

Categorical 4-Manifold Invariants from Trisection Diagrams

Joint work 2511.19384 with Catherine Meusburger (FAU Erlangen--Nürnberg)
and Fiona Torzewska (Uni Bristol)

Vincentas Mulevičius,
Uni Vienna,
2026

Plan

- Trisections of 4-manifolds
- Diagrammatic calculus
- Construction of invariants
- Examples and discussion

Recall: Heegaard splittings

Notation:

- $\Sigma_g := \#^g(S^1 \times S^1)$
- $H_g := \natural^g(S^1 \times \overline{B^2})$,
 $\partial H_g = \Sigma_g$



genus g surface

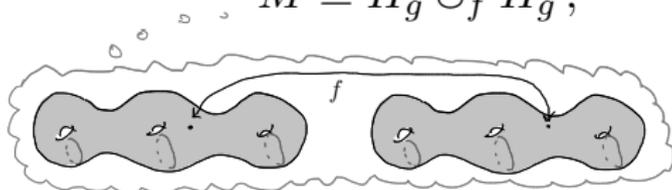


solid handlebody

Every (connected closed oriented) 3-manifold M is diffeomorphic to

$$M \cong H_g \cup_f H_g,$$

$$f: \Sigma_g \rightarrow \Sigma_g$$



Recall: Heegaard diagrams

Alternatively: glue two solid handlebody bordisms

$$[M_1: \emptyset \longrightarrow \Sigma_g] = (H_g, \quad \kappa_1: \Sigma_g \rightarrow \Sigma_g = \partial H_g)$$

$$[M_2: \Sigma_g \longrightarrow \emptyset] = (H_g, \quad \kappa_2: \Sigma_g \rightarrow \Sigma_g = \partial H_g)$$

$$M \cong [\emptyset \xrightarrow{M_1} \Sigma_g \xrightarrow{M_2} \emptyset]$$

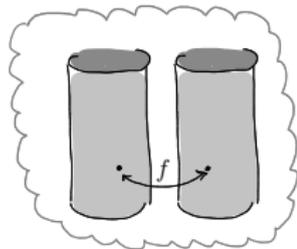
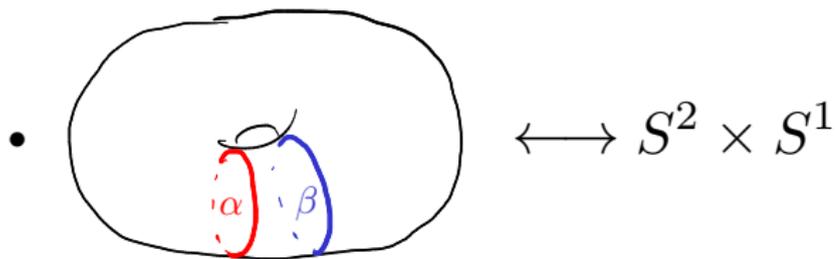
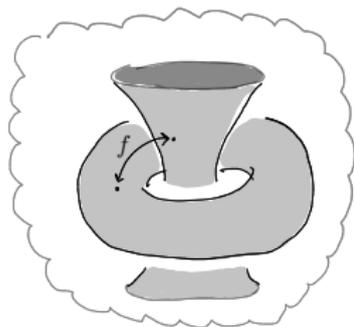
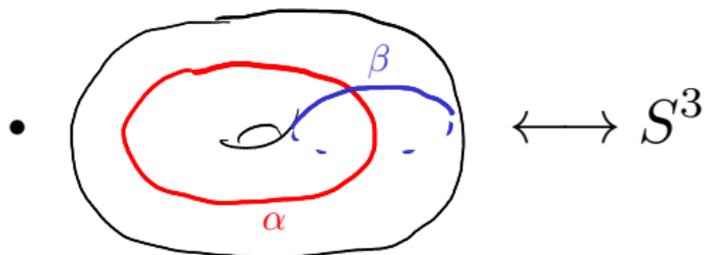
Heegaard diagram = two sets of curves on Σ_g

$$\alpha = \{\alpha_1, \dots, \alpha_g\} = \kappa_1(\text{meridians of } H_g)$$

$$\beta = \{\beta_1, \dots, \beta_g\} = \kappa_2(\text{meridians of } H_g)$$

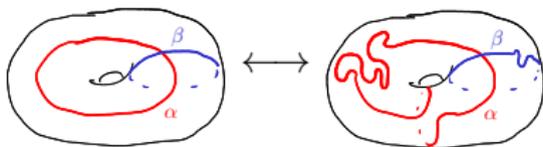
Recall: Heegaard diagrams

Examples:

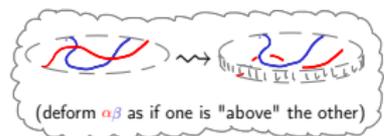
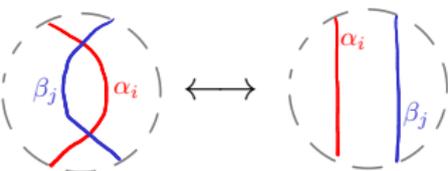


Recall: moves Heegaard diagrams

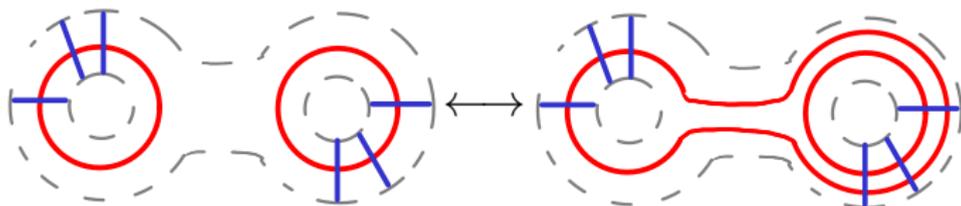
i) diffeomorphisms
(\Rightarrow isotopies)



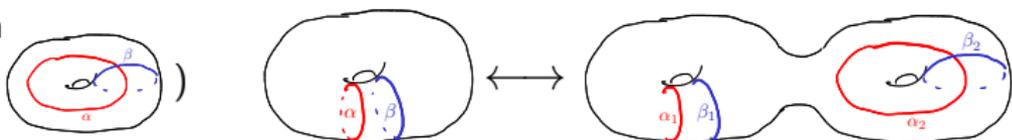
ii) two-point move
(a.k.a. Reidemeister II)



iii) handle slide



iv) stabilisation
($\#$ with



Theorem: Two Heegaard diagrams yield the same 3-manifold
iff they are related by a fin. sequence of i) -- iv)

Trisections of 4-manifolds

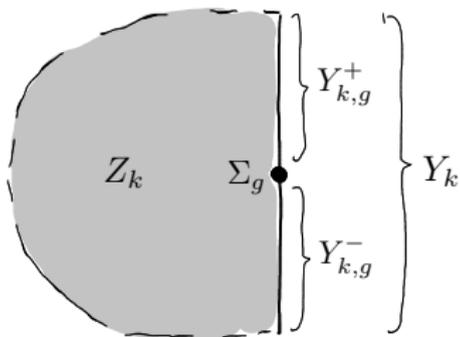
[Gay--Kirby]

Set:

- $Y_k := \#^k(S^1 \times S^2)$
- $Z_k := \natural^k(S^1 \times \overline{B^3})$,
 $\partial Z_k = Y_k$

- $\Sigma_g := \#^g(S^1 \times S^1)$
- $H_g := \natural^g(S^1 \times \overline{B^2})$,
 $\partial H_g = \Sigma_g$

- $Y_k = [\emptyset \xrightarrow{Y_{k,g}^-} \Sigma_g \xrightarrow{Y_{k,g}^+} \emptyset]$ -- Heegaard splitting

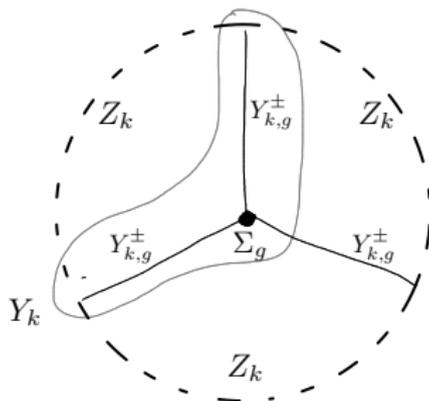


Definition A (g,k) -trisection of a ^{Surfaces} (closed connected oriented) 4-manifold X consists of:

- i) decomposition $X = X_1 \cup X_2 \cup X_3$ ii) diffeomorphisms $\phi_i: X_i \rightarrow Z_k$

such that

$$\phi_i(X_i \cap X_{i+1}) = Y_{k,g}^- \quad \phi_i(X_{i-1} \cap X_i) = Y_{k,g}^-$$



- $Y_k := \#^k(S^1 \times S^2)$
- $Z_k := \natural^k(S^1 \times \overline{B^3})$,
 $\partial Z_k = Y_k$
- $Y_k = [\emptyset \xrightarrow{Y_{k,g}^-} \Sigma_g \xrightarrow{Y_{k,g}^+} \emptyset]$ -- Heegaard splitting

Theorem Every X has a trisection!

Note: can encode the three Heegaard splittings $Y_k = Y_{k,g}^- \cup_{\Sigma_g} Y_{k,g}^+$

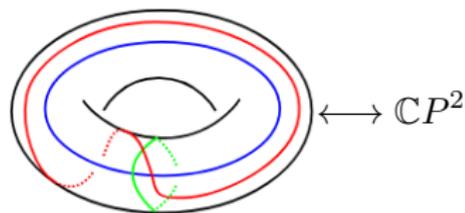
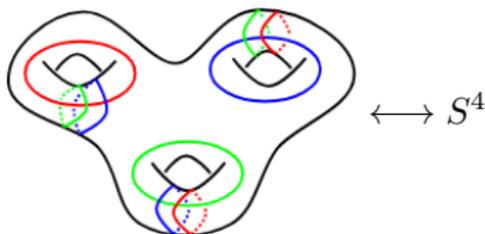
by 3 sets of curves on Σ_g , fillings $\partial Z_k = Y_k$ are canonical!

Definition (g,k) -trisection diagram -- 3 sets of curves α, β, γ on Σ_g

such that $(\Sigma_g, \alpha, \beta)$, $(\Sigma_g, \beta, \gamma)$, $(\Sigma_g, \gamma, \alpha)$

are Heegaard diagrams for $Y_k = \#^k(S^1 \times S^2)$

Examples:



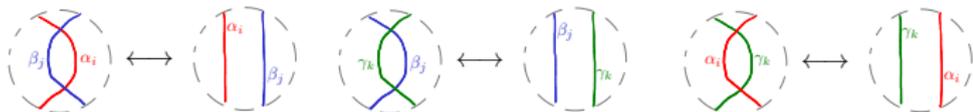
Moves on trisections diagrams

[Gay--Kirby]

[Chaidez--Cotler--Cui]

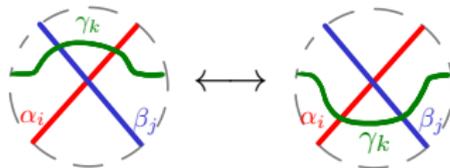
i) diffeomorphisms

ii) two-point moves

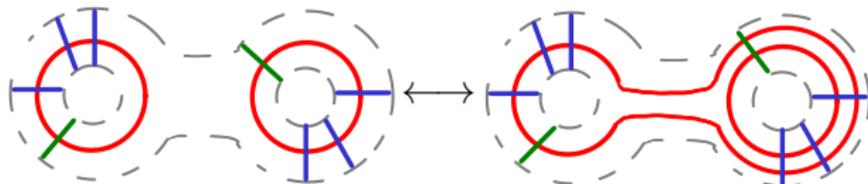


iii) three-point moves

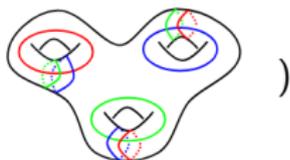
(a.k.a. Reidemeister III)



iv) handle slide

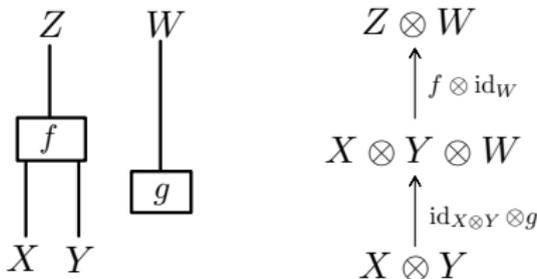


v) stabilisation ($\#$ with

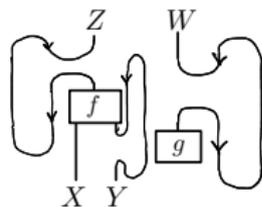


Theorem: Two trisection diagrams yield the same 4-manifold iff they are related by a fin. sequence of i) -- v)

- String diagrams for monoidal categories:



- String diagrams for pivotal categories



- can deform the diagrams using evaluation/coevaluation \bowtie -morphisms



Will work in the pivotal (multifusion) category $\text{End}(\mathcal{M})$ of endofunctors of a certain finitely semisimple linear category \mathcal{M} with trace

Specifically, let \mathcal{B} and \mathcal{C} be spherical fusion categories and \mathcal{M} an indecomposable finitely semisimple \mathcal{B} - \mathcal{C} bimodule category with bimodule trace

$\Theta = \{\Theta_X: \text{End}_{\mathcal{M}}(X) \rightarrow \mathbb{C}\}_{X \in \mathcal{M}}$ such that [Schaumann]

- $\Theta_X(- \circ -): \text{Hom}_{\mathcal{M}}(Y, X) \otimes \text{Hom}_{\mathcal{M}}(X, Y) \rightarrow \mathbb{C}$ - non degenerate pairing
- $\Theta_X([X \xrightarrow{f} Y \xrightarrow{g} X]) = \Theta_Y([Y \xrightarrow{g} X \xrightarrow{f} Y])$ - cyclicity
- $\Theta_{b \triangleright X}(f) = \Theta_X(\text{tr}_{\mathcal{B}} f)$, $\Theta_{X \triangleleft c}(g) = \Theta_X(\text{tr}_{\mathcal{C}} g)$ - partial trace identities

$\Rightarrow \text{End}_{(\mathcal{B}, \mathcal{C})}(\mathcal{M}) \subset \text{End}(\mathcal{M})$ - is another spherical fusion category

In addition, pick one more spherical fusion category \mathcal{A}

and a pivotal functor $\Phi: \mathcal{A} \rightarrow \text{End}_{(\mathcal{B}, \mathcal{C})}(\mathcal{M})$

Altogether have a datum $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{M}, \Phi)$

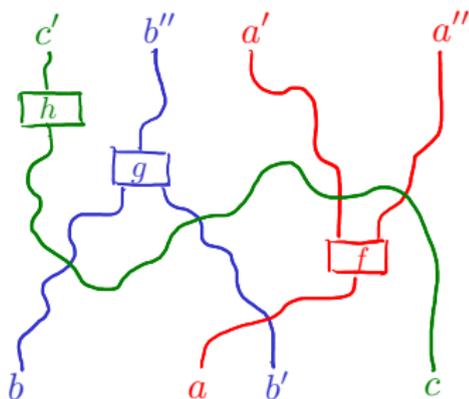
Can build "tri-layered" string diagrams in $\text{End}(\mathcal{M})$

$$\mathcal{A} \xrightarrow{\Phi} \text{End}_{(\mathcal{B}, \mathcal{C})}(\mathcal{M}) \longrightarrow \text{End}(\mathcal{M})$$

\swarrow 1) \mathcal{B}
 \searrow 2) \mathcal{C}

$$1,2) \quad \mathcal{B} \longrightarrow \text{End}(\mathcal{M}) \quad \mathcal{C} \longrightarrow \text{End}(\mathcal{M})$$

$$b \mapsto b \triangleright (-) \quad c \mapsto (-) \triangleleft c$$



a, a', a'', f - image of Φ in $\text{End}(\mathcal{M})$

b, b', b'', g — " — \mathcal{B} — " —

c, c', c'', h — " — \mathcal{C} — " —

Need crossings



Diagrammatic calculus

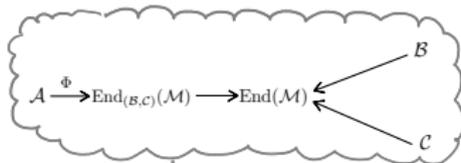


$$(b \triangleright (-)) \triangleleft c$$

 \rightsquigarrow

 \rightsquigarrow take coherence
iso. from \mathcal{M}

$$b \triangleright ((-) \triangleleft c)$$



$$b \triangleright \Phi_a(-)$$

 \rightsquigarrow

 \rightsquigarrow take coherence
iso. from $\Phi_a: \mathcal{M} \rightarrow \mathcal{M}$

$$\Phi_a(b \triangleright (-))$$



$$\Phi_a((-) \triangleleft c)$$

 \rightsquigarrow

 \rightsquigarrow ditto

$$\Phi_a(-) \triangleleft c$$

Diagrammatic calculus

Automatically have:

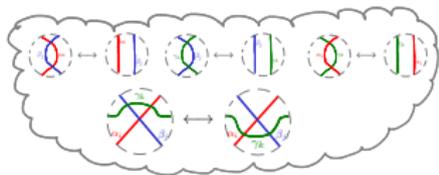
$$\begin{array}{c} \text{blue} \\ \text{green} \end{array} \text{ crossing} = \begin{array}{c} | \\ | \end{array} \begin{array}{c} | \\ | \end{array} \quad , \quad \begin{array}{c} \text{red} \\ \text{blue} \end{array} \text{ crossing} = \begin{array}{c} | \\ | \end{array} \begin{array}{c} | \\ | \end{array} , \dots$$

Also have:

$$\begin{array}{c} \text{red} \\ \text{blue} \\ \text{green} \end{array} \text{ crossing} = \begin{array}{c} \text{blue} \\ \text{red} \\ \text{green} \end{array} \text{ crossing}$$

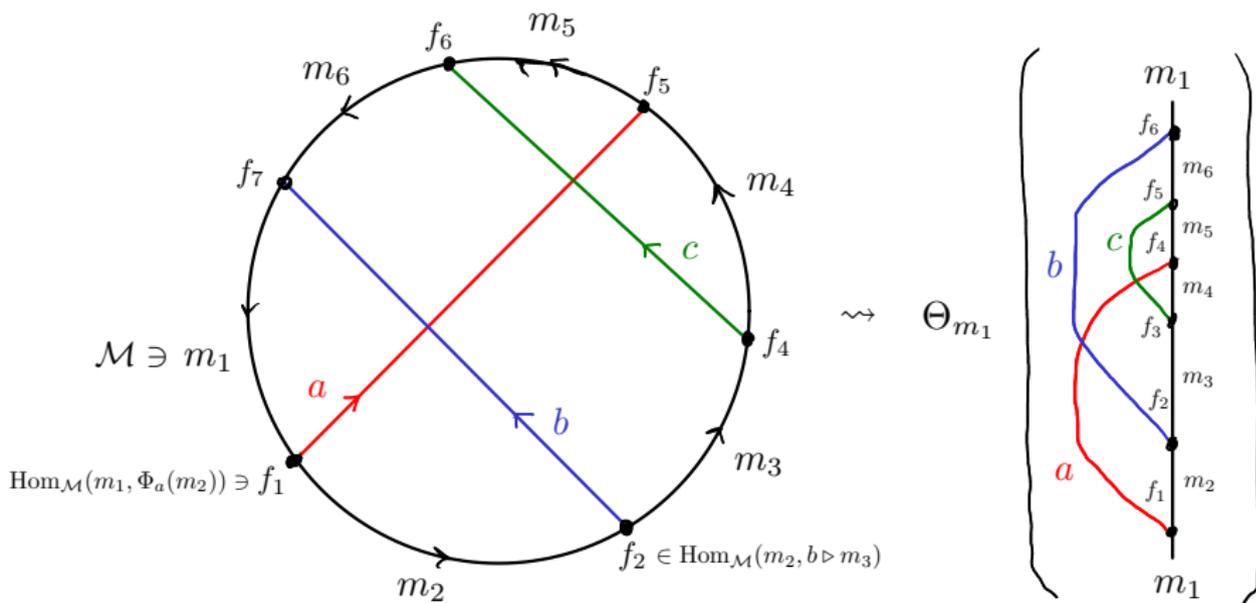
$$\begin{array}{ccc}
 \Phi_a(b \triangleright ((-) \triangleleft c)) & \Rightarrow & \Phi_a((b \triangleright (-)) \triangleleft c) \\
 \Downarrow & & \Downarrow \\
 b \triangleright \Phi_a((-) \triangleleft c) & \cong & \Phi_a(b \triangleright (-)) \triangleleft c \\
 \Downarrow & & \Downarrow \\
 b \triangleright (\Phi_a((-) \triangleleft c)) & \Rightarrow & (b \triangleright \Phi_a(-)) \triangleleft c
 \end{array}$$

(Φ_a is a bimodule functor)



Diagrammatic calculus

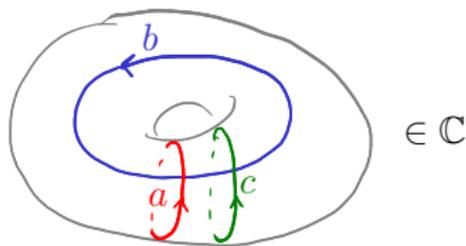
Evaluation of trilayered diagrams on a disc -- use the bimodule trace:



Diagrammatic calculus

Evaluation of trilayered diagrams on a surface

-- cut into discs using identities



$$\text{Disc} = \sum_{m \in \text{Irr}(\mathcal{M})} \dim(m) \cdot \text{Disc}(m)$$

- sum over simple objects of $\dim(m) = \Theta_m(\text{id}_m)$

$$\text{Disc}(m) \otimes \text{Disc}(n) = \sum_{\alpha} \text{Disc}(m, n, \alpha)$$

$\{\alpha\} \subset \text{Hom}_{\mathcal{M}}(m, F(n))$

$\{\bar{\alpha}\} \subset \text{Hom}_{\mathcal{M}}(F(m), n)$

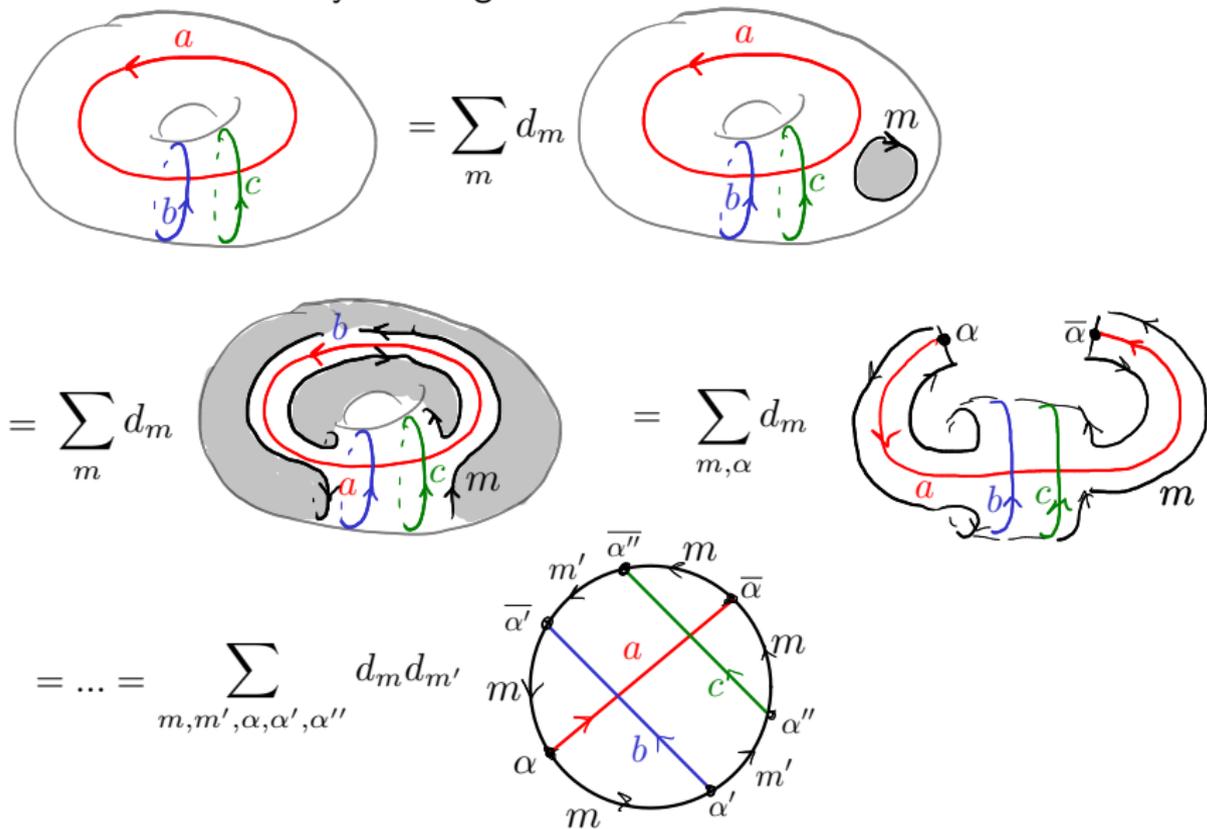
- basis -- dual basis pair

$$\Theta_m(\bar{\alpha}' \circ \alpha) = \delta_{\alpha, \alpha'}$$

Theorem The result does not depend on the cutting.

Diagrammatic calculus

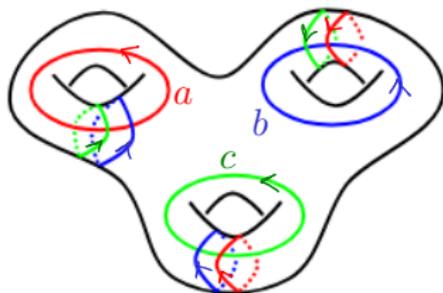
Evaluation of trilayered diagrams on a surface



In summary, a datum

$$(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{A}\mathcal{M}_{\mathcal{B}}, \Phi: \mathcal{A} \longrightarrow \text{End}_{(\mathcal{B}, \mathcal{C})}(\mathcal{M}))$$

allows one to evaluate trisection-looking diagrams like



$$a \in \mathcal{A}, b \in \mathcal{B}, c \in \mathcal{C}, \dots$$

Evaluation is automatically invariant under

- i) diffeomorphisms
- ii) two-point moves
- iii) point moves

Next: how to also ensure the invariance under

- iv) handle slides
- v) stabilisation
- ?

Categorical trisection invariants

Handle slides: use the standard construction with Kirby loops

$$\begin{aligned}
 \sum_{a \in \text{Irr}(\mathcal{A})} d_a \left(\text{Diagram 1} \right) &= \sum_{a, a', \alpha} d_a d_{a'} \left(\text{Diagram 2} \right) \\
 &= \sum_{a, a', \alpha} d_a d_{a'} \left(\text{Diagram 3} \right) \\
 &= \sum_{a' \in \text{Irr}(\mathcal{A})} d_{a'} \left(\text{Diagram 4} \right)
 \end{aligned}$$

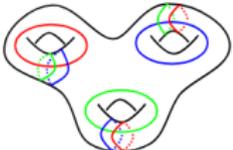
\Rightarrow Label the lines with the simples of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and take weighted sums

$$\sum_{a \in \text{Irr}(\mathcal{A})} d_a \cdots, \quad \sum_{b \in \text{Irr}(\mathcal{B})} d_b \cdots, \quad \sum_{c \in \text{Irr}(\mathcal{C})} d_c \cdots \quad (\text{this also eliminates orientations of lines})$$

Categorical trisection invariants

Stabilisation: add some normalisation factors

- Denote: $\text{av}(T)$ - above weighted sum of a trisection diagram
 $\text{av}_m(T)$ - same but with $m \in \text{Irr}(\mathcal{M})$ -labelled boundary component
($\Rightarrow \text{av}(T) = \sum_m d_m \text{av}_m(T)$)
- Have: $\text{av}(T \# T') = \sum_{m \in \text{Irr}(\mathcal{M})} \text{av}_m(T) \text{av}_m(T')$

Definition Let $\Sigma_{\text{st}} =$  and pick a datum $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{M}, \Phi)$

\mathcal{M} is called stabilising with respect to Φ if $\text{av}_m(\Sigma_{\text{st}}) = C \cdot d_m$ for some $C \neq 0$

Lemma If $\text{av}_m(\Sigma_{\text{st}}) \neq 0$ for all m then \mathcal{M} is stabilising with respect to Φ

Proof: for indecomposable \mathcal{M} , map $m \mapsto \text{av}_m(\Sigma_{\text{st}})$ defines a bimodule trace, which is unique up to scalar.

[Schaumann]

Categorical trisection invariants

Putting everything together:

- $\mathcal{A}, \mathcal{B}, \mathcal{C}$ -- spherical fusion categories
- \mathcal{M} -- \mathcal{B} - \mathcal{C} bimodule category
- $\Phi: \mathcal{A} \rightarrow \text{End}_{(\mathcal{B}, \mathcal{C})}(\mathcal{M})$ -- pivotal functor
- \mathcal{M} -- stabilising with respect to $\Phi: \text{av}_m(\Sigma_{\text{st}}) = C \cdot d_m, C \neq 0$

Theorem For a (g, k) trisection diagram T the scalar

$$|T| := C^{-g/3} \cdot \text{av}(T)$$

is invariant under i) diffeomorphisms,

ii) two-point moves,

iii) three-point moves,

iv) handle slides,

v) stabilisation

and hence gives an invariant of (smooth closed connected oriented) 4-manifolds.

Example: Hopf triplets

[Chaidez--Cotler--Cui]

- For finite dimensional Hopf algebras A, B a skew pairing $\tau: A \times B \rightarrow \mathbb{C}$ satisfies

$$\tau(aa', b) = \tau(a, b_{(1)})\tau(a', b_{(2)}) \quad \tau(a, bb') = \tau(a_{(2)}, b)\tau(a_{(1)}, b')$$

$$\tau(a, 1) = \varepsilon(a) \quad \tau(1, b) = \varepsilon(b)$$

- A Hopf triplet consists of three fin. dim. semisimple Hopf algebras together with three skew pairings A, B, C

$$\tau_{AB}: A \times B \rightarrow \mathbb{C} \quad \tau_{BC}: B \times C \rightarrow \mathbb{C} \quad \tau_{CA}: C \times A \rightarrow \mathbb{C}$$

satisfying a compatibility relation.

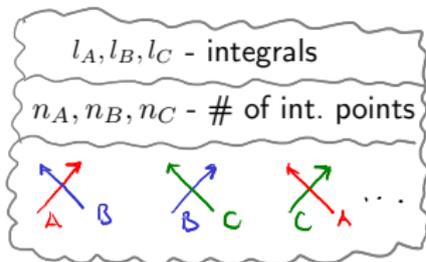
- A Hopf triplet can evaluate a (g, k) -trisection diagram

$$\mathbb{C} \xrightarrow{l_A^{\otimes g} \otimes l_B^{\otimes g} \otimes l_C^{\otimes g}} A^{\otimes g} \otimes B^{\otimes g} \otimes C^{\otimes g}$$

$$\xrightarrow{\Delta^{(n_A-1)} \otimes \Delta^{(n_B-1)} \otimes \Delta^{(n_C-1)}} A^{\otimes n_A} \otimes B^{\otimes n_A} \otimes C^{\otimes n_A}$$

$$\xrightarrow{\text{permute}} \bigotimes_{\text{int. points}} X_\lambda \otimes X_\mu$$

$$\xrightarrow{\tau\text{'s and their inverses}} \mathbb{C}$$



Example: Hopf triplets

Theorem i) The Hopf triplet invariant is a special case of the categorical trisection invariant with the datum

$$\mathcal{A} = A^{*\text{cop}}\text{-mod} \quad \mathcal{B} = B^*\text{-mod} \quad \mathcal{C} = C^{*\text{op}}\text{-mod} \quad \underbrace{\mathcal{M} = \text{vect} \quad \Phi}_{\text{use } \tau\text{'s}}$$

ii) Conversely, if $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{M}, \Phi)$ is a datum with $\mathcal{M} = \text{vect}$ then the categorical invariant is given by a Hopf triplet

Proof uses reconstruction of a Hopf algebra from the fiber functor.

More examples

- Graded vector spaces: $\mathcal{B} = \text{vect}_G$ $\mathcal{C} = \text{vect}_H$

$$\mathcal{M} = \text{vect}_M \quad M = H \times G^{\text{op}} \text{ - set}$$

- Crane--Yetter: \mathcal{E} - modular fusion category

$$\mathcal{A} = \mathcal{B} = \mathcal{M} = \mathcal{E}, \quad \mathcal{C} = \mathcal{E}^{\text{rev}}$$

$$\Phi: \mathcal{E} \longrightarrow \text{End}_{(\mathcal{E}, \mathcal{E}^{\text{rev}})}(\mathcal{E}) \simeq \mathcal{Z}(\mathcal{E}) \simeq \mathcal{E} \boxtimes \tilde{\mathcal{E}}$$

- Bärenz--Barrett: \mathcal{E} - modular fusion category

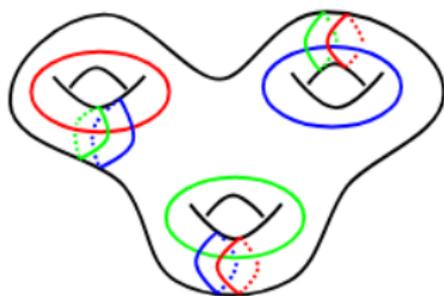
input - pivotal functor

$$\mathcal{B} = \mathcal{M} = \mathcal{E}, \quad \mathcal{C} = \mathcal{E}^{\text{rev}}$$

$$\Phi: \mathcal{A} \xrightarrow{\phi} \mathcal{E} \longrightarrow \text{End}_{(\mathcal{E}, \mathcal{E}^{\text{rev}})}(\mathcal{E}) \simeq \mathcal{Z}(\mathcal{E}) \simeq \mathcal{E} \boxtimes \tilde{\mathcal{E}}$$

Case of \mathcal{E} non-modular?

Relation to defect TQFTs



diagrammatic calculus $\longrightarrow \mathbb{C}$

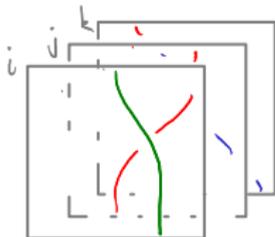
maybe also $\xrightarrow{\text{2d defect TQFT}} \mathbb{C}$

In fact the diagrammatic calculus seems to be a 2d defect TQFT of state-sum type

[Davydov--Kong--Runkel]

Relation to 4d TQFTs

- Douglas--Reutter invariants (TQFT) -- 4d state-sum construction from a spherical (pre-)fusion 2-category \mathfrak{S}
- To do: obtain DR via trisections
already have a version of trilayered diagrams from 3d graphical calculus



can e.g. start with $\mathcal{B} = \mathcal{C} = \boxplus_i \text{End}_{\mathfrak{S}}(i)$, $\mathcal{M} = \boxplus_{i,j} \text{Hom}_{\mathfrak{S}}(i,j)$

(might need to modify categorical trisection invariants for multifusion categories)

- Trisection invariants to DR?

Thank you!