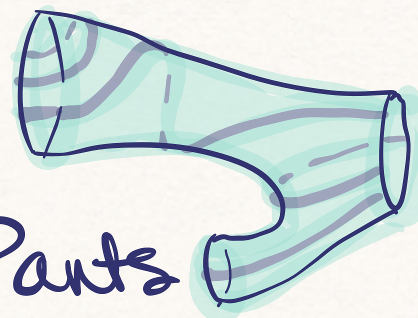
TQFT club Seminar

Nested Cobordisms & TQFTs



or: Putting Stripes on Pants

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jt. w/ R. Hoekzema, L. Murray, N. Pacheco-Tallaj, C. Rovi, and S. Sridhar-Shapiro (WIT IV + AIM)

§1 : Motivation (the folklore theorem)

Defn - $n \geq 1$, Cob_n is the (ordinary) cobordism category

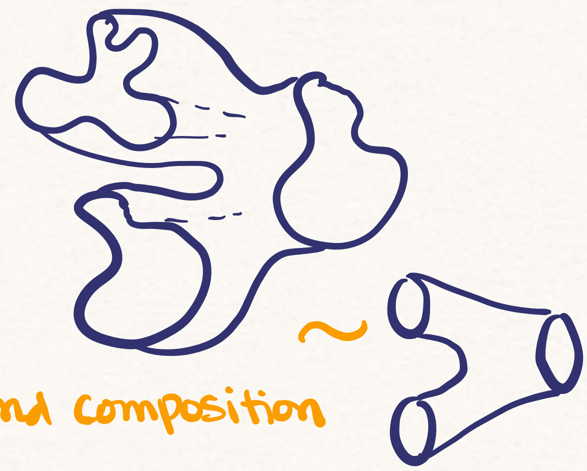
Ob = closed $(n-1)$-mflds

Hom = Cobordisms $/\sim$

where $W \sim W'$ if $\exists$ diffeo rel ∂

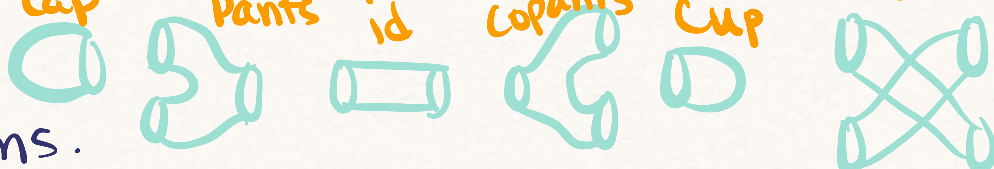
- Composition = gluing
- $\otimes = \sqcup$, unit $= \emptyset$ } Sym. mon. structure

Ex.  b/c no diffeo rel ∂



Thm. The category Cob_2 is generated by (under $\sqcup$ and composition)

cap, pants, id, copants, cup, "tw".



+ relations:

- identity relns

e.g.

- co/unit relns

e.g.

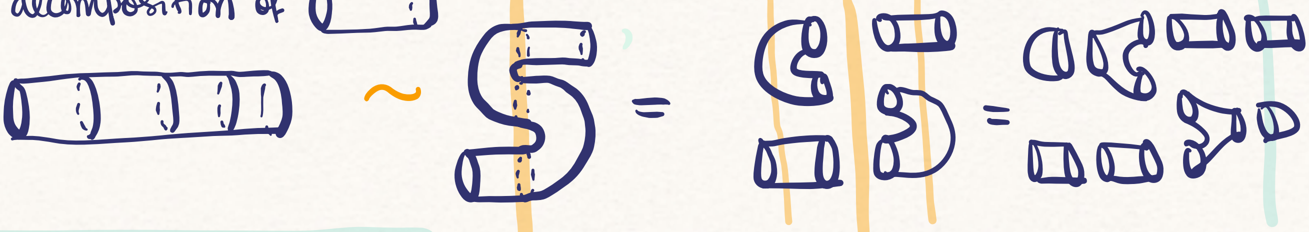
- co/associativity relns

e.g.

- co/commutativity relns

- Frobenius reln

Ex Snake decomposition of 



§1: Motivation (the folklore theorem)

Defn. n-TQFT is a symmetric monoidal functor

$$(n\mathrm{Cob}, \sqcup, \emptyset, \mathrm{tw}) \xrightarrow{F} (\mathrm{Vect}_k, \otimes, k, \sigma)$$

more generally, any symmetric monoidal category

$\sigma: V \otimes W \xrightarrow{\cong} W \otimes V$

Unpacking the defn: A TQFT F is an assignment

$$\begin{array}{llcll} (n-1)\text{-mfld} & M & \longmapsto & V & \text{v.s.} \\ \text{cobordism class} & \Downarrow W & \mapsto & \downarrow f & \text{linear map} \\ & M' & \longmapsto & V' & \end{array}$$

compatible w/ composition

s.t. $F(M_1 \sqcup M_2) = F(M_1) \otimes F(M_2)$ (and same for maps) and $F(\emptyset) = k$.

Observations for $n=2$:

- Every closed 1-mfld is $(S^1)^{\sqcup n}$ so $F(M) \cong F((S^1)^{\sqcup n}) = F(S^1)^{\otimes n} = V^{\otimes n}$
- V must be dualizable, i.e. finite dim'l
- $\overset{F}{\mapsto}$ $V \otimes V \xrightarrow{\mu} V$ "multiplication" w/ "unit" $\overset{F}{\mapsto}$ $k \xrightarrow{\eta} V$
- $\mapsto$ $V \xrightarrow{\delta} V \otimes V$ "comultiplication" w/ "counit" $\mapsto$ $V \xrightarrow{\varepsilon} k$

interactions of μ and δ describe a Frobenius algebra structure on V.

Ex. $H^*(M, k) \in \mathrm{FrAlg}(k)$ for M cpt mfld

multiplication: $H^*(M) \otimes H^*(M) \xrightarrow{\cup} H^*(M)$ cup product

comultiplication: $H^*(M) \longrightarrow H^*(M \times M) \simeq H^*(M) \otimes H^*(M)$

$\Delta: M \to M \times M$ diagonal

Ex. Group rings $\mathbb{Z}[G]$. Comultiplication = diagonal

Thm (folklore). $\mathrm{Fun}^{\otimes}(\mathrm{Cob}_2, \mathrm{Vect}_k) \cong$ Frobenius algebras over k

Pf/ main ingredient: gens + relns description of Cob_2

§2: Adding stripes

Now what?

- change dimensions $Cob_1^{unor.} \rightsquigarrow$ f.d. v.s. w/ non-degen symm bilin form

- take subcategories $(Cob_2)_{operad} \rightsquigarrow E_2$-algebras (e.g. in spaces, $\Omega^2 X$)

- different notions of cobordism

Cobordism hypothesis (Baez-Dolan, Lurie)

∞-category $Bord_\infty \rightsquigarrow$ "fully dualizable" objects

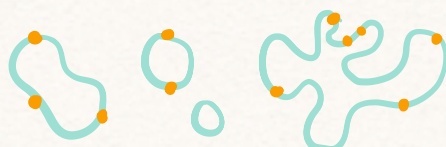
(Landa-Pfeiffer)

"open-closed" $Cob_2^{ext} \rightsquigarrow$ "Knowledgeable" Frobenius algebras

★ oriented just means top-dim'l mfld is oriented ★

Defn. The (unoriented) nested cobordism category $Cob_{1<2}$:

Objects: nested $(0<1)$-manifolds

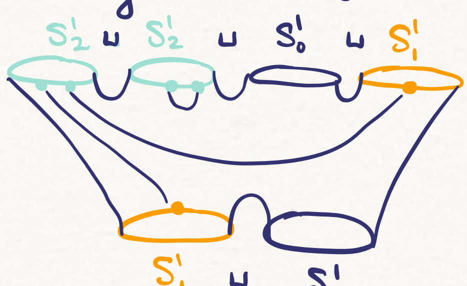


Morphisms: nested cobordisms $/\sim$

$\left.\begin{array}{l}\text{Composition} = \text{gluing} \\ \otimes = \sqcup \text{ w/ unit } \varnothing\end{array}\right\}$ sym mon

Prop. $Cob_{1<2}$ is a "colored PROP", i.e. objects are generated from collection of "colors" under $\otimes = \sqcup$.

# of marked points



operad:	PROP:
n-to-1	n-to-m

§2: Adding stripes

Defn (not super useful) A nested $(1<2)$ TQFT is a PROP-algebra,

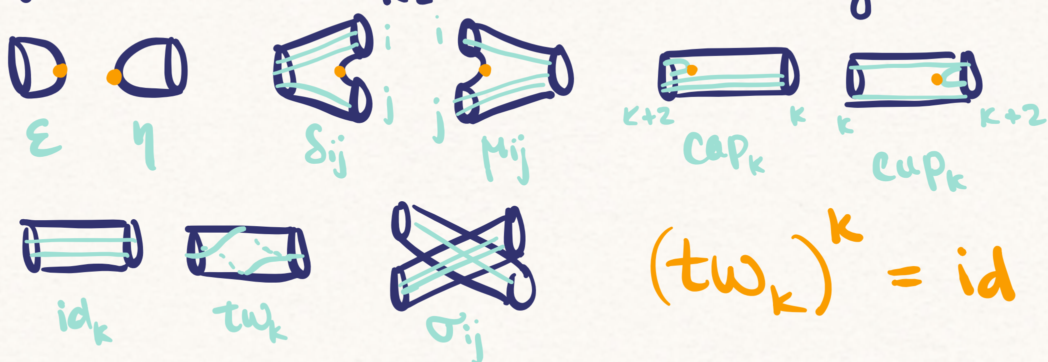
$$TQFT_{1<2} := Fun^{\otimes}(Cob_{1<2}, Vect_{k}).$$

Goal: Get a better description

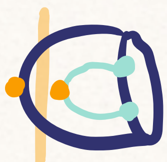
↳ Find gens-+-relns for $Cob_{1<2}$

Step ①: generators

Thm. The generators of $Cob_{1<2}$ are the "elementary nested cobordisms" [CHMPTSS]



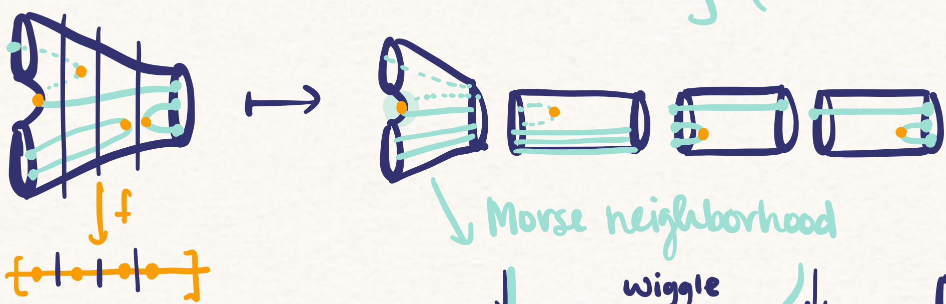
ex:

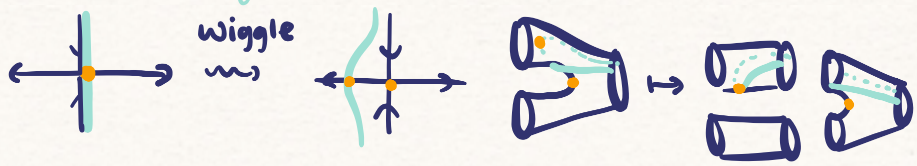


tw_3:

$$(tw_k)^k = id$$

Pf idea. "Nested" Morse theory ($\subseteq$ Stratified Morse theory)



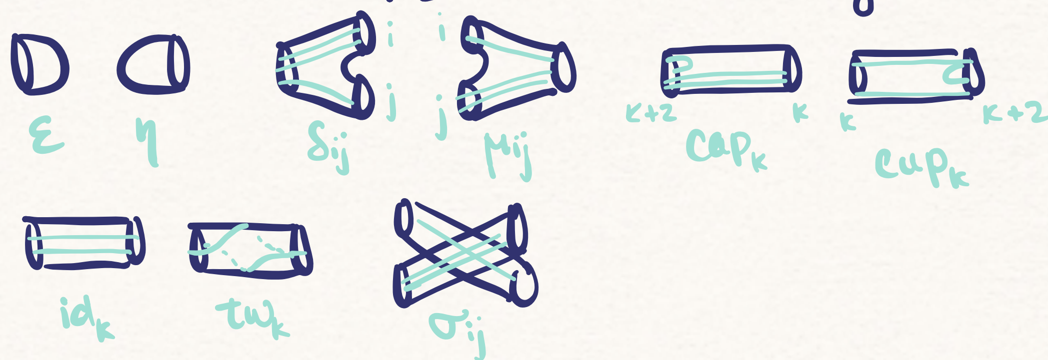


Cor. a $TQFT_{1<2}$ $F: Cob_{1<2} \to Vect_k$ is determined by

- $F(S^1_n) = V_n$
- $F(cap_k): V_k \to V_{k-2}$
- $F(cup_k): V_k \to V_{k+2}$
- $F(tw_k): V_k \xrightarrow{\cong} V_k$
- $F(\eta): V_0 \to k$
- $F(\varepsilon): k \to V_0$
- $F(\mu_{ij}): V_i \otimes V_j \to V_{i+j}$
- $F(\delta_{ij}): V_{i+j} \to V_i \otimes V_j$
- $F(\sigma_{ij}): V_i \otimes V_j \to V_j \otimes V_i$

§2: Adding stripes

Thm. The generators of $Cob_{k\le 2}$ are the "elementary nested cobordisms"



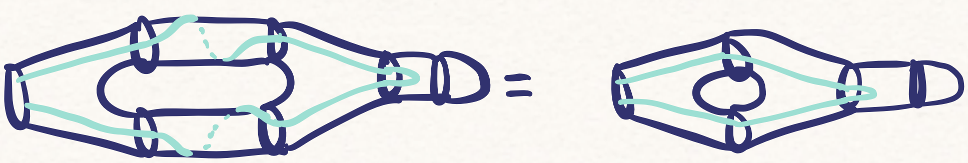
Step ②: Relations

We have necessary relns:

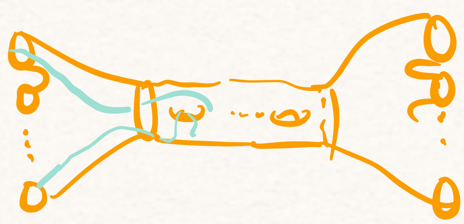
Cob_1 relations

" striped Cob_2 relations"

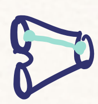
and more ... twists are evil



sufficiency is <u>hard</u>! normal form: ??



Rmk - our necessary relations imply:

- $V_0 \in FrAlg(k)$, V_1 is a V_0-module 
- $\bigoplus_{n\ge 0} V_{2n}$ is a <u>graded</u> $FrAlg(k)$, $\bigoplus_{n\ge 0} V_{2n+1}$ is a <u>graded</u> module
- $V_n \circlearrowleft C_n$

"super Frob. alg"

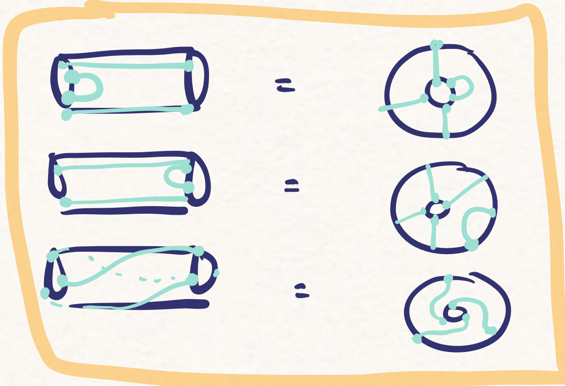
§2: Adding stripes

Thm [CHMPTSS]. Have all relns on subcategory $Cyl \subseteq Cob_{1<2}$ "cylinder background"

Cor A functor $Cyl \xrightarrow{F} Vect_k$ is given by:

- $F(S^1_n) = V_n$
- $F(cap_k): V_k \to V_{k-2}$ face
- $F(cup_k): V_k \to V_{k+2}$ degen
- $F(tw_k): V_k \xrightarrow{\cong} V_k$ cyclic

w/ relns $\simeq$ simplicial + cyclic



Thm [~] Functors out of Cyl $\rightsquigarrow$ (affine / annular) Temperley-Lieb algebras

$\rightsquigarrow$ cyclic objects "with a square root"

Rmk. Similar to results of Penneys, Kaufmann-Penner

Ex (Cyl bar construction)

Let $V \in Vect^{f.d.}(k)$, choose iso $V \cong V^*$. Define $B^{cyl}(V): Cyl \to Vect_k$ by:

- $B^{cyl}_n(V) = V^{\otimes n}$
- "cyclic" maps use $V^{\otimes n} \circlearrowleft C_n$
- face maps use $V^* \otimes V \xrightarrow{ev} k$
- degeneracy maps use $k \xrightarrow{coev} V \otimes V^*$

Note. Should be related to $B^{cyc}(End(V))$....

34: Ongoing work

Goal: Complete list of relations btwn gens of $\mathrm{Cob}_{1<2}$

$\rightsquigarrow$ algebraic descriptions of $\mathrm{TQFT}_{1<2}$s.

idea

Method: "Nested Cerf theory"

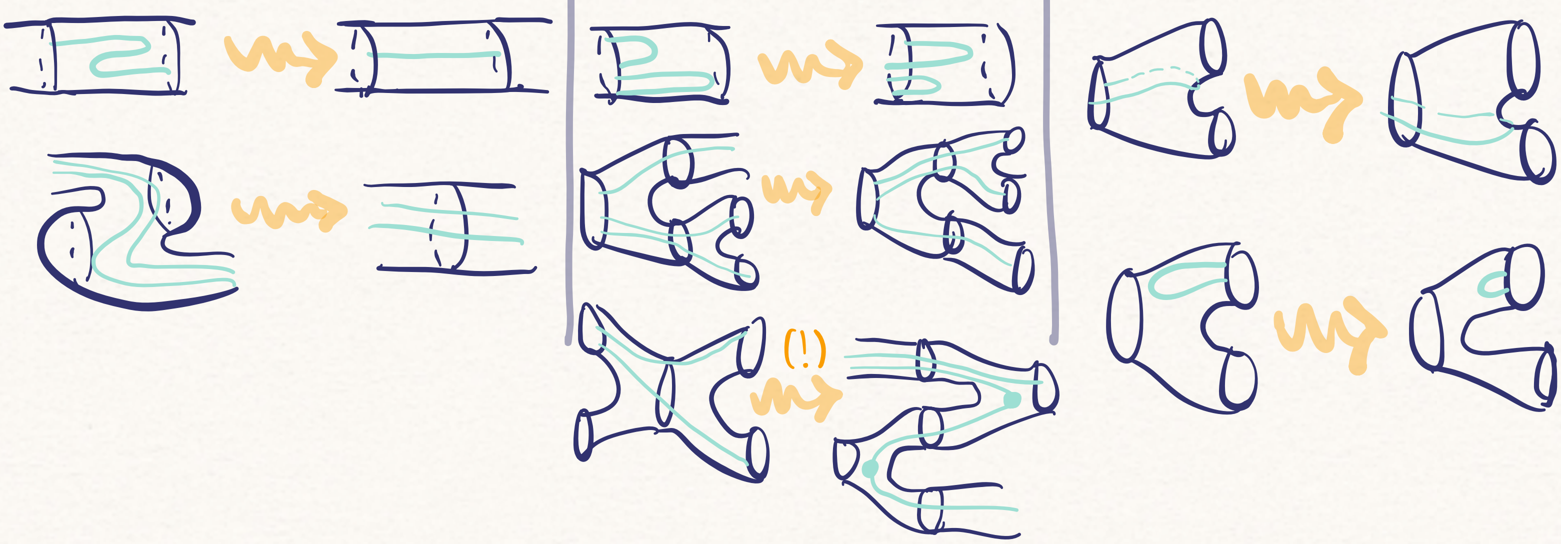
↳ $\{$"excellent" Morse fns$\} \subseteq C^\infty((W, \partial W), (I, \partial I))$

$\rightsquigarrow$ decomp of W into gens

↳ (Conj:) If f_0, f_1 excellent, then $\exists$ path f_t (*) which is excellent except at finitely many t_i, where f_{t_i} is "good"

↳ "good" Morse fns $\rightsquigarrow$ relations

type α_i (i = dimension) | type β_i | type γ_{ij}



Questions ... How to prove this "Nested Cerf Thm"?

Examples?

Relationship to "defect TQFTs"?