Defect skein theory, parabolic restriction and the Turaev coproduct

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Juan Ramón Gómez García

IMJ - PRG (Paris)

Context and motivation: skein theory

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- monoidal
- rigid
- braided
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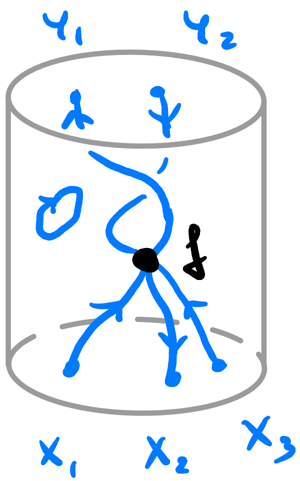

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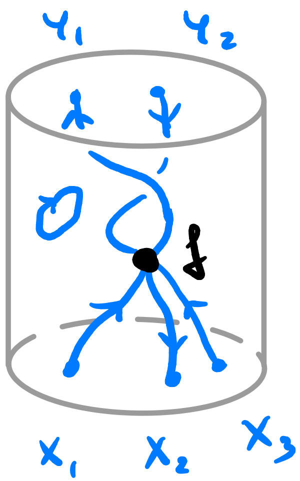

$$\longmapsto \varphi \in Hom_{\mathcal{A}}(X_1 \otimes X_2^* \otimes X_3, Y_1 \otimes Y_2^*)$$

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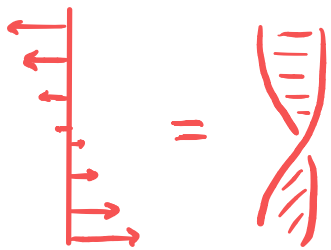

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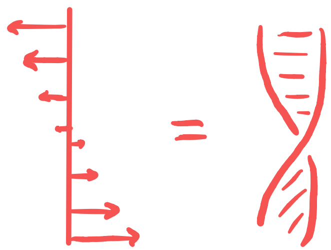
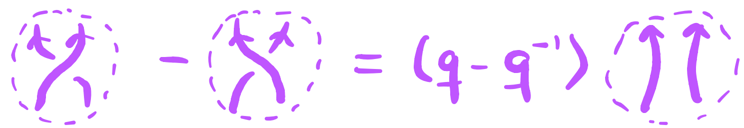


Kernel of the evaluation functor



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Skein algebra: $M^3 = S \times [0,1]$ with S oriented surface

↳ multiplication = stacking cylinders

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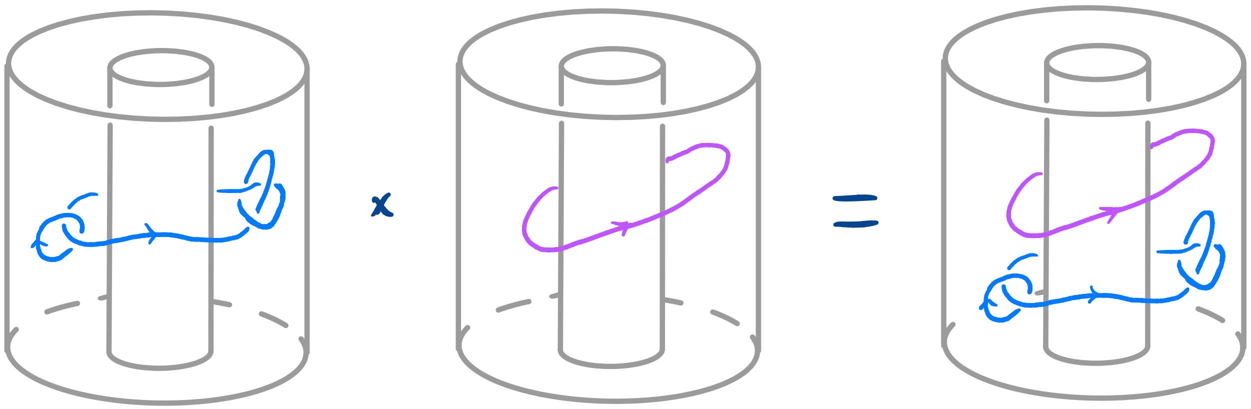

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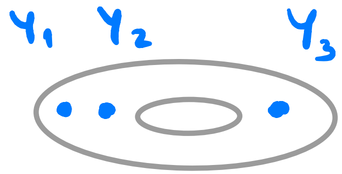

Skein categories: S oriented surface

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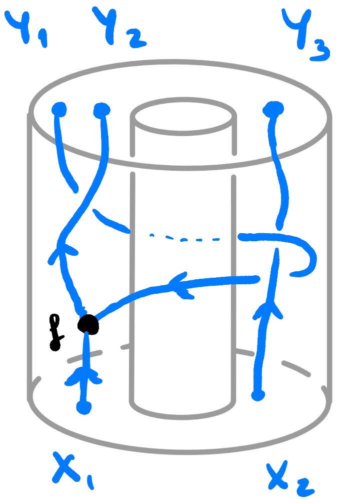

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Theorem (Walker) Skein theory forms a 3D-categorified

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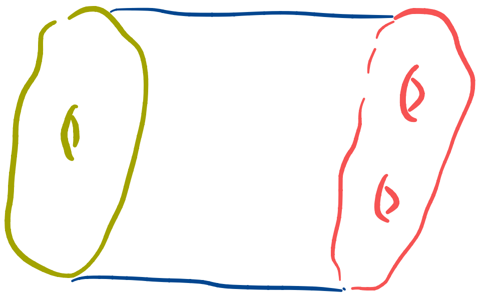

S_{int} S_{ext}

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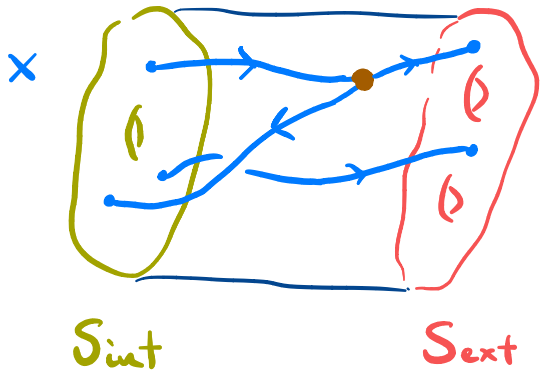

$$\longmapsto \Big(SKCat_A(S_{int}) \boxtimes SKCat_A(S_{ext})^{op} \longrightarrow Vect_{\mathbb{K}} \Big)$$

$$(X, Y) \longmapsto SK_A(M^3, X \sqcup Y)$$

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Ribbon categories: G reductive group, $q \in \mathbb{K}^\times$ generic

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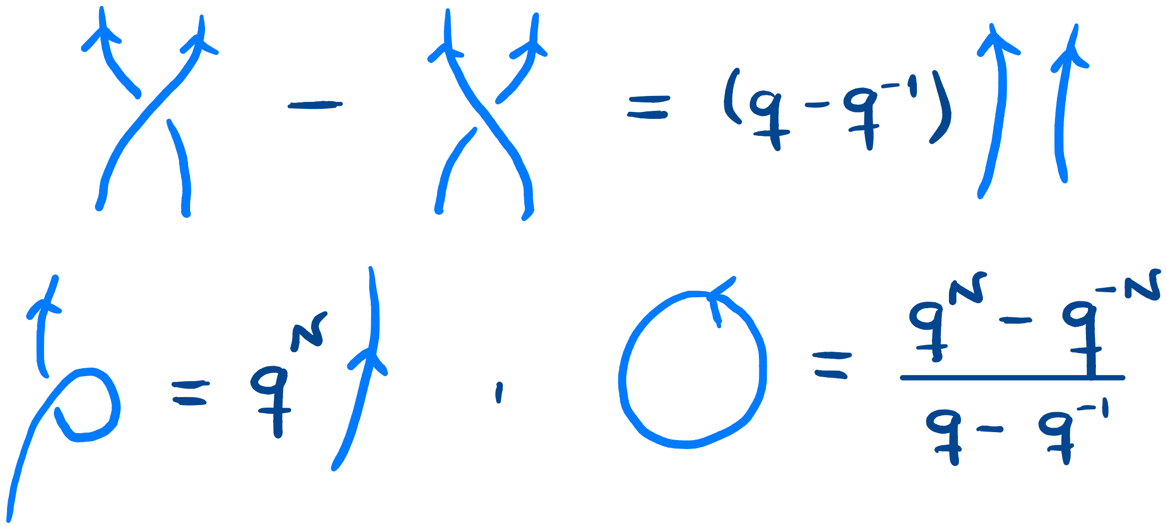

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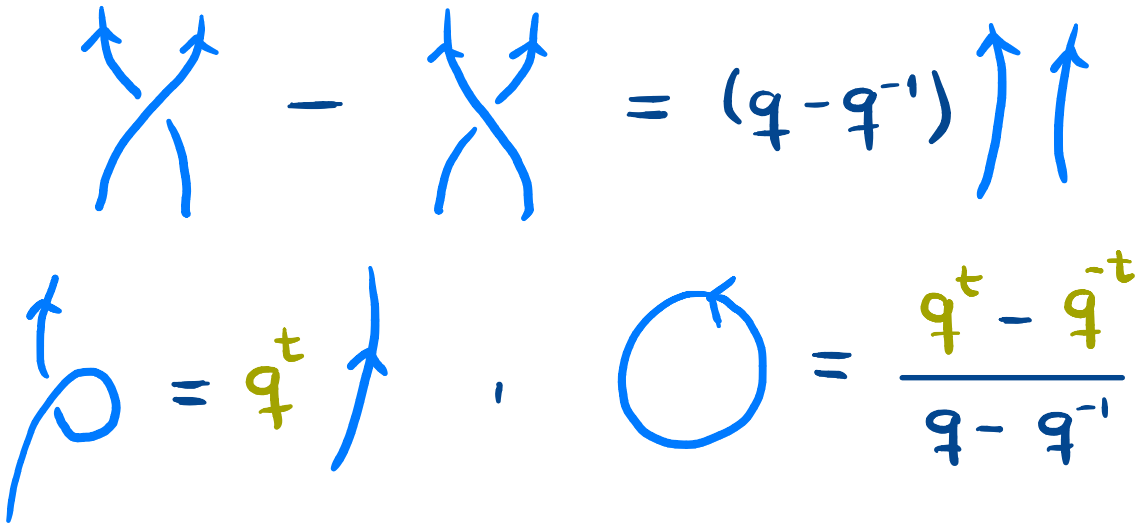

HOMFLY skein relations

Replace N by a formal parameter t

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HOMFLY category (Deligne, Brundan)

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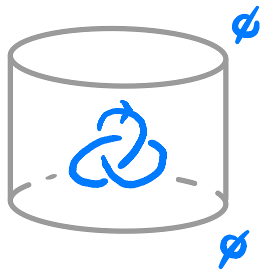
↳ we can recover $\operatorname{Rep}_q GL_N$ as a quotient (+ some completions) of its specialisation at $t=N$

Context and motivation: Turaev coproduct

HOMFLY polynomial :



$$\longmapsto \quad \begin{tikzcd} \mathbb{C}(q,q^t) \\ \mathbb{C}(q,q^t) \arrow[u] \end{tikzcd} \quad \in \mathbb{C}(q,q^t)$$

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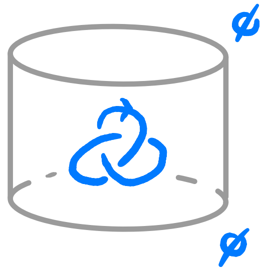

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Jaeger : composition formula for the HOMFLY polynomial

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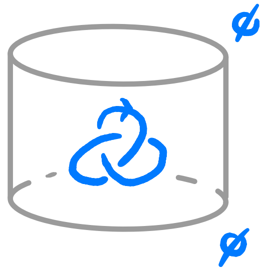

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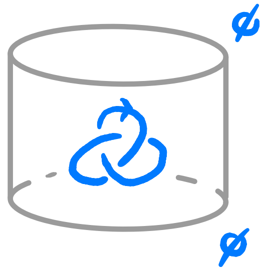

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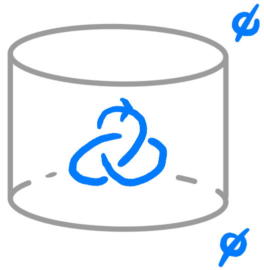
HOMFLY polynomial :  $\longmapsto$ $\mathbb{C}(q,q^t) \xleftarrow{} $ …

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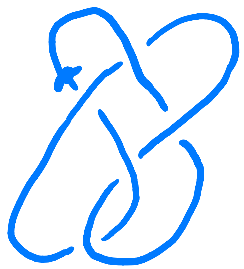
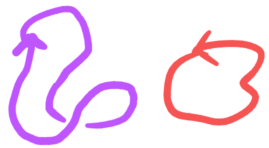
 $\xrightarrow[\sim\sim\sim]{\text{Jaeger}}$ $\sum$ 

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \mathrm{Rep}_q\, GL_{m+n} & \longrightarrow & \mathrm{Rep}_q\, GL_m \boxtimes \mathrm{Rep}_q\, GL_n \end{array}$$

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Theorem (Turaev) Jaeger's formula extends to a morphism of algebras

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- Compatibility with cutting/gluing surfaces
- Local formula for Δ $\overset{?}{\rightsquigarrow}$ extension to $\mathrm{SkCat}_{GL_t}(S)$

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combinatorial structure (Necklace bialgebra)

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Schedler: combinatorial description of $\mathrm{gr}(\mathrm{SK}_{GL_t}(S))$

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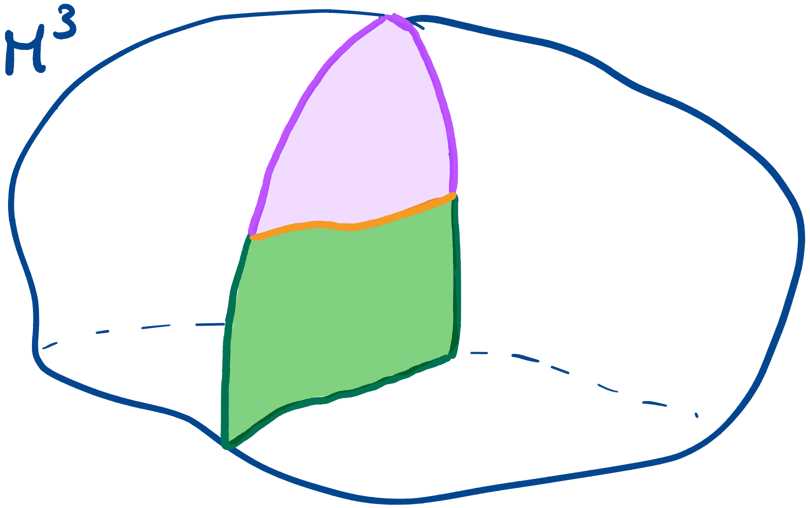
Schedler: combinatorial description of $gr\,(SK_{GL_t}(S))$

? Quantum formality theorem

Defect skein theory



Defect skein theory: skein modules / algebras / categories for stratified manifolds

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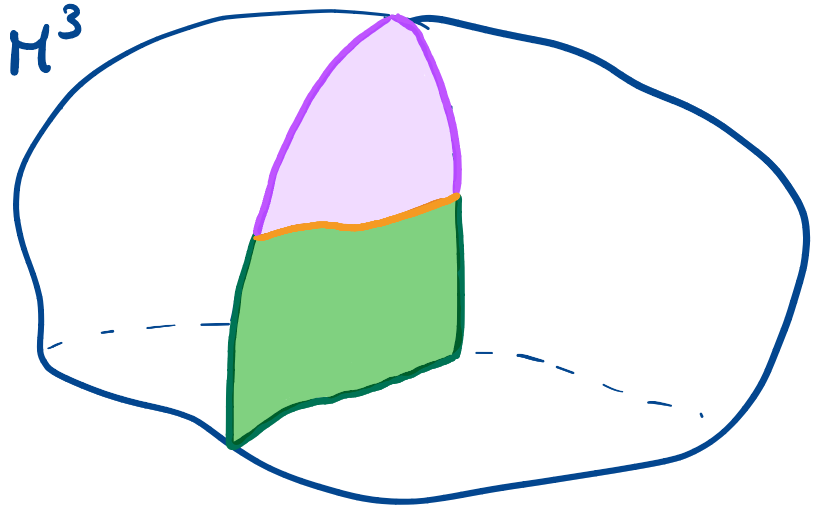

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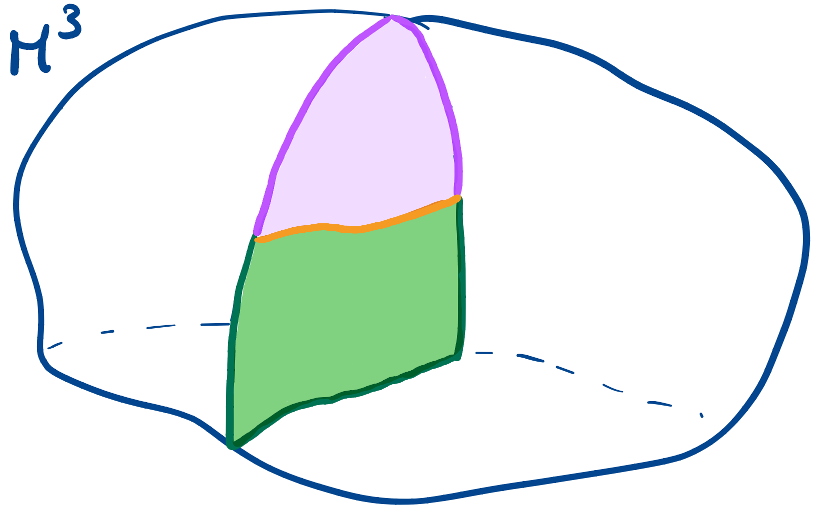

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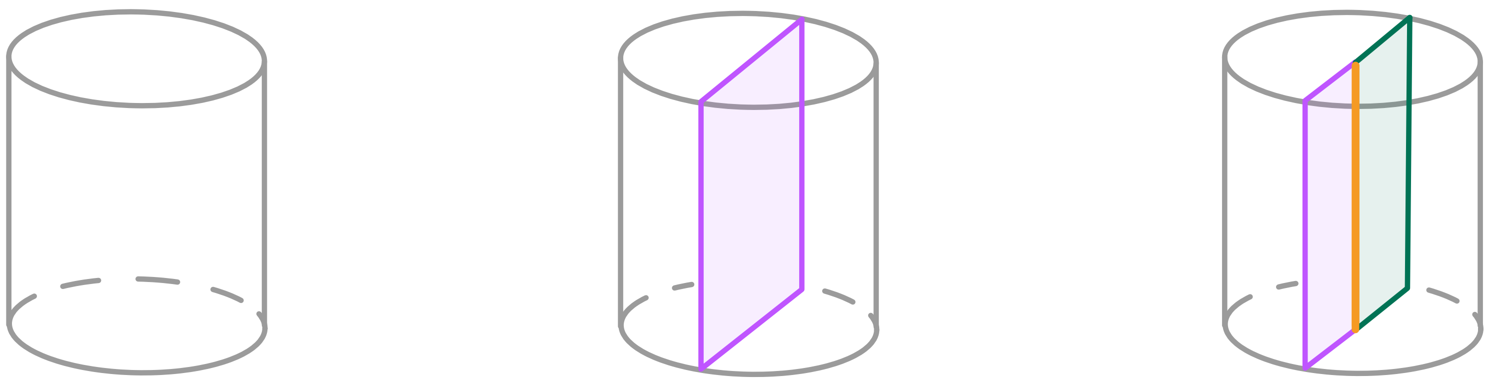
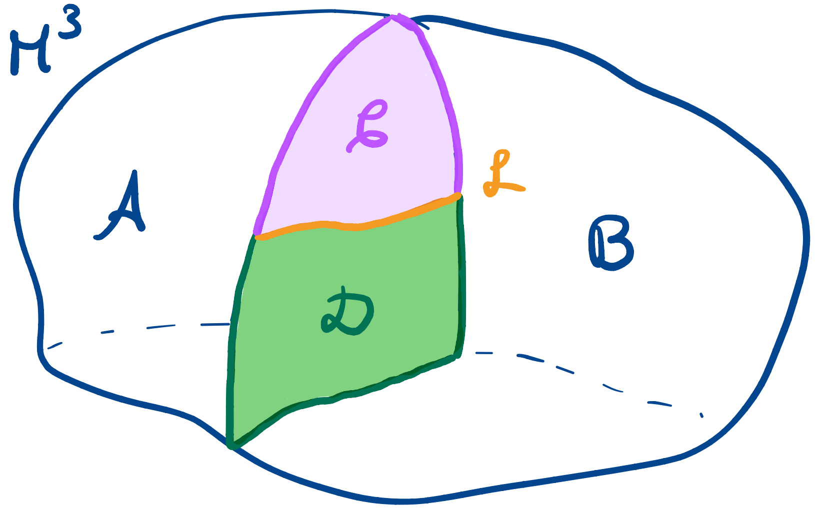
* Three different local models:



Defect skein theory



Defect skein theory : skein modules / algebras / categories for stratified manifolds

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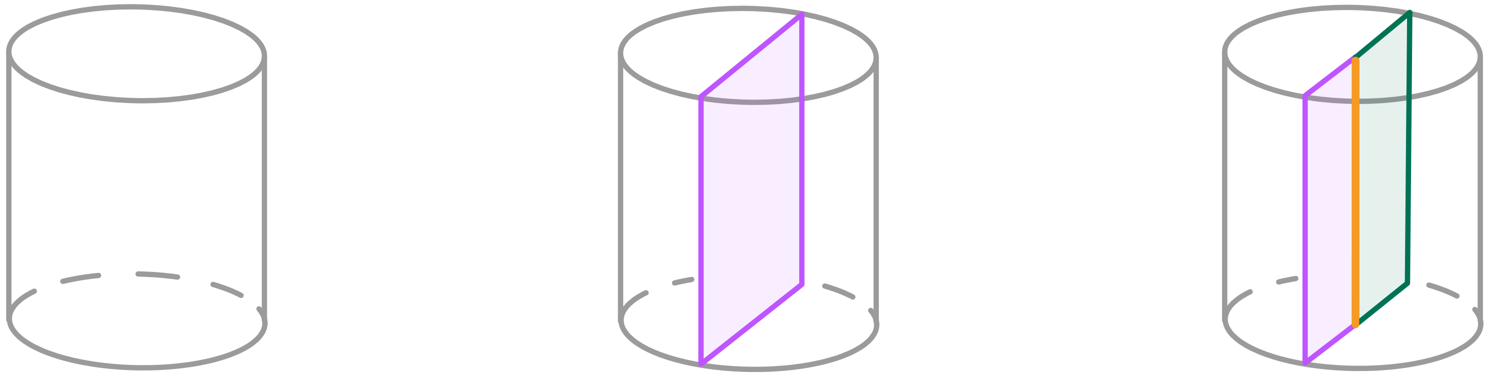

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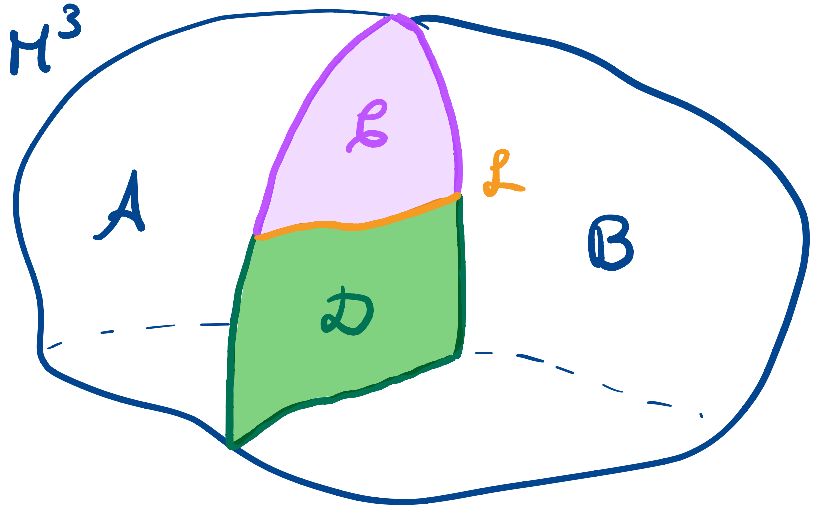

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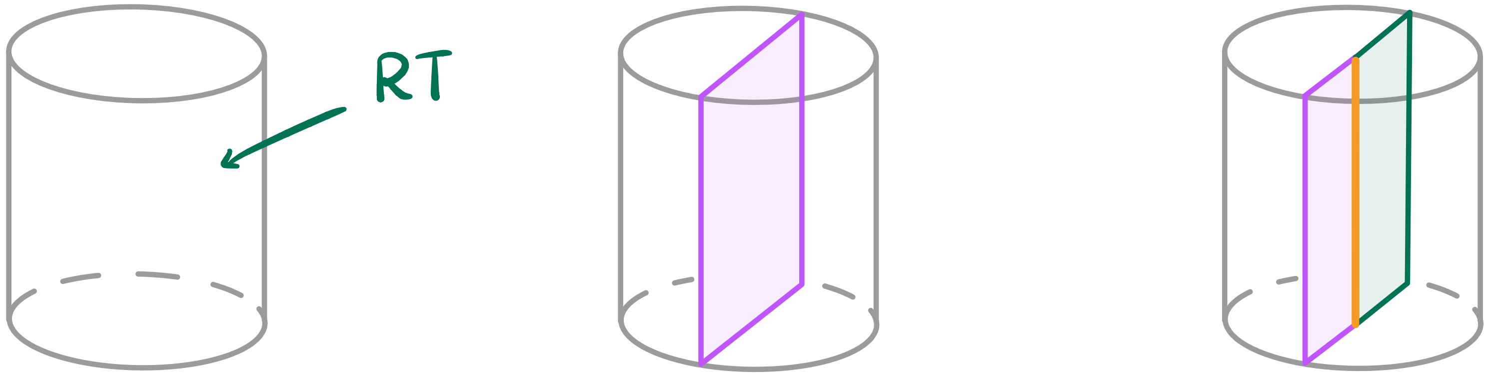

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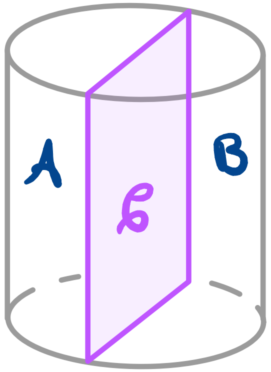
Defect skein theory

Surface defects (Brown - Jordan '25)



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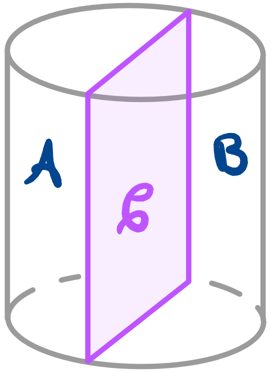

$\mathcal{C}$ is an (A, B) - central algebra

Defect skein theory

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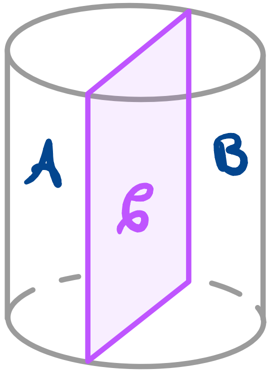

$\mathcal{E}$ is an $(\mathcal{A}, \mathcal{B})$ - central algebra

Algebra object in $\mathcal{A}$ - $\mathcal{B}$ - bimod

<=> braided monoidal functor

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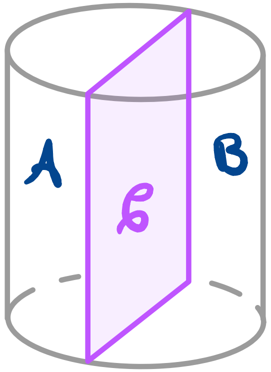

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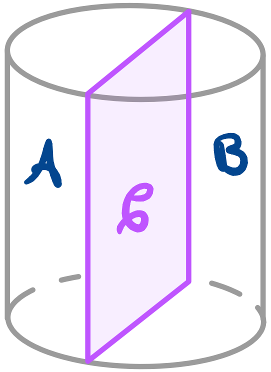

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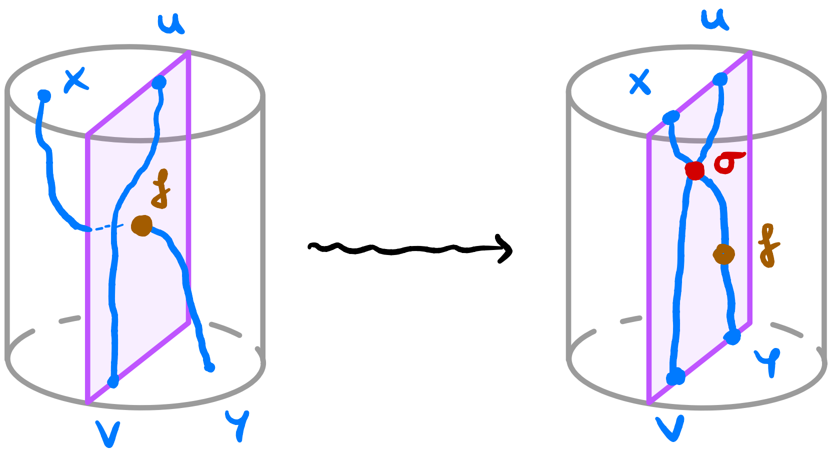

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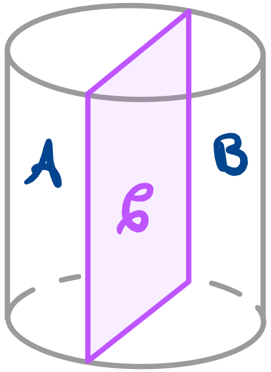

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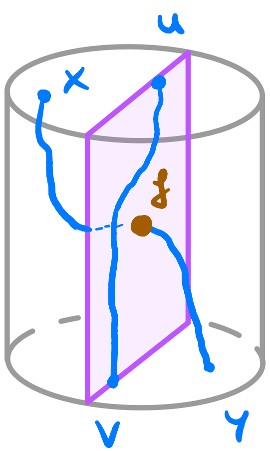
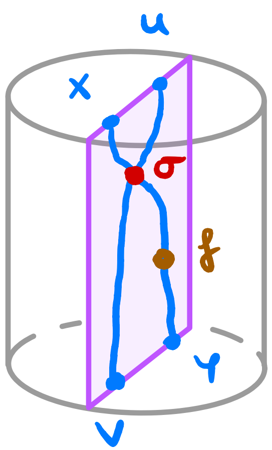


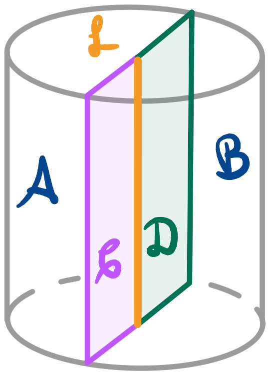
$\rightsquigarrow$



$\longmapsto \mathrm{Hom}_{\mathcal{E}}(V \otimes H(Y), H(X) \otimes u)$

Defect skein theory

Line defects (GG '26, Brown - GG - Jordan - Vancraeynest)



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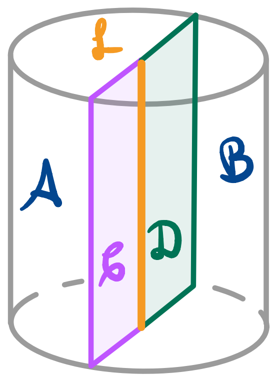

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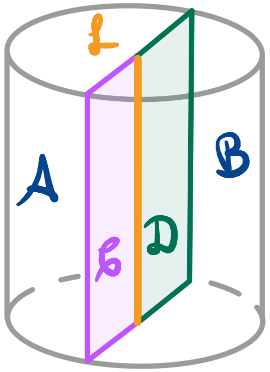

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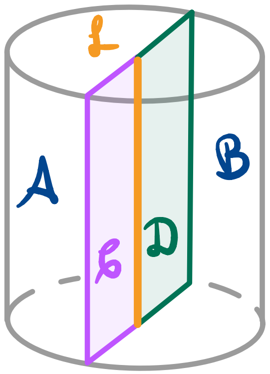

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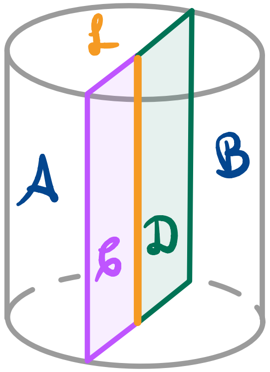

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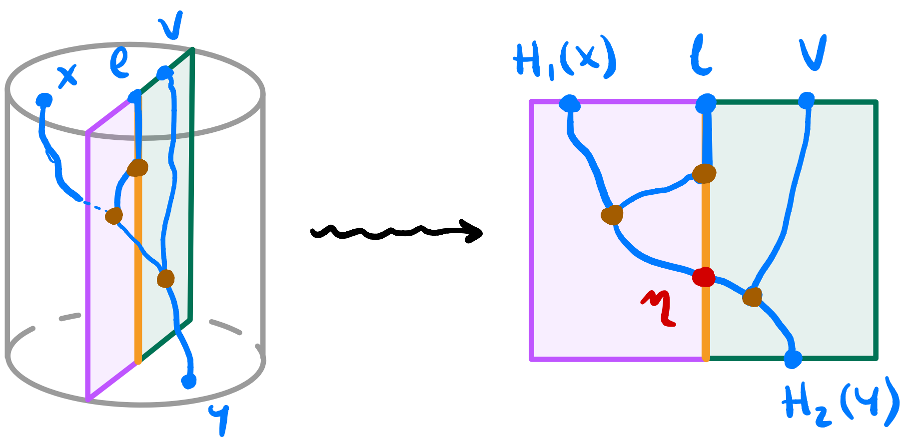

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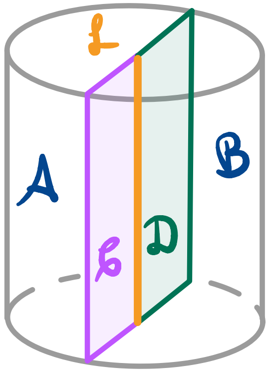

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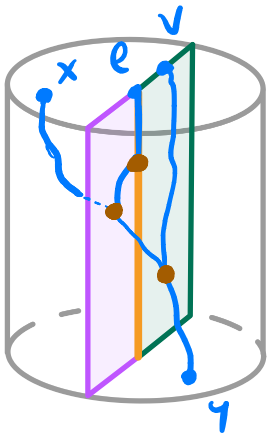
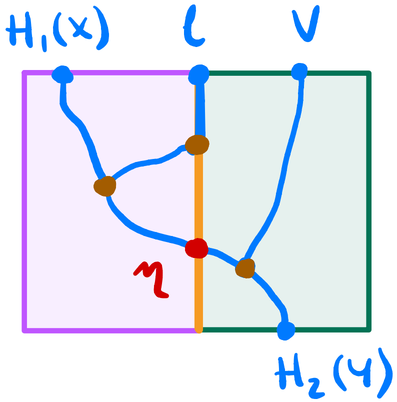


$\rightsquigarrow$



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HOMFLY parabolic restriction

$G = GL_{m+n}$

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$$G = GL_{m+n} \quad , \qquad L = \left(\begin{array}{c|c} GL_m & 0 \\ \hline 0 & GL_n \end{array}\right) \xhookrightarrow{i} G$$

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Parabolic restriction

$$\begin{tikzcd} & \mathrm{Rep}_q P & \\ \mathrm{Rep}_q GL_{m+n} \arrow[ur, "i^*"] & & \mathrm{Rep}_q L \cong \mathrm{Rep}_q GL_m \boxtimes \mathrm{Rep}_q GL_n \arrow[ul, "\pi^*"'] \end{tikzcd}$$

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$\boxed{\text{HOMFLY version}}$

Parabolic restriction

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planar diagrammatic subcategory of $\mathrm{Rep}_q\, L_t$

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planar diagrammatic subcategory of $\mathrm{Rep}_q L_t$

Theorem (GG'26)

- $\mathrm{Rep}_q P_t$ is a central $(\mathrm{Rep}_q GL_t, \mathrm{Rep}_q L_t)$-algebra
- $\mathrm{Rep}_q L_t$ is a $\mathrm{Rep}_q L_t$-centered $(\mathrm{Rep}_q L_t, \mathrm{Rep}_q P_t)$-bimod.

HOMFLY parabolic restriction

(S, ϕ, f, M) marked bipartite framed surface

HOMFLY parabolic restriction

$(S, \phi, f, \mathcal{M})$ marked bipartite framed surface

marked boundary components

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trivialization of TS

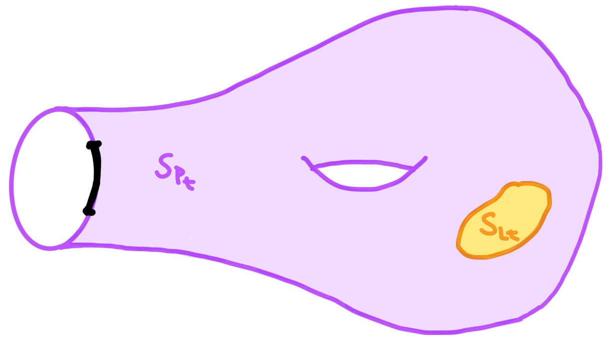
HOMFLY parabolic restriction

(S, ϕ, f, M) marked bipartite framed surface

marked ↔ marked boundary components

bipartite ↔ two open 2-dim. strata

framed ↔ trivialization of TS



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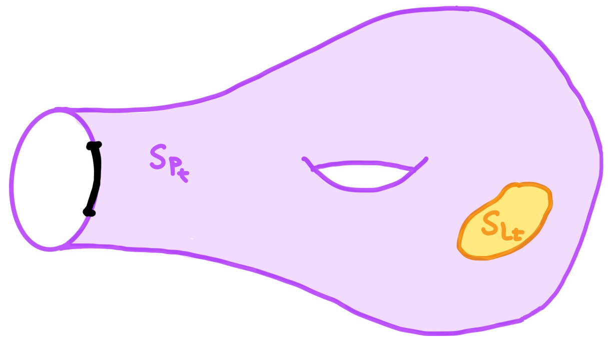
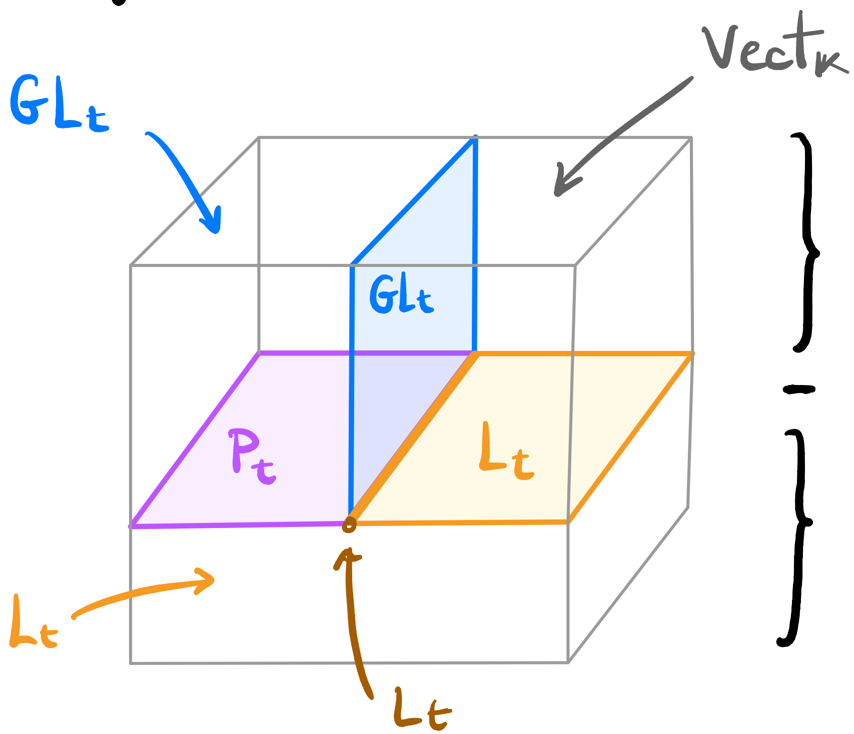


Stratification on $S \times I$:



$\phi \times (1/2, 1]$

$(S, \phi) \times \{1/2\}$

$S \times [0, 1/2)$

HOMFLY parabolic restriction

$(S, \phi, f, \mathcal{M})$ marked bipartite framed surface

marked boundary components

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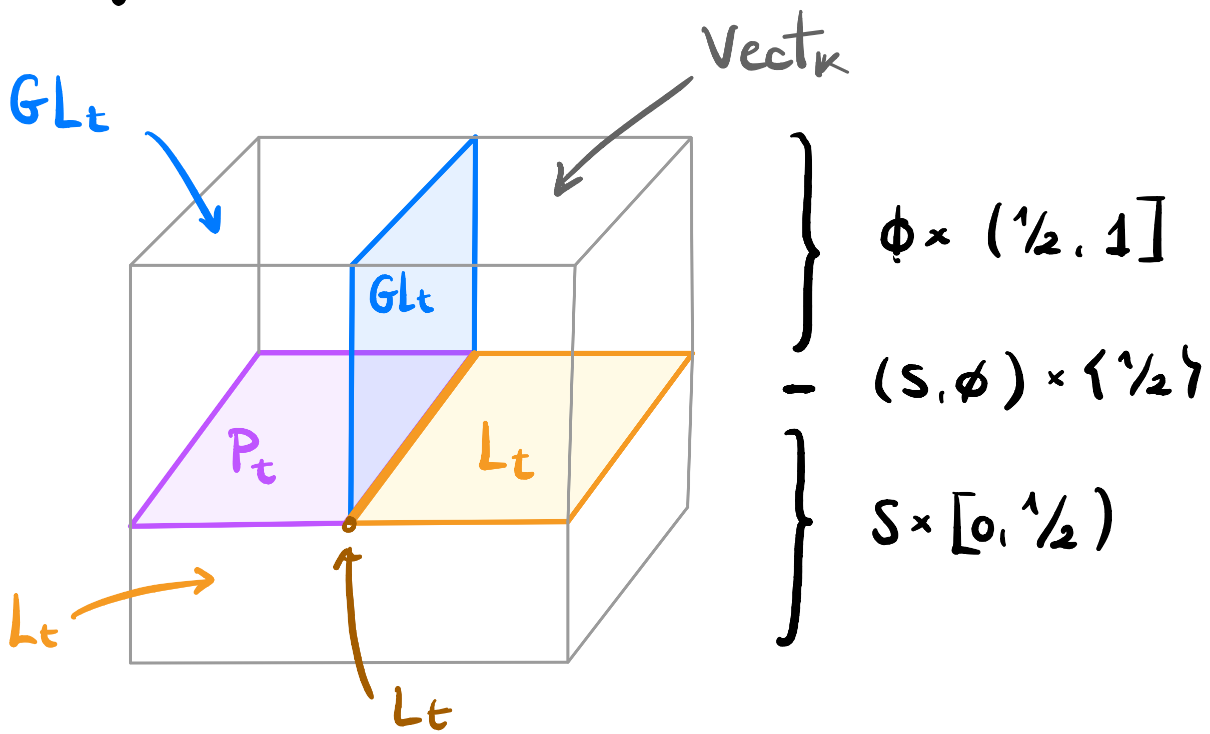

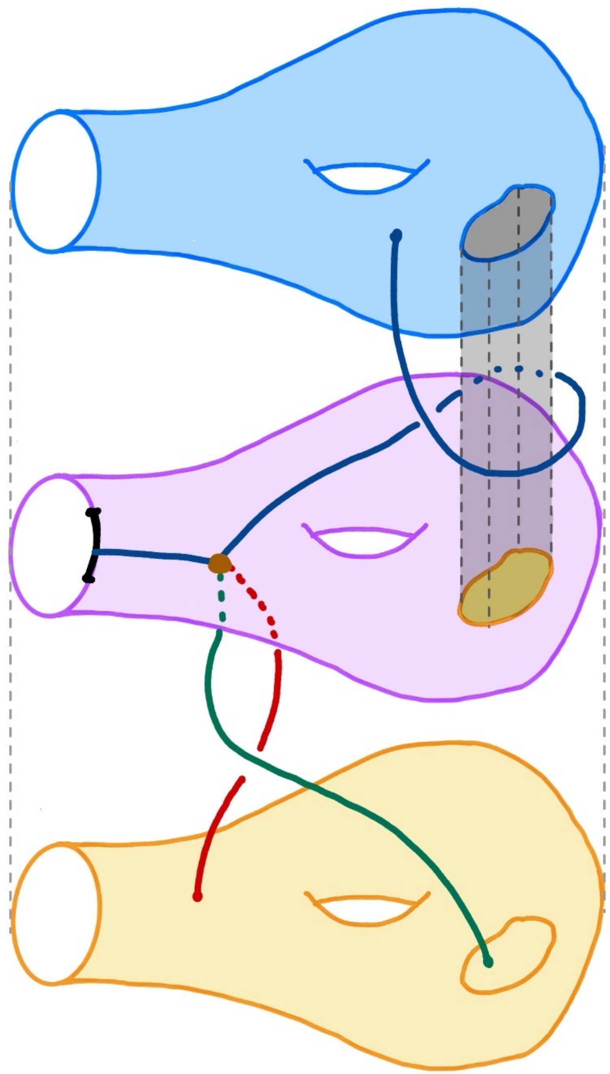

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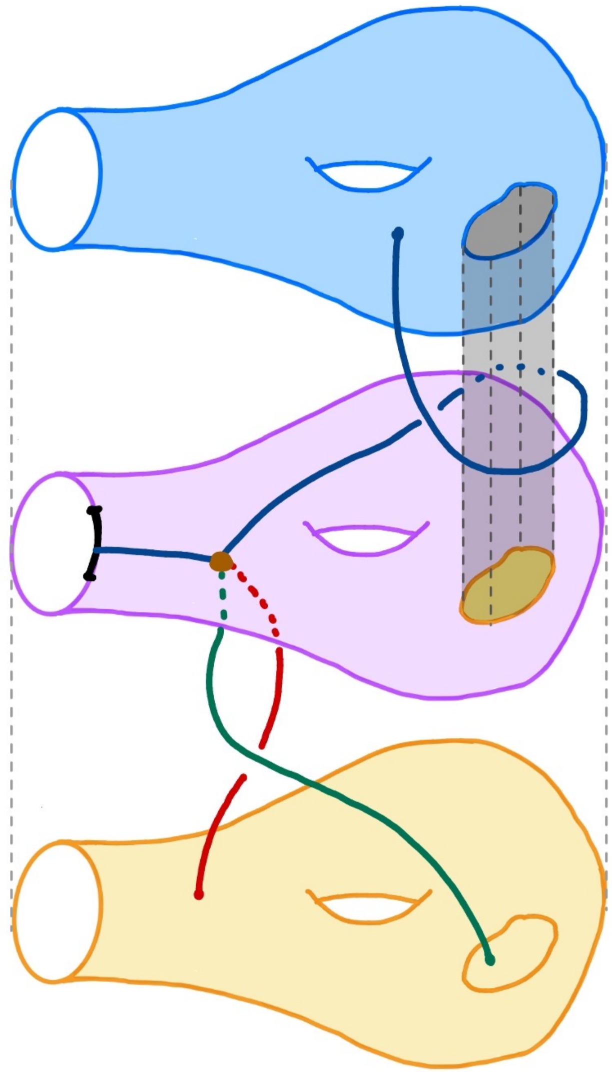


The associated TQFT:

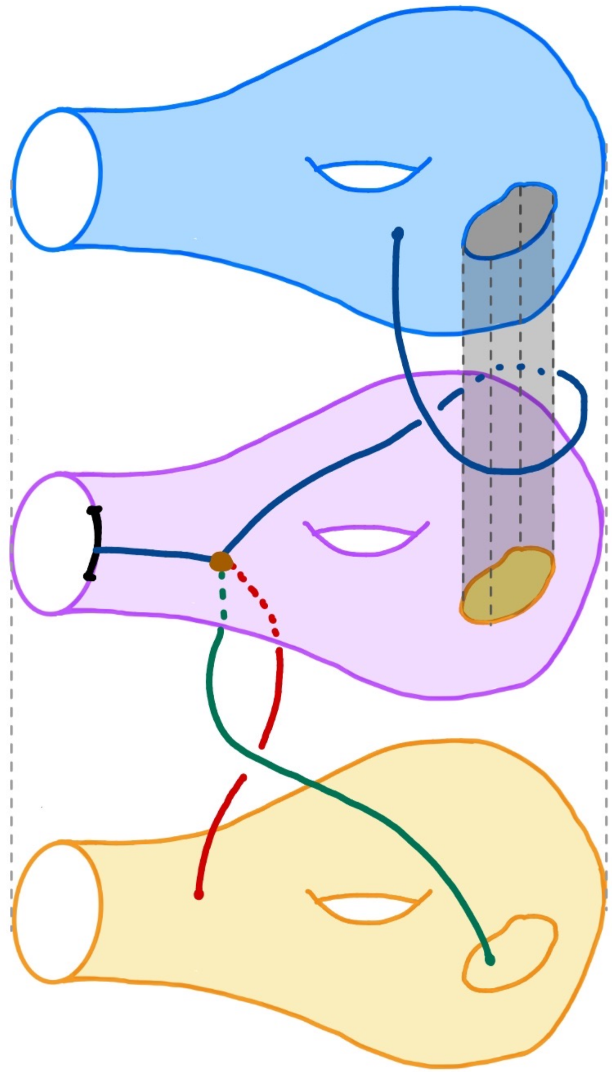
HOMFLY parabolic restriction

$(S, \phi, f, \mathcal{M})$ marked bipartite framed surface

marked $\leftrightarrow$ marked boundary components

bipartite $\leftrightarrow$ two open 2-dim. strata (P_t, L_t)

framed $\leftrightarrow$ trivialization of TS



The associated TQFT: we get

$$\mathcal{A} : \mathrm{Bord}_{2,1}^{bip, fr} \longrightarrow \mathrm{BiMod}$$

given by $\mathcal{A}(I_{P_t}) = \mathrm{Rep}_q P_t$, and

$\mathcal{A}(I_{L_t}) = \mathrm{Rep}_q L_t$,

$$\boxed{\mathcal{A}(S) : \mathcal{A}(\mathcal{M}) \boxtimes \mathrm{SkCat}_{GL_t}(S_{P_t}) \boxtimes \mathrm{SkCat}_{L_t}(S)^{op} \longrightarrow \mathrm{Vect}_k}$$

The Turaev coproduct

Empty boundary condition:

$\mathcal{A}(S)(\emptyset,\emptyset,\emptyset)$ is a $\left(\mathrm{SkAlg}_{GL_t}(S_{P_t}),\ \mathrm{SkAlg}_{L_t}(S)\right)$-bimodule

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Theorem (GG'26) We can choose a stratification Φ so that

$S_{P_t} \cong S$ and we have an isomorphism

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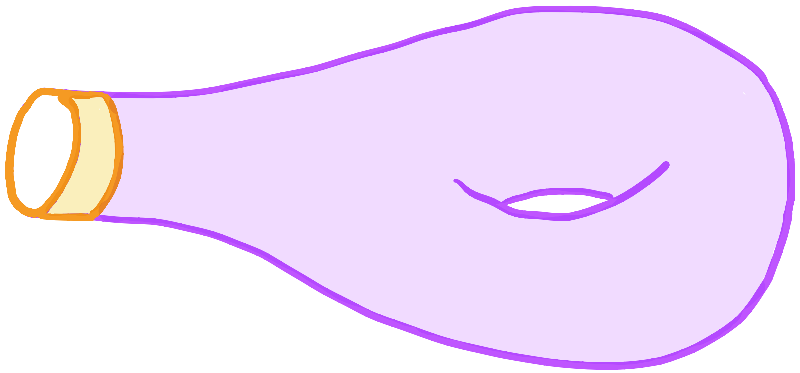

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Theorem (GG'26) We can choose a stratification Φ so that $S_{P_t} \cong S$ and we have an isomorphism

$$\mathcal{A}(S)(\emptyset,\emptyset,\emptyset) \cong \mathrm{SkAlg}_{GL_t}(S) \otimes \mathrm{SkAlg}_{GL_t}(S)$$

Acting on the empty diagram induces the Turaev cop.

$$\Delta_f : \mathrm{SkAlg}_{GL_t}(S) \longrightarrow \mathrm{SkAlg}_{GL_t}(S) \otimes \mathrm{SkAlg}_{GL_t}(S)$$