Full Locality in 3D TQFTs

- Two results pertaining to fully local 3D TQFTs inspired by the distinction between
  Turaev-Viro and Reshetikhin-Turaev theories

- Application to tangential structures, in particular Spin and orientation in relation to modularity

- "No-Go" theorem (w/ Dan Freed, CMP 2021)

  Fusion categories $\overset{\Longleftarrow \text{(new)}}{\Longleftrightarrow}$ fully local TQFTs with a Dirichlet ∂ condition
  ($\Longrightarrow$ Douglas, Schommer-Pries, Snyder 2013)

  "No topological ∂ conditions for RT theories unless they are really TV theories"

- "Go" theorem (w/ Freed and C. Scheimbauer)
  "RT theories are fully local"
  (+ spin versions for super-categories)

- Open questions:
  - Role of unitarity? ("preferred orientations")
  - Poincaré duality for TQFTs:
    "lower half can be reconstructed from the upper half"

1 Fully local TQFTs and boundary conditions

Relies on Lurie's Cobordism Hypothesis

Symmetric monoidal functors $\mathcal{J}: \mathrm{Bord}_n^{\mathrm{framed}} \longrightarrow$ (n-category $\mathcal{C}$)

$\longleftrightarrow$ fully dualizable objects $\in \mathcal{C}$, $\mathcal{J}(\text{point})$.

Boundary conditions for $\mathcal{J}$ $\Leftrightarrow$ f.d. $\mathrm{Hom}(\mathbb{1}, \mathcal{J}(\text{point}))$
(interfaces with the trivial theory).

Model: 2D, $\mathcal{J}(pt)$ = Algebra, Linear Category; ∂ cond. = module object

Fact: Unless working in a derived context, the algebra/category must be finite semisimple (Segal 1990's; FT 2021)

3 Dimensions: algebra object in linear cats
$\leftarrow$ tensor category F
∂ conds $\leftrightarrow$ module cats

If: F is a fusion category$/\mathbb{C}$
$\hookrightarrow$ finite, semisimple, internal duals
M is a finite semisimple module

3D TQFT: Turaev-Viro theory.
(originally: 1, 2, 3 "state sum construction"
$S^1 \mapsto Z(F)$, from triangulations

2 Reshetikhin-Turaev TQFTs from Modular Tensor Cats

MTC $M = \mathcal{J}(S^1)$: fusion category

non-degenerately braided β

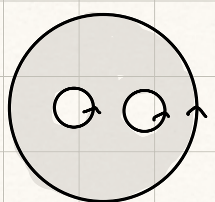
$\hookrightarrow$ central object $\equiv$ $\mathrm{Vect}\cdot 1 \subset M$.

Ribbon $\theta \in \mathrm{Aut}(\mathrm{Id}_M)$

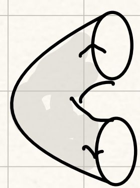
$\theta_a \in \mathbb{C}^\times$ for each simple $a \in \mathbb{C}$

"homogeneous quadratic refinement of β": $\theta_{ab}\theta_a^{-1}\theta_b^{-1} = \beta_{a,b} \circ \beta_{b,a}$.

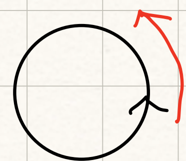
Explanation from pictures:



Braiding



internal duals



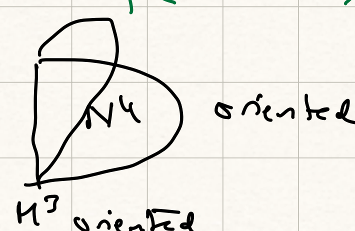
Ribbon

$\int p_1$ on a 3-mfld: a point on $K(\mathbb{Z},1) = S^1$

Anomalous Oriented theory:

has a scalar framing, p_1- or signature-dependence via a central charge c $\exp(2\pi i c \cdot p_1/24) = \exp(2\pi i c \cdot \sigma/8)$

(Witten; Walker)



No assignment $\mathcal{J}(pt)$ unless it happens that

$$M = Z(F)$$

Müger
Ostrik, Etingof, Nykshych
Davydov

In lucky cases (spherical) $Z(F)$ acquires a ribbon

then, one can take $c = 0 \bmod 24$

but that is not the only choice ($c = 4 \bmod 24$ works too)

3. No Go Theorem

Assume: $\mathcal{J} : \mathrm{Bord}_3^{fr} \longrightarrow \mathcal{C}$ is a fully local TQFT

where $\mathcal{C}$ is a symmetric $\otimes$ 3-category

$\Omega\mathcal{C} = \mathrm{End}(\mathbb{1}) \subset \mathbb{L}$,

(linear categories closed under finite colimits)

$\mathcal{J}$ has a boundary condition β

which detects every summand of $\mathcal{J}$.

Note: $Z(\mathcal{J}) := \mathcal{J}(S^1_{\text{boundary framing}}) \in \mathbb{L}$

Then, in dims (1,2,3), the pair

$$(\mathcal{J}, \beta) \equiv TV(F, \text{regular module})$$

for some fusion category F.

$F = \mathrm{End}_{\mathcal{J}}(\beta)$.

picture proof: it's a BFC

$= \oplus$ nondegenerate BFC

$\downarrow$

breaks up $\mathcal{J}$

(get projections onto $\mathcal{J}$

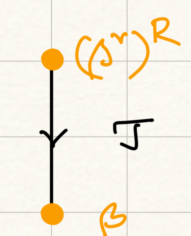
from summands

of $1 \in Z(\mathcal{J})$)

algebra of point ops in $\mathcal{J}$.

Remark If you ask in addition that $\mathcal{C}$ also has finite colimits then $\mathcal{J}(pt) \equiv F$

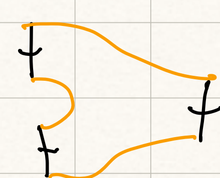
Idea of proof When does F come from



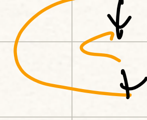
$\in \Omega\mathcal{C}$

$\Rightarrow$ ss category

2D TQFT



multiplication



(a little tricky)

4. "Go" Theorem

Let $\mathbb{F} :=$ linear 3-category of fusion categories (over $\mathbb{C}$)

<u>Thm (D, SP, S)</u> This is a category with full duals (every k-morphism is (3−k)-dualizable)

$\Longleftrightarrow$ "everything defines a framed TQFT, an interface, embedded defect, etc, according to its nature"

[Some conjectures about sphericality vs SO_3 invariance
Spin invariance]

$\Longrightarrow$ TV theories are fully local.

<u>Thm</u> (FST) There exists $\mathbb{EF}$ such that:

(0.4) *Results.* In this paper, we confirm this suspicion, and enlarge the 3-category $\mathbb{F}$ of fusion categories to a universal *symmetric monoidal 3-category* $\mathbb{EF}$ ("enlarged $\mathbb{F}$"), containing the point generators of Reshetikhin-Turaev theories, with the following properties:

(i) $\mathbb{EF}$ *has full duals*: all k-morphisms are $(3-k)$-dualizable, $0 \le k \le 3$;

(ii) $\mathbb{EF} = \bigoplus_{w \in \widetilde{W}} \mathbb{F}_w$, where $\widetilde{W} \to W$ is an extension of the Witt group by $\boldsymbol{\mu}_6$;

(iii) In particular, there are no non-zero morphisms relating objects in $\mathbb{F}_w$ and $\mathbb{F}_{w'}$ when $w \neq w'$;

(iv) Each $\mathbb{F}_w$ is an invertible module over $\mathbb{F}_1 = \mathbb{F}$; specifically, choosing a representative braided category $T(w)$ for w gives an isomorphism $\mathbb{F}_w \equiv \mathbb{F}_{T(w)}$, the 3-category of fusion categories with central action of $T(w)$ (called *fusion categories over* $T(w)$);

(v) When $\zeta = \exp(2k\pi i/6) \in \boldsymbol{\mu}_6 \subset \widetilde{W}$, we have $\mathbb{F}_\zeta \equiv \mathbb{F}$ as a module, generated by an invertible object $U^{\otimes k}$, unique up to (Morita) isomorphism;

(vi) The units $U^{\otimes k}$ generate the six invertible framed TQFTs valued in $\mathbb{EF}$, and factor uniquely through the category of oriented manifolds with p_1-structure;

(vii) Every symmetric monoidal 3-category whose looping contains $\mathbb{L}$ and where all 1/2/3 framed TQFTs have point generators that are unique up to isomorphism receives a unique symmetric tensor functor from $\mathbb{EF}$.

Remarks

- W = Witt group of nondegenerate BFCs (no modularity assumption)
- Known to have no 3-torsion but I don't know an explicit splitting of $\tilde{W} \to W$.
- μ_6 classifies the invertible $SO_3^{p_1}$ – TQFTs $\quad J_u : \exp\left(2\pi i \frac{p_1}{(-6)}\right)$
- $\uparrow \mu_{24}$ would classify all invertible framed TQFTs. The generator is the free topological fermion theory
- On framed manifolds, $J_U = \psi^4$
  but this does not persist on $Spin^{p_1}$ manifolds.

In fact $c(\psi) = 1/2$, $c(J_U) = -4 \qquad J_u = \psi^{-8} \otimes \nu$

ν has a dual nature $\qquad SO_3^{p_1}$ – theory with $c = 12 \bmod 24$

$\qquad$ Spin theory with $c = 0$

Only $c \bmod 6$ can be detected a priori in Spin theories,

and $-4 = 2 \bmod 6$.

$\pi_1 MTSpin_2^{p_1/4} = \pi_3^s \oplus \pi_1^s$	$\pi_0 MTSpin_3^{p_1/4} = \pi_3^s$	N/A
$\pi_1 MTSpin_2^{p_1/2} = \pi_3^s \oplus \mathbb{Z}/4$	$\pi_0 MTSpin_3^{p_1/2} = \pi_3^s \oplus \mathbb{Z}/2$	$\pi_3 MSpin^{p_1/2} = \pi_3^s$
$\pi_1 MTSpin_2^{p_1} = \mathbb{Z}/48 \oplus \mathbb{Z}/4$	$\pi_0 MTSpin_3^{p_1} = \mathbb{Z}/48 \oplus \mathbb{Z}/2$	$\pi_3 MSpin^{p_1} = \mathbb{Z}/48$
$\pi_1 MTSpin_2^{\mathbb{C}p_1} = \mathbb{C}/48\mathbb{Z} \oplus \mathbb{Z}/4$	$\pi_0 MTSpin_3^{\mathbb{C}p_1} = \mathbb{C}/48\mathbb{Z} \oplus \mathbb{Z}/2$	$\pi_3 MSpin^{\mathbb{C}p_1} = \mathbb{C}/48\mathbb{Z}$
$\pi_1 MTSpin_2 = \mathbb{Z}/4$	$\pi_0 MTSpin_3 = \mathbb{Z}/2$	$\pi_3 MSpin = 0$
$\pi_1 MTSO_2^{p_1} = \mathbb{Z}/12$	$\pi_0 MTSO_3^{p_1} = \mathbb{Z}/6$	$\pi_3 MSO^{p_1} = \mathbb{Z}/3$
$\pi_1 MTSO_2^{\mathbb{C}p_1} = \mathbb{C}/12\mathbb{Z}$	$\pi_0 MTSO_3^{\mathbb{C}p_1} = \mathbb{C}/6\mathbb{Z}$	$\pi_3 MSO^{\mathbb{C}p_1} = \mathbb{C}/3\mathbb{Z}$

Table 2: Some relevant bordism groups

= group completion of bordism categories

Eg: $\pi_1 MTSO_3 \cong \mathbb{Z}$ but p_1 is 6 on the generator

$\Rightarrow \sigma/2 = p_1/6$ is defined on combed 4-manifolds

$\Rightarrow$ get 6 invertible 3D TQFTs on p_1-manifolds from the values of $\sigma/2$ on a "bulking" 4-mfold.

5. Action of SO(3) and orientability

<u>Thm</u> (FST) A spherical structure on a fusion category $\mathcal{F}$

$\Longleftrightarrow$ orientation-invariance of the pair ($\mathcal{F}$, regular module)