Monotone Symplectic Manifolds with a Torus Action of Complexity One

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A **smooth Fano variety** X is a compact complex manifold, whose anti-canonical line bundle $\mathcal{L} = -\det(T^*X)$ is ample, i.e.,

$$\exists \text{ holomorphic embedding } \; i \colon X \hookrightarrow \mathbb{C}P^N \;\; \text{s.t.} \;\; \mathcal{L}^k = i^*(\mathcal{O}(1))$$

for some $N, k > 0$.
By pulling back the Fubini-Study symplectic form

$$\omega_{FS} \in \Omega^2(\mathbb{C}P^N),$$

X together with $\omega_x \colon = i^*(\omega_{FS})$ is a symplectic manifold .

Smooth Fano Varieties

Since, the first Chern class of a line bundle equals the first Chern class of its top exterior power, we have that

$$k \cdot c_1(TX) = k \cdot c_1(\mathcal{L}) = c_1(\mathcal{L}^k) = c_1(i^*(\mathcal{O}(1)) = [\omega_X].$$

Monotone Symplectic Manifolds

- (M^{2n}, ω) : **compact** symplectic manifold
- $J\colon TM \to TM$: almost complex structure
  compatible with ω, i.e. $\omega(\cdot, J\cdot)$ is a Riemannian metric
- $c_1(M)$: first Chern class of (TM, J)

Definition

A symplectic manifold is **monotone** if

$$c_1(M) = \lambda\,[\omega] \qquad \text{with } \lambda > 0.$$

Monotone vs. Fano

Question

In which context monotone symplectic manifolds are different from smooth Fano varieties?

Fact:

If a monotone symplectic manifold (M, ω) is Kähler , i.e., there exists an integrable almost complex structure J s.t.

$$\omega(\cdot, J\cdot)$$

is a Riemannian metric, then, by the **Kodaira Embedding Theorem**, M with the complex atlas that induces J is a Fano variety.

Monotone vs. Fano: What is known?

Given a monotone symplectic manifold (M, ω) :

- $\mathbb{R}$-dim $= 2$: $M = \mathbb{C}P^1$ which is Fano.
- $\mathbb{R}$-dim $= 4$:
  Ohta-Ono ('96): M is Kähler, hence Fano.

$$M = \mathbb{C}P^2, \mathbb{C}P^1 \times \mathbb{C}P^1, \text{ or}, \mathbb{C}P^2 \# k\overline{CP^2},\ k \leq 8.$$

- $\mathbb{R}$-dim ≥ 12 :
  Fine–Panov ('10): Examples, where M is even not homotopy equivalent to a Fano variety.
- dimension $6, 8, 10$: Open?

What is about monotone symplectic manifolds with "nice" symmetries ?

Hamiltonian T-spaces

- (M^{2n}, ω) : **compact** symplectic manifold
- $T^d \cong (S^1)^d \curvearrowright (M, \omega)$ effective and Hamiltonian

  with moment map $\phi : M \to \left(\mathrm{Lie}(T^d)\right)^*$, i.e., for $\xi \in \mathrm{Lie}(T^d)$

$$\omega(X_\xi, \cdot) = -d \langle \phi, \xi \rangle$$

- the **complexity** of this action is $k := n - d \geq 0$
- (M, ω, T, ϕ) is called a **complexity k space**

What is known about monotone Hamiltonian T-spaces? Complexity Zero and Dimesion Four

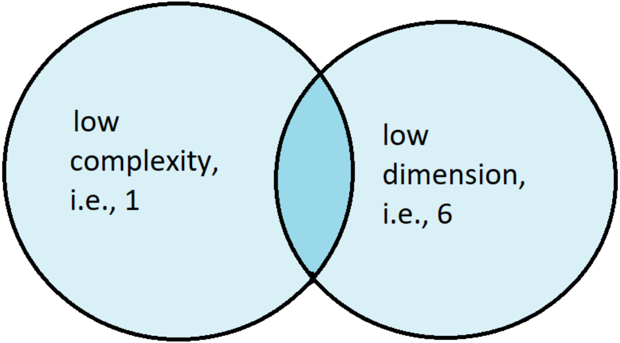
- Delzant ('88): Classification of complexity zero spaces
- Karshon ('99): Classification of 4-dimensional complexity one space

In both cases the space admits a T-invariant Kähler structure.

Hence, for that cases monotone implies that the space is equivaiently symplectomorphic to a smooth Fano variety with a holomorphic T-action.

What is known about monotone Hamiltonian T-spaces? Beyond Complexity Zero and Dimesion Four

Where to start beyond complexity zero and dimension four?



What is known about monotone Hamiltonian T-spaces? Dimension 6

Conjecture (Fine, Panov '10)

A monotone Hamiltonian S^1-space of dimension six is diffeomorphic to a Fano threefold.

Lindsay, Panov '19:
Results, which are supporting this conjecture:

- $\pi_1(M) = 0$,
- any connected component of M^{S^1} is a point, a 2-sphere, or a Del Pezzo surface

What is known about monotone Hamiltonian T-spaces? Dimension 6

Conjecture (Fine, Panov '10)

A monotone Hamiltonian S^1-space of dimension six is diffeomorphic to a smooth Fano threefold.

In two cases this is confirmed. Namely, if

- $b_2(M) = 1$ (Tolman '07, McDuff '08), or
- the action is semifree
  (i.e, any $p \notin M^{S^1}$ has trivial stabilizer).
  In this case these spaces are classified and they admit an invariant Kähler structure (Cho '19).

What is known about monotone Hamiltonian T-spaces? Complexity One

Sabatini, Sepe '19:

Monotone complexity one spaces share topological properties with Fano varities:

- $\pi_1(M) = 0$,
- any connected component of M^T is a point,or a 2-sphere.

What is known about Hamiltonian T-spaces? Complexity One

Given a complexity one space (M, ω, T, ϕ), for each $x \in \phi(M)$ $M_x \colon = \phi^{-1}(x)/T$ is a point or a topological surface.

Definition

A complexity one space (M, ω, T, ϕ) is **tall** if M_x is a topological surface for all $x \in \phi(M)$.

Karshon-Tolman '03: Classification of (non-compact) **tall** complexity one spaces

Results

Main Theorem (C.-Sabatini-Sepe '25)

Let $(M^{2n}, \omega, T^{n-1}, \phi)$ *be a tall complexity one space with* $c_1(M) = [\omega]$.
Then:

- *(Uniqueness)* (M, ω, T, ϕ) *is determined by its Duistermaat-Heckman function.*
- *(Finiteness) There exist just finitely many of such spaces up to twisted equivariant symplectomorphisms.*
- *(Extension) The* T^{n-1}*-action can be extended to a complexity zero action.*

Results : Tall and Monotone Complexity One Spaces

Theorem (C.-Sabatini-Sepe '25)

Let $(M^{2n}, \omega, T^{n-1}, \phi)$ be a tall complexity one space with $c_1(M) = [\omega]$.

The space admits an invariant Kähler structure J.

In particular, the space is equivariantly symplectomorphic to a smooth Fano variety with a holomorphic T-action.

First step for the proof of the Main Theorem:

Proposition (C.-Sabatini-Sepe '25)

Let $(M^{2n}, \omega, T^{n-1}, \phi)$ be a tall complexity one space with $c_1(M) = [\omega]$. Then $\phi(M) = \Delta_M$ is -up to translation- a ***reflexive Delzant polytope*** *of dimension $n-1$.*

The Convexity Theorem

Theorem (Atiyah, Guillemin-Sternberg '82)

Let (M, ω, T, ϕ) be a Hamiltonian T-space. Then $\phi(M) = \Delta_M$ is a convex polytope and the fibers of ϕ are connected.
In particular,

$$\phi(M) = Conv\left\{\phi(C_1), ..., \phi(C_r)\right\},$$

where $C_1, ..., C_r$ are the connected components of M^T.

Given $p \in M^T$:
$\exists$ complex coordinates $(z_1, ..., z_n)$ at p and $\alpha_1, ..., \alpha_n \in \ell_T^*$, s.t. for $\xi \in \mathrm{Lie}(T)$

$$T \ni \exp(\xi)(z_1, ..., z_n) = (e^{i\alpha_1(\xi)} z_1, ..., e^{i\alpha_n(\xi)} z_n,)$$

$$\phi(z) = \frac{1}{2} \sum_{j=1}^{n} \alpha_j \left| z_j \right|^2 + \phi(p).$$

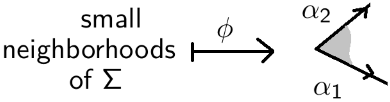
Effectiveness $\rightsquigarrow \mathbb{Z} \langle \alpha_1, ..., \alpha_n \rangle = \ell_T^* \, (\cong \mathbb{Z}^{\dim(T)})$
$\rightsquigarrow$ for any connected component $C \subset M^T$:

$$\dim_{\mathbb{R}}(C) \leq 2 \cdot \textit{complexity}$$

If $\dim_{\mathbb{R}}(C) = 2 \cdot \textit{complexity} \rightsquigarrow$ The normal weights of C form a $\mathbb{Z}$-basis of ℓ_T^*.

For complexity one spaces: Given a fixed surface $\Sigma \subset M^T$ with normal weights $\alpha_{\Sigma,1}, ..., \alpha_{\Sigma,n-1}$.

Small neighborhoods of Σ are mapped to the cone $\phi(\Sigma) + \mathbb{R}_{\geq 0} \langle \alpha_{\Sigma,1}, ..., \alpha_{\Sigma,n-1} \rangle$



Since the fibers of ϕ are connected:
$\rightsquigarrow$ $\phi(\Sigma)$ is a vertex of Δ_M and the edges meeting at $\phi(\Sigma)$ are of the form

$$e_i \subset \phi(\Sigma) + \mathbb{R}_{\geq 0}\alpha_{\Sigma,i} \quad \text{for } i = 1, ..., n-1.$$

In particular, $\phi(\Sigma)$ is a **"Delzant vertex"**.

The preimage of a vertex: Complexity One Spaces

Let (M, ω, T, ϕ) be a complexity one space, and let $\Delta_M = \phi(M)$ be its moment map polytope.

Facts:

- For each vertex v of Δ_M, $\phi^{-1}(v)$ is a connected component of M^T, so it is a point or surface.
- (M, ω, T, ϕ) is tall if and only if $\phi^{-1}(v) = \phi^{-1}(v)/T$ is a surface for any vertex v of Δ_M.

The Moment Map Polytope of a Tall Complexity One Space

Lemma

The moment map polytope of a tall complexity one space is ***Delzant*** *and*

$$v \mapsto \phi^{-1}(v)(=: \Sigma_v)$$

defines a bijection between the vertices of Δ_M and the fixed surfaces.
The weights of $T \curvearrowright N\Sigma_v$ are the same as the ones of the vertex v.

Note: So far we did not use the monotone condition.

The Weight Sum Formula: A powerful Tool

Given a Hamiltonian T-space (M, ω, T, ϕ) with $c_1(M) = [\omega]$.

By results in equivariant cohomology [Kirwan '84], (after a shift) the moment map satisfies the

weight sum formula:

$$\forall p \in M^T: \qquad \phi(p) = -\alpha_1 - ... - \alpha_n,$$

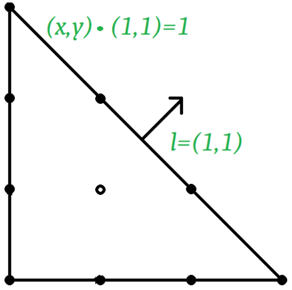
where $\alpha_1, ..., \alpha_n$ are the weights of $T \curvearrowright T_pM$.

Reflexive Polytopes

A d-dimensional polytope $\Delta \subset \mathbb{R}^d$ is **reflexive**, if it is **integral** and if for each facet F

$$F \subset \left\{ x \in \mathbb{R}^d | \, \langle x, l \rangle = 1 \right\},$$

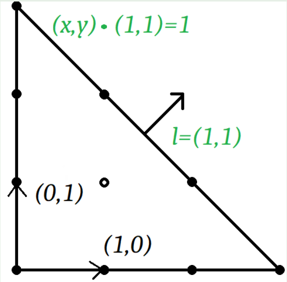
where $l \in \mathbb{Z}^d$ is the primitive outward normal vector to F.



Fact: There exist just finitely many reflexive (Delzant) polytopes in any dimension up to $GL(\mathbb{Z}, n)$-transformations.

When is a Delzant Polytope reflexive?

Example (of a reflexive Delzant Polytope)



This polytope satisfies the **vertex Fano condition,** i.e., $v = -$sum of the weights at v for all vertices.

Fact: For a Delzant polytope:

vertex Fano condition $\iff$ reflexive

The moment map polytope

Proof of:

Proposition (C.-Sabatini-Sepe '25)

Let $(M^{2n}, \omega, T^{n-1}, \phi)$ be a tall complexity one space with $c_1(M) = [\omega]$. Then $\phi(M) = \Delta_M$ is -up to translation- a ***reflexive Delzant polytope*** *of dimension $n-1$.*

Proof.

Given a tall complexity one space (M, ω, T, ϕ), such that $c_1(M) = [\omega]$.

- Δ_M is Delzant and the weights of $T \curvearrowright N\Sigma_v$ are the same the ones v for vertex v.

- ϕ satisfies the **weight sum formula**, $\implies$ vertex Fano condition ($\iff$ reflexive)

$\square$

Second step for the proof of the Main Theorem:

The Minimum of the Duistermaat-Heckman Function

The DH-function

Given a Hamiltonian T-space (M, ω, T, ϕ) its Duistermaat-Heckman measure is

$$m_{DH}(U) = \int_{\phi^{-1}(U)} \frac{\omega^n}{n!},$$

for any Borel set $U \subset \Delta_M \subset (\mathrm{Lie}(T))^* \cong \mathbb{R}^d$.

The DH-function

By work of Duistermaat-Heckman ('82) and Guillemin-Lerman-Sernberg ('88), there exists a unique continuous function

$$DH_M \colon \Delta_M \to \mathbb{R}_{\geq 0}$$

-called the Duistermaat-Heckman function- s.t.

$$m_{DH}(U) = \int_U DH_M \, \mathrm{d}\, x^d,$$

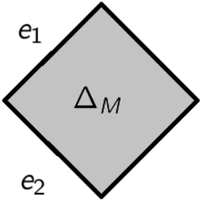
where $\mathrm{d}\, x^d$ is the Lebesgue measure on $\mathrm{Lie}(T))^* \cong \mathbb{R}^d$.

Example: For complexity zero $DH_M = 1$.

The DH-function near its minimum

(M, ω, T, ϕ) : tall complexity one space
$DH_M \colon \Delta_M \to \mathbb{R}_{\geq 0}$: DH-function
v : vertex of Δ_M with edges $e_i \subset v + t\alpha_i$

$\phi(\Sigma_v) = v$



$c_1(L_i)$: first Chern class of the normal bundle of Σ_v in $\phi^{-1}(e_i)$

near v :

$$DH_M(v + t_1\alpha_1 + \ldots + t_{n-1}\alpha_{n-1}) = -\sum_{i=1}^{n-1} t_i \cdot c_1(L_i)[\Sigma_v] + \int_{\Sigma_v} \omega$$

(Cho-Kim '12) $\implies$ DH_M is log-concave $\implies$ DH_M attains its minimum at a vertex of Δ_M.

Lemma (c.f. Sabatini-Sepe)

In the situation as above: Assume $c_1(M) = [\omega]$ *and that* DH_M *attains its minimum at* v*, then* Σ_v *is 2-sphere and either*

- $\int_{\Sigma_v} \omega = 2$ *and* $c_1(L_i)[\Sigma_v] = 0$ *for all* $i = 1, ..., n-1$

or

- $\int_{\Sigma_v} \omega = 1$ *and* $c_1(L_j)[\Sigma_v] = -1$ *for one* $j = 1, ..., n-1$ *and* $c_1(L_i)[\Sigma_v] = 0$ *for* $i \neq j$*.*

Proof.

- DH_M attains its minimum at $v \implies \forall$ i: $c_1(L_i)[\Sigma_v] \leq 0$
- $c_1(M) = [\omega] \implies c_1(M)[\Sigma_v] = \int_{\Sigma_v} \omega > 0$:

  $1 \leq c_1(M)[\Sigma_v] = c_1(\Sigma_v)[\Sigma_v] + \sum_i c_1(L_i)[\Sigma_v]$
- Hence, $c_1(\Sigma_v)[\Sigma_v] = 2$.

$\square$

Consequences of this Lemma

- Results of [Sabatini-Sepe '19]:
  For $x \in \phi(M)$, $M_x = \phi^{-1}(x)/T$ is a 2-sphere and M is simply connected.
  (By using a result of [Li'07])

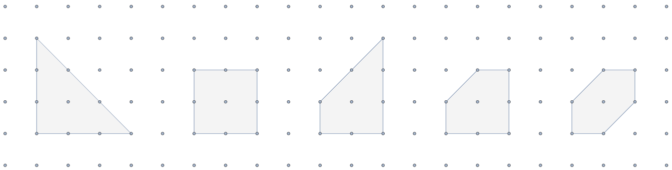
- Results for the isolated fixed points:
  Let v be a vertex on which the DH- function attains its minimum, then for an edge e_i meeting v ,s.t.

$$c_1(L_i)[\Sigma_v] = 0$$

  $\nexists$ isolated $p \in M^T$ with $\phi(p) \in e_i$
  by using an explicit formula for the DH-functions of complexity one spaces of dimension four
  [Guillemin-Lerman-Sternberg '88]

The remaining steps for the proof of the Main Theorem:

Verification in an Example in Dimension 6

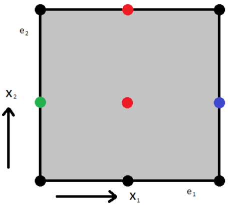


Let (M^6, ω, T^2, ϕ) be a tall complexity one space, s.t.

• $c_1(M) = [\omega]$

• ϕ satisfies the weight sum formula

• $\Delta_M =$ "*picture*".



W.l.o.g.:

• v=(-1,-1) minimum of DH_M

• near v : $DH_M = 2$ or $2 + x_2$

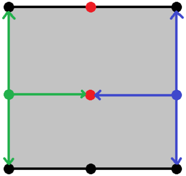
$\rightsquigarrow \nexists\ p \in M^T$ isolated with $\phi(p) \notin e_1$.

$\rightsquigarrow$ for $p \in M^T$ isolated : $\phi(p) \in \{(-1,0), (1,0), (0,0), (1,0)\}$

"Interaction between the isolated fixed points" and the weight sum formula imply :

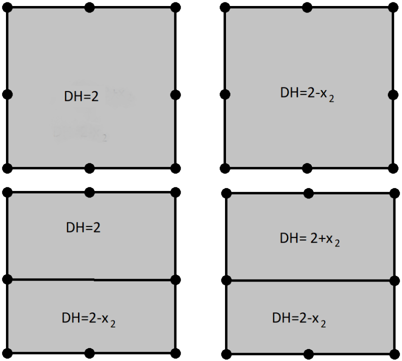
$\bullet \phi(p) \in \{(0,0),(0,1)\}$ is impossible for $p \in M^T$.

$\bullet$If $\phi(p) = (-1,0) or (1,0)$ for $p \in M^T$ the weights at p are as in the picture:



The number of $p \in M^T$ with $\phi(p) = (-1,0) or (1,0)$ is equal and at most two (formula for the DH function in dimension four) .

The Duistermaat-Heckman function is one of the following:



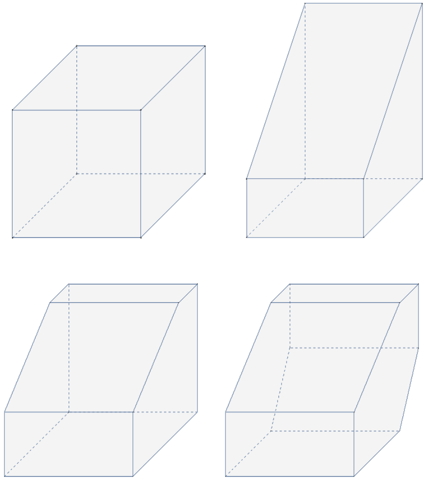
In particular, the Duistermaat-Heckmann function determines the space of exceptional orbits M_{exc}, which is as topological space the empty set or a resp. two compact intervals.

Uniqueness:

- Since monotonicity implies that the genus is zero (Sabatini-Sepe '19):
- (Karshon-Tolman '03) $\implies$ In this example the DH-function determines the space.

Existence and Extension:

These spaces are coming from the monotone symplectic toric manifolds of dimension six (M, ω, T^3, ϕ) given by the following Delzant polytopes (by forgetting one S^1-factor.)



Beyond the Tall Case?

Beyond the Tall Case

- The proofs for the tall case rely strongly on the classification of (non-compact) tall complexity one spaces by [Karshon-Tolman '03].
- Given a non-tall compact complexity one space (M, ω, T, ϕ), by erasing its short parts, i.e., the fibers of the moment map where $\phi^{-1}(x)$ is a single orbit, we obtain a non-compact tall complexity one space -called the tall part of (M, ω, T, ϕ)-.

Beyond the Tall Case

Conjecture (Karshon-Tolman '03)

Given two compact complexity one spaces, these spaces are equivariantly symplectomorphic if and only if their tall parts are equivariantly symplectomorphic.

Beyond the Tall Case

Theorem (C.-Kessler, Kessler-Wardenski, works in progress)

Given a compact complexity one space (M, ω, T, ϕ) of dimension 6, then M is diffeomorphic to a smooth Fano threefold X endowed with a holomorphic T-action.

Moreover, the tall part of (M, ω, T, ϕ) is equivariantly symplectomorphic to the part of a smooth Fano threefold endowed with a holomorphic T-action.

Here, we use again the classification of (non-compact) complexity one spaces by Karshon-Tolman('03). For the case that are only isolated fixed points, we used a computer program based on an algorithm by Godinho-Sabatini ('12).

Beyond the Tall Case

Conjecture

A compact complexity one space of dimension 6 is equivariantly symplectomorphic to a smooth Fano threefold X endowed with a holomorphic T-action.

Thank you for your attention!