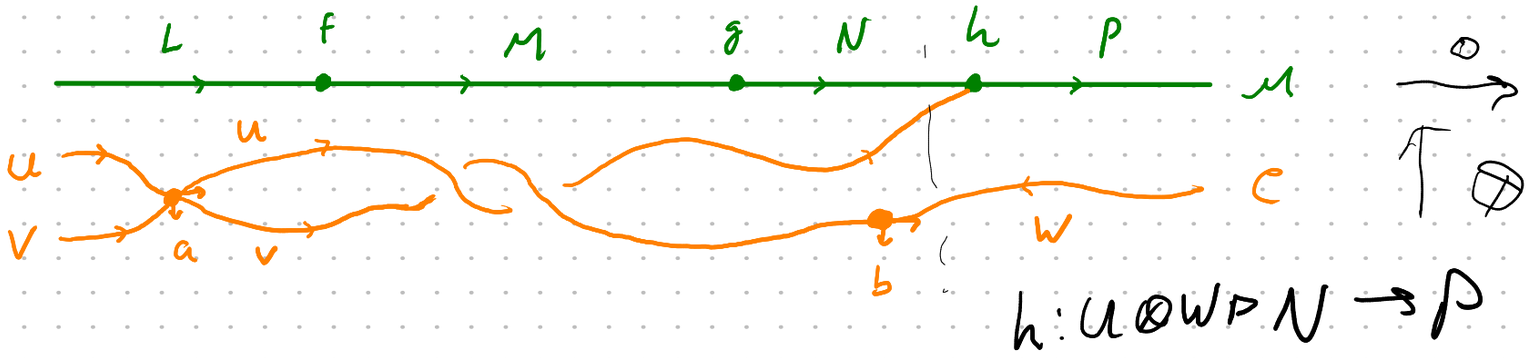
Defect shein modules

0, 1, ... Data Line, point, ...
Strata ⇝ Defects

Line defects : $\mathcal{C}$-module category $\mathcal{M}$

Def: A $\mathcal{C}$-module category is a category $\mathcal{M}$ equipped with $\mathcal{M} \times \mathcal{C} \xrightarrow{\triangleleft} \mathcal{M}$ and $\alpha : (- \triangleleft -) \circ - \Rightarrow - \triangleleft (- \otimes -)$ ...

Graphically:



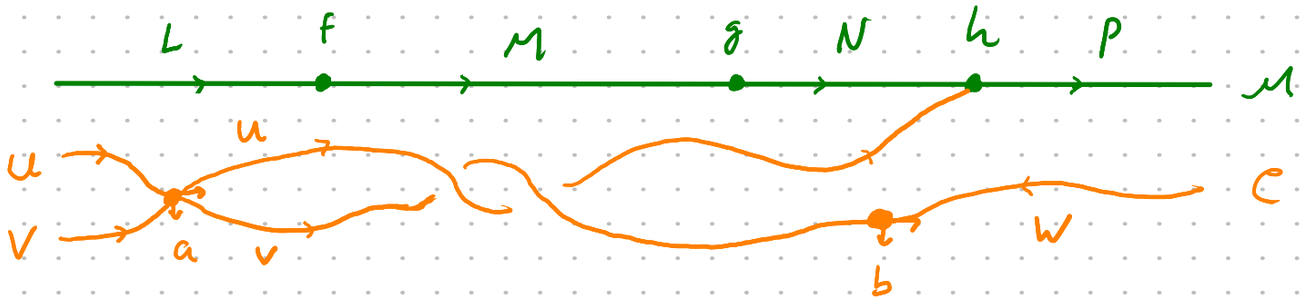
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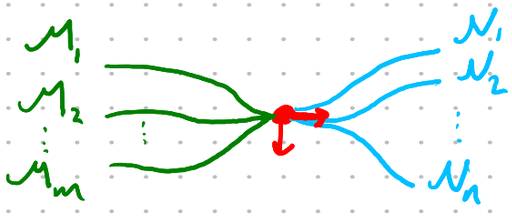
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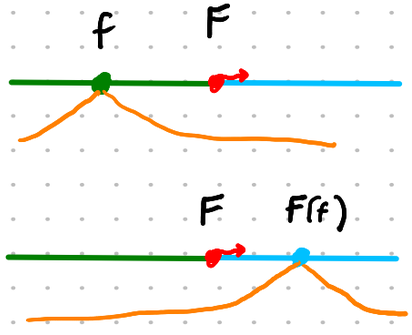
Point defects:



$\mathcal{C}$-module functor $F : \mathcal{M} \to \mathcal{N}$

$\mathcal{M} = \mathcal{M}_1^{\varepsilon_1} \boxtimes_{\mathcal{C}} \dots \boxtimes_{\mathcal{C}} \mathcal{M}_m^{\varepsilon_m}$ etc.

Graphically:



$\mathcal{M}_i^{\varepsilon_i} = \begin{cases} \mathcal{M}_i : \text{oriented with framing} \\ \mathcal{M}_i^{op} : \text{oriented against.} \end{cases}$ etc.

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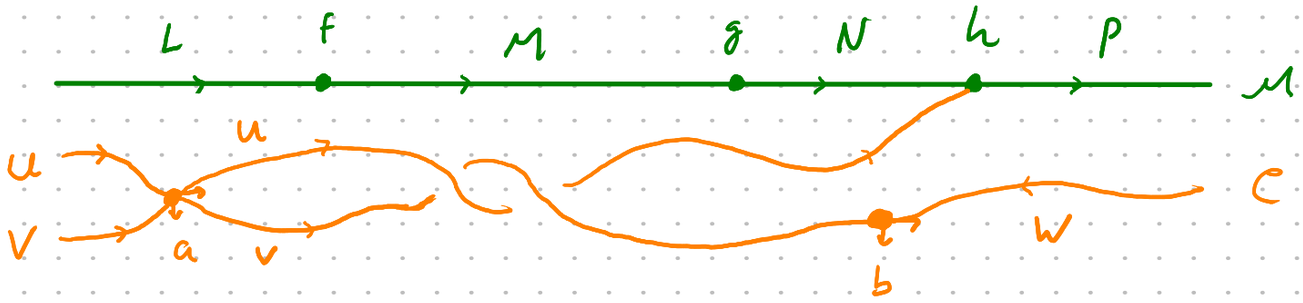
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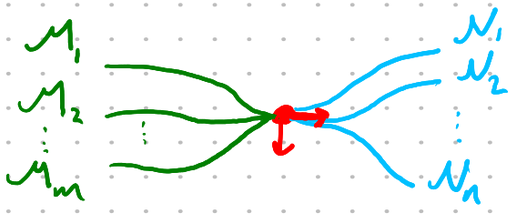
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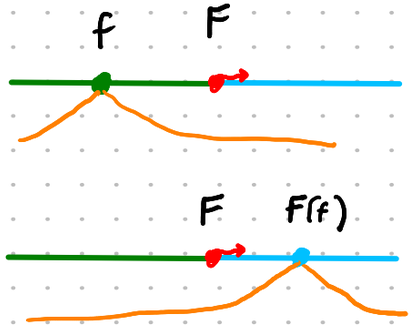
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Notation Σ, $\mathbb{M}$, etc...

Let M a 3-manifold with $\partial M \simeq \Sigma$.

Def A <u>compatible</u> ribbon graph is $\Gamma \hookrightarrow M$

such that $\Gamma \cap \partial M$ <u>refines</u> the defect graph.

A <u>colouring</u> of Γ is an assignment

edges $\rightsquigarrow$ objects (of category labelling stratum)

vertices $\rightsquigarrow$ morphisms from input $\to$ output edges

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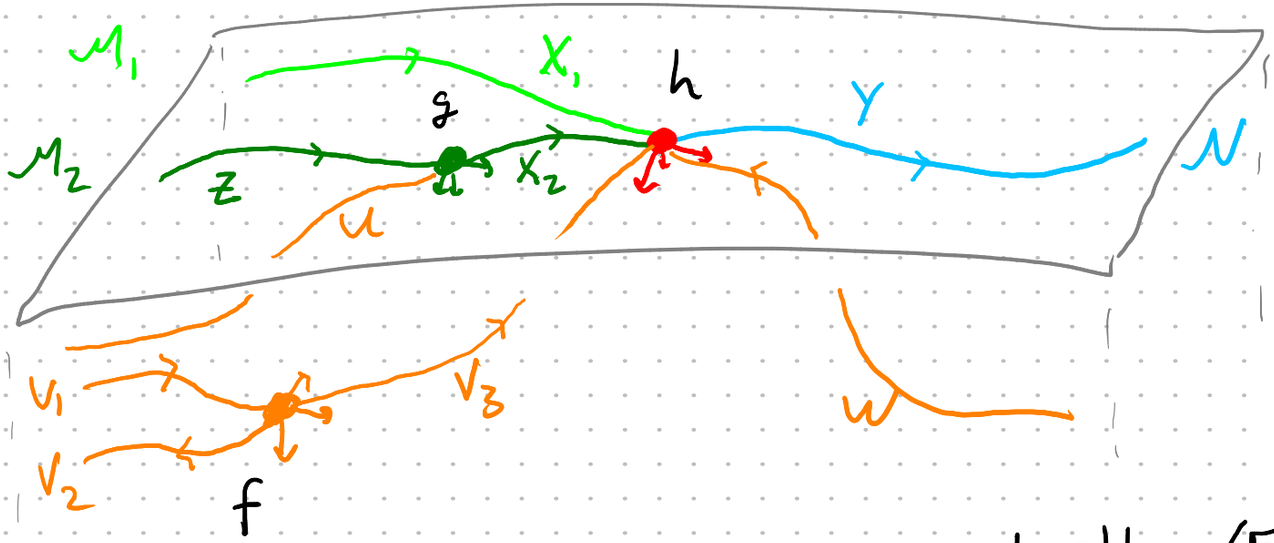
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E.g.:



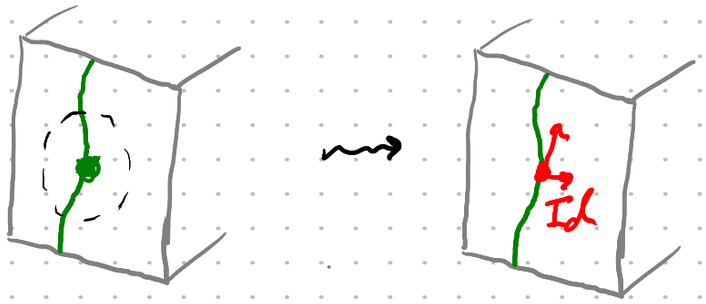
$f \in \mathrm{Hom}_{\mathcal{C}}(V_1 \otimes V_2^{\vee}, V_3)$

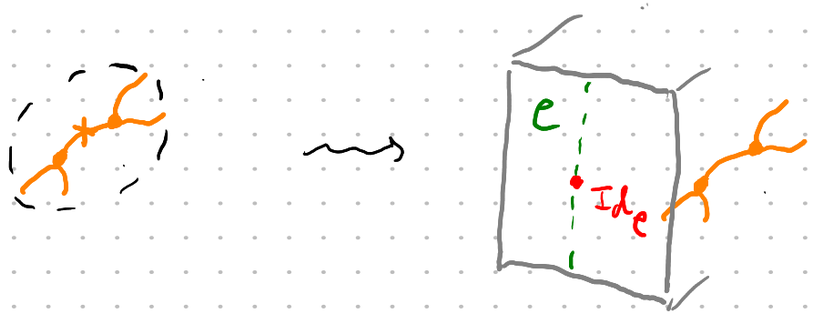
$g \in \mathrm{Hom}_{M_2}(z \triangleleft u, X_2)$

$h \in \mathrm{Hom}_{N}(F(X_1 \boxtimes X_2) \triangleleft V_3, Y \triangleleft W^{\vee})$

Skein relations

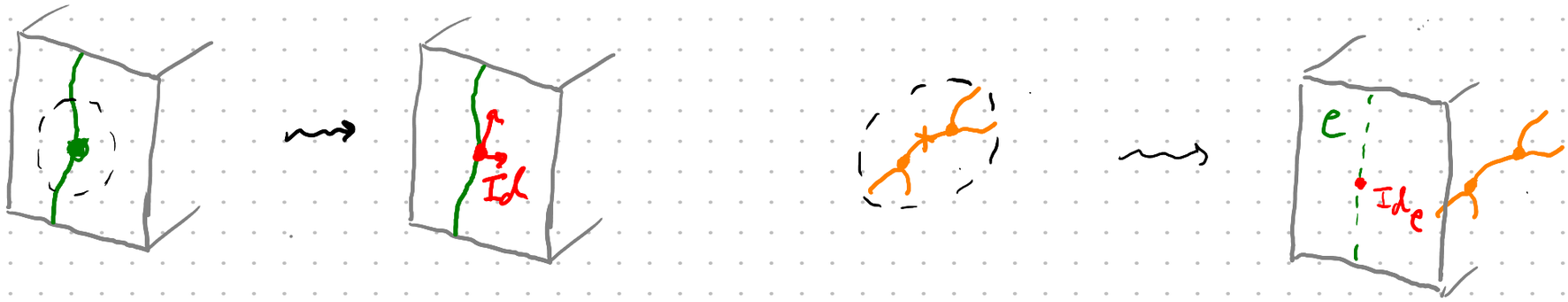
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- Can get all links as special cases of links of point objects





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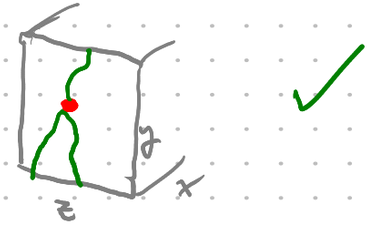
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$\Gamma \hookrightarrow M$ coloured graph.

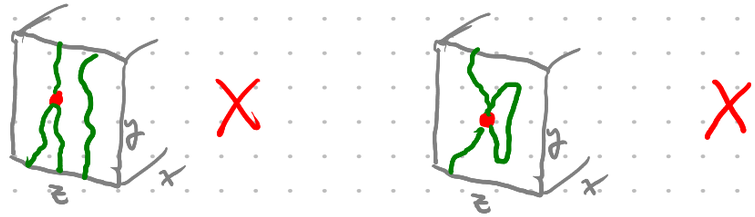
$C : [0,1] \times [-\frac{1}{2}, \frac{1}{2}] \times [0,1] \hookrightarrow M$ cube, called <u>valid</u> if:

- C meets a unique point defect P at $C(0,0,0)$.
- C only meets line defects incident to P.
  (which meet $\{y = \pm \frac{1}{2}\}$ transversely).



✓

- $\partial C \cap \Gamma \cap \operatorname{int} M \subseteq \{y = \pm \frac{1}{2}\}$,
  with distinct x-coords.

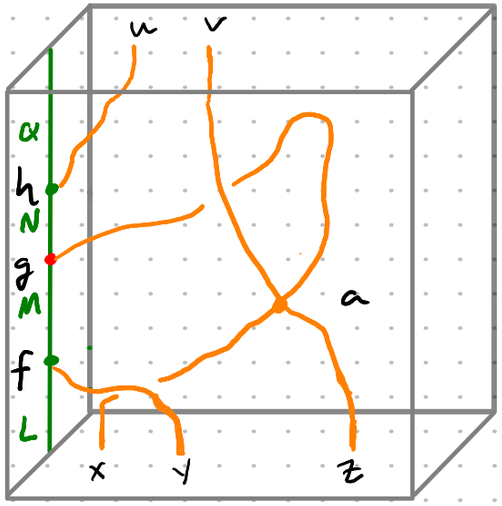


X X

Assume ρ Labelled by $F: \mathcal{M} \longrightarrow \mathcal{N}$. (possibly $F = Id$)

Project $\Gamma \cap C$ to xy-plane, apply F. Define:

$\langle \Gamma, C \rangle =$



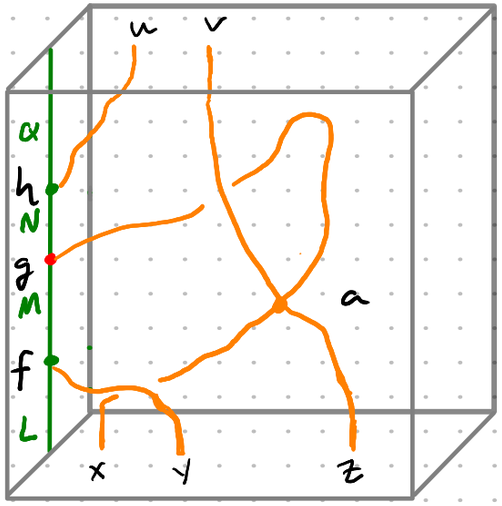
$\in Hom_{\mathcal{N}}\left(F(L) \triangleleft (X \otimes Y \otimes Z),\ Q \triangleleft (U \otimes V)\right)$

$(1 \triangleleft (1 \otimes ev)) \circ (h \triangleleft (c \otimes 1)) \circ g \circ (1 \triangleleft a) \circ F(f) \circ (1 \triangleleft (c \otimes 1))$

Assume Γ labelled by $F: \mathcal{M} \to \mathcal{N}$. (possibly $F = Id$)

Project $\Gamma \cap C$ to xy-plane, apply F. Define:

$$\langle \Gamma, C \rangle = $$



$$\in \mathrm{Hom}_{\mathcal{N}}\left(F(L) \triangleleft (X \otimes Y \otimes Z),\ Q \triangleleft (U \otimes V)\right)$$

$$(\mathrm{id}(\mathrm{id} \otimes \mathrm{ev})) \circ (h \triangleleft (\sigma \otimes \mathrm{id})) \circ g \circ (\mathrm{id} \triangleleft \alpha) \circ F(f) \circ (\mathrm{id} \triangleleft (\sigma \otimes \mathrm{id}))$$

Def Say $\sum_i \lambda_i [\Gamma_i] \sim 0$ if $\exists$ a valid cube C such that $\Gamma_i \setminus C = \Gamma_j \setminus C \ \forall i,j$ and $\sum_i \lambda_i \langle \Gamma_i, C \rangle = 0$

Def (K-Kuntzel) The defect shein module of $\mathcal{M}$ is

$$Sh(\mathcal{M}) = k\left\langle \begin{array}{c} \text{Colored graphs} \\ \Gamma \hookrightarrow \mathcal{M} \end{array} \right\rangle / \sim$$

RT with defects

[Carqueville - Runkel - Schaumann]

$\widehat{\mathrm{Bord}}_3^{\mathrm{def}}$: defect surfaces + 3-bordisms with $\left\{\begin{array}{c}\text{surface}\\ \text{line}\\ \text{point}\end{array}\right\}$ defects, where...

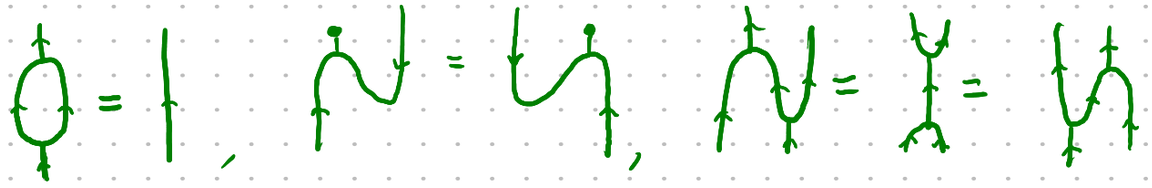
RT with defects

[Carqueville - Runkel - Schaumann]

$\widehat{\text{Bord}}_3^{\text{def}}$: defect surfaces + 3-bordisms with $\left\{\begin{matrix}\text{surface}\\ \text{line}\\ \text{point}\end{matrix}\right\}$ defects, where...

- surface defects in 3d : Label by
  (line defects in 2d)

Δ-separable symmetric Frobenius algebras



$(A \in \Delta\text{ssFrob} \overset{(\dagger)}{\rightsquigarrow} A\text{-mod}_C \ni C)$

$E.g.\ C_G^* = \oplus C_g$

$C \circlearrowright C_g \ni x,\ \underline{\text{End}}_C(X).$

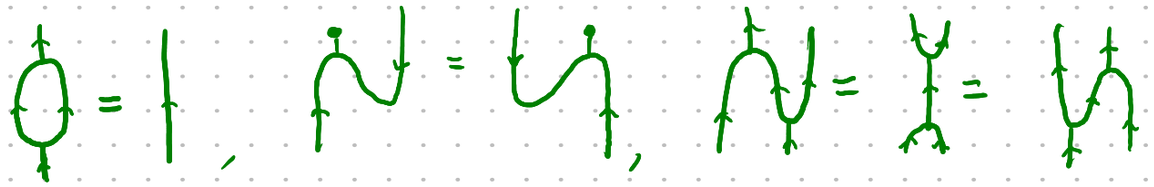
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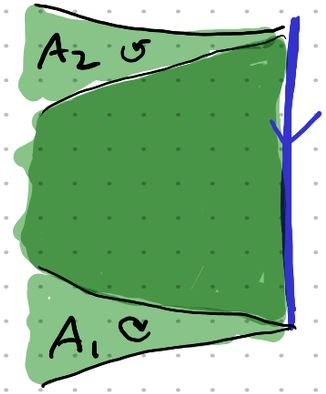


$(A \in \Delta\text{ssFrob} \overset{(\dagger)}{\rightsquigarrow} A\text{-mod}_{\mathcal{C}} \circlearrowright \mathcal{C})$

  E.g. $\mathcal{C}_G^{\times} = \bigoplus_{g\in G} \mathcal{C}_g$
  $\mathcal{C} \circlearrowright \mathcal{C}_g \ni X$, $\underline{\mathrm{End}}_{\mathcal{C}}(X)$.

- line defects in 3d: only incoming surfaces,
  (point defects in 2d) Label by multimodules

$$(X_{A_1 \otimes \dots \otimes A_n} \overset{(\dagger)}{\rightsquigarrow} A_1\text{-mod} \boxtimes_{\mathcal{C}} \dots \boxtimes_{\mathcal{C}} A_n\text{-mod} \xrightarrow{(X \otimes_A -)} \mathcal{C})$$



$X_{A_1 \otimes A_2^{\mathrm{op}}}$

RT with defects

[Carqueville - Runkel - Schaumann]

$\widehat{\text{Bord}}_3^{\text{def}}$: defect surfaces + 3-bordisms with $\left\{\begin{array}{c}\text{surface}\\ \text{line}\\ \text{point}\end{array}\right\}$ defects, where...

- surface defects in 3d: Label by
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  Δ-separable symmetric Frobenius algebras

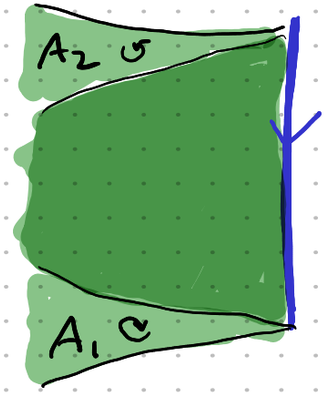


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$X_{A_1 \otimes A_2^{\text{op}}}$

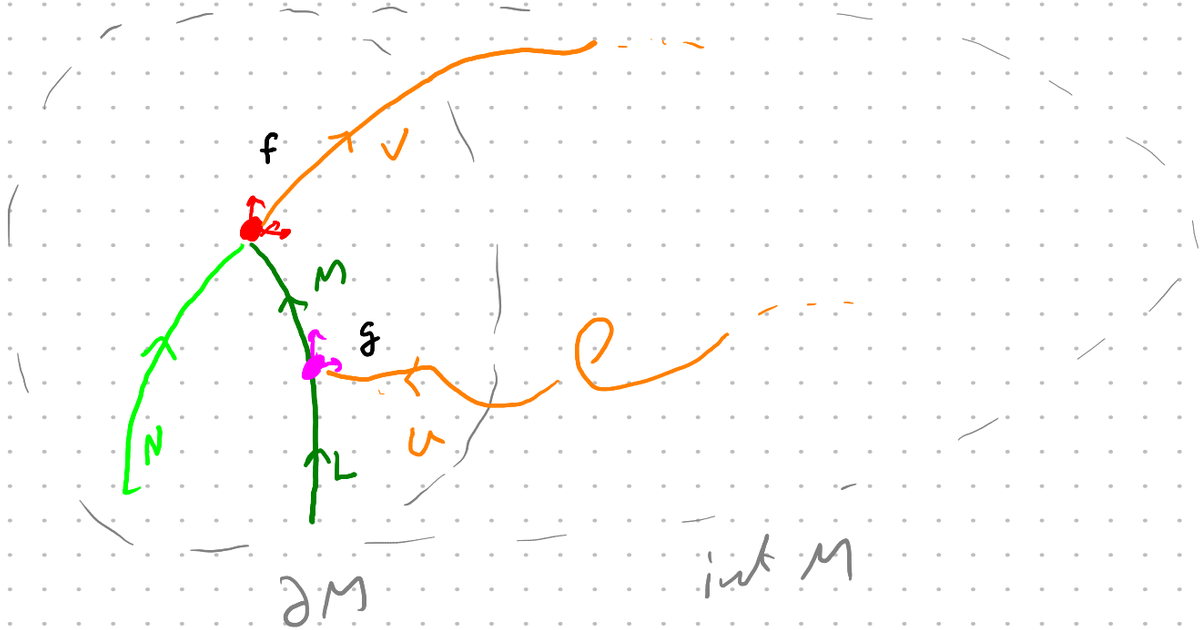
- point defects in 3d: multimodule maps
  (do not exist in 2d!)

Sheins vs State Spaces

(†): $A \rightsquigarrow A\text{-Mod}$ etc.

$\mathcal{E}$ with defects in image of (†), $s: \mathcal{E} \xrightarrow{\sim} \partial M$. Then:

$$\begin{array}{rccc} \Phi: & Sh(M) & \longrightarrow & Z_e^d(\mathcal{E}) \\ & [\Gamma] & \longmapsto & Z_e^d\left((M,\Gamma) \underset{s}{\cup} \mathcal{E}\times I\right)(1) \end{array}$$

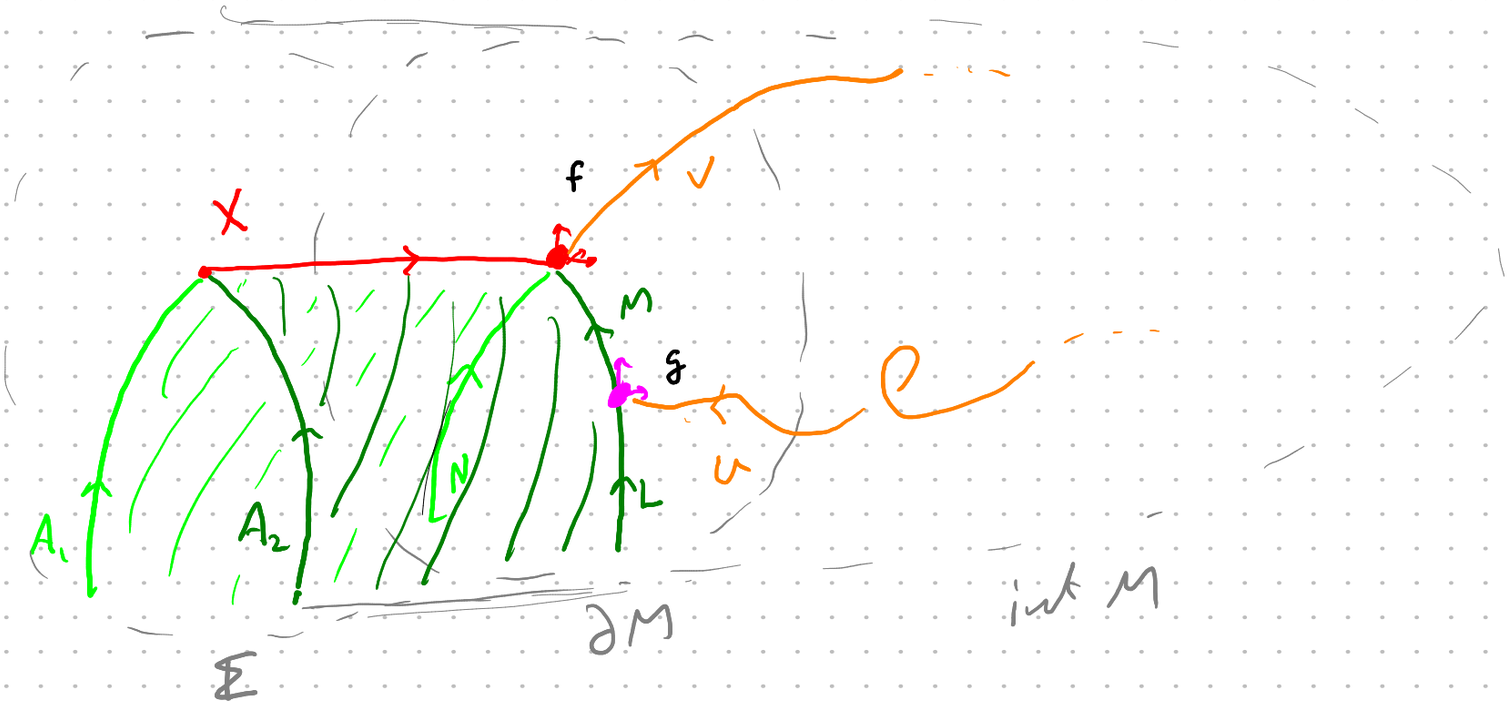


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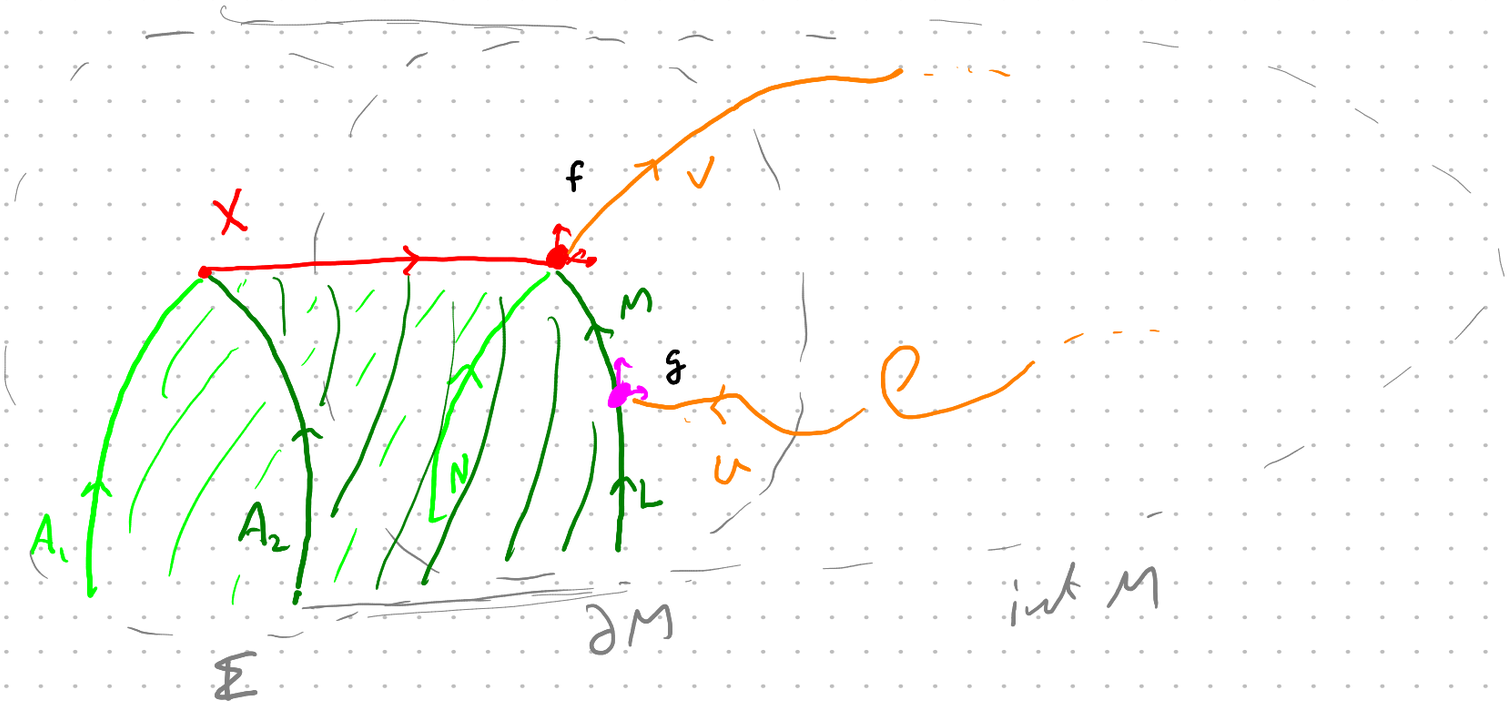


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Theorem (K-Runkel): the above map is an isomorphism.

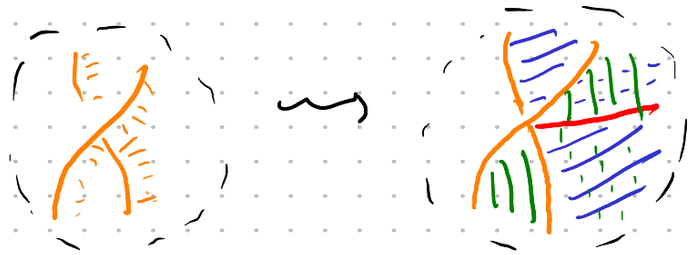
Remarks

✡ Shein theory: adding (semisimple) defects - is "linear-algebra hard".

✡ Universal construction with non-ssi surface defects

- Shein relations: relations in $\mathcal{C}$-Mod

(line defect shein relations are relations in $\mathcal{C} \simeq \mathrm{End}_{\mathcal{C}\text{-Mod}}(\mathcal{C})$)



✡ Extension to 4d? 2d?

↪ non-ssi defect versions of CYK, WRT.

{Blanchet, Costantino, Dellenzi, Gainutdinov, Haioun, Geer, Patureau-Mirand, Runkel}

✡ Techniques from factorization homology.

↪ organization of defect systems

↪ E_n-alg $\rightsquigarrow$ n-cat.

(α) (β)

[Cooke, Braun-Haioun, Runkel-Scheimbauer-Thaun, Ayala-Francis-Rozenblyum]