Bigraded cohomology for real algebraic varieties and its arithmetic variant

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Context: cohomology of real algebraic varieties

Let X be a real algebraic variety, we denote by $C_2 = \{1, \sigma\}$ the Galois group $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ which naturally acts on the topological space of complex points $X(\mathbf{C})$ such that the real points are the fixed points of the action: $X(\mathbf{R}) = X(\mathbf{C})^{C_2}$.

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The naturally cohomology groups to consider (Krasnov **[Kra91]**, **[Kra94]**, van Hamel **[Ham97]**, Benoist-Wittenberg **[BW20a]**, **[BW20b]**) are the C_2-equivariant (Borel-Grothendieck) ones

$$H^*_{C_2}(X(\mathbf{C}), \mathbf{Z}\{q\})$$

where $\mathbf{Z}\{q\} \coloneqq \left(\sqrt{-1}\right)^q \mathbf{Z} \hookrightarrow \mathbf{C}$ are the twisted coefficients "à la Tate" equipped with the natural action of the group $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$.

Context: cohomology of real algebraic varieties

By construction, there exist natural morphisms to the singular cohomology groups

$$\begin{array}{ccc} & H^{2p}_{C_2}\left(X(\mathbf{C}), \mathbf{Z}\{p\}\right) & \\ \swarrow^{\mathrm{res}} & & \searrow^{f} \\ H^p\left(X(\mathbf{R}), \mathbf{Z}/_2\right) & & H^{2p}\left(X(\mathbf{C}), \mathbf{Z}\right) \end{array}$$

such that if E is a vector bundle on X the following equalities hold **[Kah87]**, **[Kra91]**, **[Kra94]**:

$$f\left(\mathrm{c}_p^{C_2}(E(\mathbf{C}))\right) = \mathrm{c}_p(E(\mathbf{C})) \text{ and } \mathrm{res}\left(\mathrm{c}_p^{C_2}(E(\mathbf{C}))\right) = \mathrm{w}_p(E(\mathbf{R})).$$

A refinement of the C_2-equivariant cohomology

Based on ideas of J. Huisman **[Hui02]**, **[Hui]**, D. Gleuher constructed in his PhD thesis **[Gle19]** *bigraded cohomology* groups $H^{p,q}(X)$ attached to a real algebraic variety such that:

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Theorem (Huisman [Hui], [Hui24])

Let X be a real algebraic variety. After identifying X with

$$X(\mathbf{C})/C_2 \simeq \{\text{closed points of the scheme } X\}$$

we can construct bigraded cohomology groups $H^{p,q}(X)$ attached *to the quotient* such that there are (non trivial) *refinement* morphisms

$$r : H^{p,q}(X) \longrightarrow H^p_{C_2}(X(\mathbf{C}), \mathbf{Z}\{q\})$$

and natural morphisms

$$\rho : H^{2p,p}(X) \longrightarrow H^p\left(X(\mathbf{R}), \mathbf{Z}/2\right), \quad \widetilde{r} : H^{2p,p}(X) \longrightarrow H^{2p}(X(\mathbf{C}), \mathbf{Z})$$

A refinement of the C_2-equivariant cohomology

and if E is a real vector bundle over the quotient $X = X(\mathbf{C})/C_2$, one can construct characteristic classes $\mathrm{cw}_p(E) \in H^{2p,p}(X)$, called *Chern–Stiefel–Whitney classes*, such that in the natural commutative diagram,

$$\begin{array}{ccccc}
H^{2p,p}(X) & \xrightarrow{\;r\;} & H^{2p}_{C_2}(X(\mathbf{C}), \mathbf{Z}\{p\}) & & \\
\downarrow{\scriptstyle \rho} & \swarrow{\scriptstyle \mathrm{res}} & & \searrow{\scriptstyle f} & \\
H^p(X(\mathbf{R}), \mathbf{Z}/2) & & & & H^{2p}(X(\mathbf{C}), \mathbf{Z})
\end{array}
\qquad H^{2p,p}(X) \xrightarrow{\;\widetilde{r}\;} H^{2p}(X(\mathbf{C}), \mathbf{Z})$$

we have the equalities

$$r(\mathrm{cw}_p(E)) = \mathrm{c}_p^{C_2}(E(\mathbf{C})),\ \widetilde{r}(\mathrm{cw}_p(E)) = \mathrm{c}_p(E(\mathbf{C})),\ \rho(\mathrm{cw}_p(E)) = \mathrm{w}_p(E(\mathbf{R})).$$

The notion of real space

Definition

A *real space* is a triple $X := (X, \widehat{X}, \gamma)$ with X a topological space, $\widehat{X}$ a C_2-topological space and $\gamma : \widehat{X} \longrightarrow X$ a continuous map that descends to the quotient and the induced map $\overline{\gamma}$ is a homeomorphism. A morphism of real spaces

$$\left(f, \widehat{f}\right) : \left(X, \widehat{X}, \gamma_X\right) \longrightarrow \left(Y, \widehat{Y}, \gamma_Y\right)$$

is a commutative square

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{f}} & \widehat{Y} \\ {\scriptstyle \gamma_X}\downarrow & & \downarrow{\scriptstyle \gamma_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

with f a continuous map and $\widehat{f}$ a continuous C_2-equivariant map.

The notion of real space

We denote by $X(\mathbf{R}) := \gamma\left(\widehat{X}^{C_2}\right)$ the *real points* of the real space X and, if $\widehat{X}$ is Hausdorff, $X(\mathbf{R}) \xhookrightarrow{i} X$ and $X \setminus X(\mathbf{R}) \xhookrightarrow{j} X$ refer to the closed and open inclusions of the real and non-real points of X.

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Example

- If X is a real algebraic variety, a typical example of real space is $\left(X(\mathbf{C}) / C_2, X(\mathbf{C}), \pi\right)$. The real points of the real space coincide with the real points of the scheme X.

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Example

- If X is a real algebraic variety, a typical example of real space is $\left(X(\mathbf{C})/_{C_2}, X(\mathbf{C}), \pi\right)$. The real points of the real space coincide with the real points of the scheme X.
- If Y is a topological space, we can consider the diagonal action on the product space

$$\sigma \times \mathbf{1} \curvearrowright C_2 \times Y \simeq Y \coprod Y \curvearrowleft \sigma \text{ by swapping}$$

and the associated real space, the *double space*, is $(Y, Y \coprod Y, \mathrm{pr})$. There are only non-real points.

Some sheaves on real spaces

Let X be a real space, we turn the constant sheaf $\mathbf{Z}$ on $\widehat{X}$ (viewed as an étalé space) $p : \mathbf{Z} \times \widehat{X} \to \widehat{X}$ into a C_2-equivariant sheaf $\mathbf{Z}^{\mathrm{tw}}$ by letting σ act by multiplication by -1 : $\sigma.(m, \widehat{x}) := (-m, \sigma\widehat{x})$.

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Definition

The *sheaf of pure imaginary integers* on X is $J := \gamma_*^{C_2} \mathbf{Z}^{\mathrm{tw}}$.

Remark

- If $x \in X(\mathbf{R})$, we have $J_x = \{0\}$,
- The sheaf $J_{|X \setminus X(\mathbf{R})}$ is locally isomorphic to the constant sheaf $\mathbf{Z}$ on $X \setminus X(\mathbf{R})$.

Some sheaves on real spaces

Alternative construction of J : let $\mathcal{O}_X \coloneqq \gamma_* \left(\mathcal{C}_{\widehat{X}}\right)^{C_2}$ be the sheaf of complex-valued "continuous functions" on X where $\mathcal{C}_{\widehat{X}}$ is the sheaf of complex-valued continuous functions on $\widehat{X}$, equipped with the C_2-action $(\sigma.f)(\widehat{x}) \coloneqq \overline{f(\sigma.\widehat{x})}$ with $f \in \mathcal{C}_{\widehat{X}}(\mathcal{U})$. Denoting by $\widehat{J}$ the subsheaf of $\mathcal{C}_{\widehat{X}}$ with values in $i\mathbf{Z} \hookrightarrow \mathbf{C}$ we then obtain $J \simeq \gamma_*^{C_2} \widehat{J} \hookrightarrow \mathcal{O}_X$.

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Proposition

Let X be a real space,

- The quotient sheaf $\gamma_*\mathbf{Z}/\mathbf{Z}$ is isomorphic to J. In particular, we have a short exact sequence of abelian sheaves on X

$$0 \longrightarrow \mathbf{Z} \longrightarrow \gamma_*\mathbf{Z} \xrightarrow{1-\sigma} J \longrightarrow 0,$$

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- There is a canonical isomorphism $J/2J \simeq j_!\mathbf{Z}/2$,
- There is a canonical isomorphism $J^{\otimes 2} \simeq j_!\mathbf{Z}$.

Reduction complexes

We get a *reduction morphism* of abelian sheaves on X

$$J \longrightarrow J/2J \simeq j_! \mathbf{Z}/2 \hookrightarrow \mathbf{Z}/2, \qquad \rho : J \to \mathbf{Z}/2.$$

that we choose to view as a complex:

Definition

The *reduction complex* $\mathbf{Z}(1)_X$ on a real space X is the complex of abelian sheaves on X

$$\cdots \longrightarrow 0 \longrightarrow J \xrightarrow{\rho} \mathbf{Z}/2 \longrightarrow 0 \longrightarrow \cdots$$

concentrated in degrees 0 and 1.

Construction of the bigraded cohomology

For $q \geq 2$, we define the *higher reduction complexes* as follows: $\mathbf{Z}(q) := \mathbf{Z}(1)^{\otimes q}$ (and we set $\mathbf{Z}(0) := \mathbf{Z}[0]$).

Remark

- If X is a real space associated to a topological space with the trivial action then $J = 0$ hence $\mathbf{Z}(1) = {}^{\mathbf{Z}}/_{2}[-1]$ and $\mathbf{Z}(q) = {}^{\mathbf{Z}}/_{2}[-q]$: no non-real points!
- If X is a double space then $J \simeq \mathbf{Z}$ hence $\mathbf{Z}(1)$ is quasi-isomorphic to $\mathbf{Z}[0]$: no real points!

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Definition

The *bigraded (integral) cohomology* of a real space X is the hypercohomology of these complexes of sheaves,

$$H^{p,q}(X) := H^{p,q}(X, \mathbf{Z}) := \mathbf{H}^{p}(X, \mathbf{Z}(q)).$$

The analytic exponential triangle

Note that the following square

$$\begin{array}{ccc} J & \xrightarrow{\rho} & \mathbf{Z}/2 \\ {\scriptstyle \times\pi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathcal{O}_X & \xrightarrow[\exp^{\mathrm{an}}]{} & \mathcal{O}_X^\times, \end{array}$$

with $\varphi : \mathbf{1} \mapsto -1$, is commutative.

Theorem (Gleuher **[Gle19]**)

Let X be a real space. There is a natural exact triangle in the real space, called the *analytic exponential triangle on* X,

$$\mathbf{Z}(1) \longrightarrow \mathcal{O}_X \xrightarrow{\exp^{\mathrm{an}}} \mathcal{O}_X^\times \longrightarrow \mathbf{Z}(1)[1]$$

in the triangulated category $\mathcal{D}^b(X)$.

Exact triangles related to the reduction complexes

Theorem (Gleuher [Gle19])

Let X be a real space. For each integer $q \geq 0$, there is a natural exact triangle in the real space, called the *fundamental triangle on* X,

$$J^{\otimes q} \longrightarrow \mathbf{Z}(q) \longrightarrow i_* \mathbf{Z}/2\,[-q] \longrightarrow J^{\otimes q}[1]$$

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The induced long exact sequence in cohomology

$$\cdots \xrightarrow{\delta} H^p(X, J^{\otimes q}) \longrightarrow H^{p,q}(X) \longrightarrow H^{p-q}(X(\mathbf{R}), \mathbf{Z}/2) \xrightarrow{\delta} H^{p+1}(X, J^{\otimes q}) \longrightarrow \cdots$$

is the main tool for proving important properties of the bigraded cohomology and for performing some computations.

The Eilenberg-Steenrod axioms for $H^{*,*}(-)$

Theorem (Gleuher [Gle19])

The bigraded cohomology groups satisfy the analogue of Eilenberg-Steenrod axioms, *i.e.*

- If two morphisms of real spaces are real-homotopic then the induced morphisms in bigraded cohomology coincide.
- Let $(X, \mathcal{Z})$ be a pair of real spaces, and let $\mathcal{V} \subset X$ be an open subset such that $\overline{\mathcal{V}} \subseteq \mathring{\mathcal{Z}}$. The inclusion morphism $\iota : (X \setminus \mathcal{V}, \mathcal{Z} \setminus \mathcal{V}) \longrightarrow (X, \mathcal{Z})$ induces isomorphisms in bigraded cohomology:

$$\iota^* : H^{p,q}(X, \mathcal{Z}) \longrightarrow H^{p,q}(X \setminus \mathcal{V}, \mathcal{Z} \setminus \mathcal{V}).$$

The Eilenberg-Steenrod axioms for $H^{*,*}(-)$

- Let $(X, \mathcal{Z})$ be a pair of real spaces, the inclusions $\mathcal{Z} \overset{i}{\hookrightarrow} X$ and $\mathcal{U} := X \setminus \mathcal{Z} \overset{j}{\hookrightarrow} X$ induce long exact sequences in bi-graded cohomology:

$$\cdots \xrightarrow{\delta} H^{p,q}(X, \mathcal{Z}) \xrightarrow{j^*} H^{p,q}(X) \xrightarrow{i^*} H^{p,q}(\mathcal{Z}) \xrightarrow{\delta} H^{p+1,q}(X, \mathcal{Z}) \longrightarrow \cdots$$

- Let $(X, \mathcal{Z})$ be a pair of real spaces, if $X = \coprod_{i \in I} X_i$ and $\mathcal{Z}_i := X_i \cap \mathcal{Z}$ then there are canonical isomorphisms

$$H^{p,q}(X, \mathcal{Z}) \simeq \prod_{i \in I} H^{p,q}(X_i, \mathcal{Z}_i).$$

Some calculations of bigraded cohomology groups

- Let $\{\bullet_{\mathbf{R}}\}$ be the point in the category of real spaces, we have
  $$H^{p,q}(\{\bullet_{\mathbf{R}}\}) = \begin{cases} \mathbf{Z} & \text{if } p = q = 0 \\ \mathbf{Z}/2 & \text{if } p = q \neq 0. \end{cases}$$

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- The bigraded cohomology of finite-dimensional real linear C_2-representations agrees with the bigraded cohomology of a point: $H^{p,q}(\mathbf{A}^{n,w}) = \begin{cases} \mathbf{Z} & \text{if } p=q=0 \\ \mathbf{Z}/2 & \text{if } p=q\neq 0. \end{cases}$

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- If X is a M-curve (a real smooth projective curve of genus g for which Harnack's inequality $\#\pi_0(X(\mathbf{R})) \leq g+1$ is an equality) then we have
$$H^{p,1}(X) = \begin{cases} \mathbf{Z}^g \oplus \mathbf{Z}/2 & \text{if } p = 1 \\ \mathbf{Z} \oplus (\mathbf{Z}/2)^g & \text{if } p = 2 \\ 0 & \text{otherwise.} \end{cases}$$

The arithmetic exponential

Definition

Let A be an abelian group, consider the projective system of abelian groups indexed by the partially ordered set $(\mathbf{N}^*, |)$ given by $A_n := A$ for all $n \geq 1$, with transition morphisms $\varphi_{nm} : a \longmapsto \frac{n}{m}.a$ defined whenever m divides n, the inverse limit of this system $A_! := \lim\limits_{m|n} A$ is the *uniquely divisible co-envelope* of A.

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Definition

The *arithmetic exponential morphism of* A is the projection onto the first factor $\exp^{\mathrm{ar}} : A_! \longrightarrow A_1 = A$.

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The functor $\bullet_! : \mathbf{Ab} \longrightarrow \mathbf{Q}\mathrm{Vect}$ is a right adjoint to the forgetful functor $\mathbf{Q}\mathrm{Vect} \longrightarrow \mathbf{Ab}$ and the arithmetic exponential is the counit of the adjunction.

The arithmetic exponential

If the group A is divisible, then there is a short exact sequence

$$0 \longrightarrow \mathrm{Ker}\left(\exp^{\mathrm{ar}}\right) = \lim_{m|n} T_n\left(A\right) \longrightarrow A_! \xrightarrow{\exp^{\mathrm{ar}}} A \longrightarrow 0.$$

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of abelian groups.

Example

We obtain a morphism of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 2i\pi\mathbf{Z} & \longrightarrow & \mathbf{C} & \xrightarrow{\exp^{\mathrm{an}}} & \mathbf{C}^\times & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow {\scriptstyle \exists! \widetilde{\exp^{\mathrm{an}}}} & & \| & & \\
0 & \longrightarrow & \widehat{\mathbf{Z}} & \longrightarrow & \mathbf{C}_!^\times \simeq \mathbf{S}_!^1 \times \mathbf{R}_{>0} & \xrightarrow[\exp^{\mathrm{ar}}]{} & \mathbf{C}^\times & \longrightarrow & 0.
\end{array}$$

The arithmetic exponential

If $\mathcal{C}$ is a Grothendieck site and $\mathscr{F}$ is an object of $\mathbf{Ab}(\mathcal{C})$ one defines similarly $\mathscr{F}_!$ and the arithmetic sheaf exponential morphism $\exp^{\mathrm{ar}} : \mathscr{F}_! \longrightarrow \mathscr{F}$. The same universal property still holds.

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Example

In particular, if X is a topological space with $\mathcal{C}_X$ its sheaf of complex-valued continuous functions and $\mathcal{I}_X := \mathcal{C}_X^\times$ we have a morphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 2i\pi\mathbf{Z}_X & \longrightarrow & \mathcal{C}_X & \xrightarrow{\exp^{\mathrm{an}}} & \mathcal{I}_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \exists!\widetilde{\exp^{\mathrm{an}}}} & & \| & & \\ 0 & \longrightarrow & \widehat{\mathbf{Z}}_X & \longrightarrow & \mathcal{I}_{X!} & \xrightarrow[\exp^{\mathrm{ar}}]{} & \mathcal{I}_X & \longrightarrow & 0. \end{array}$$

The real arithmetic exponential sequence

If X is a real space, we equip the sheaf $\mathcal{I}_{\widehat{X}!}$ with the pointwise action corresponding to the C_2-action on $\mathcal{I}_{\widehat{X}}$ in order to force the equivariance of the arithmetic exponential morphism

$$C_2 \curvearrowright \mathcal{I}_{\widehat{X}!} \xrightarrow{\quad \exp^{\mathrm{ar}} \quad} \mathcal{I}_{\widehat{X}} \curvearrowleft C_2$$

and to obtain a morphism

$$\mathcal{J}_{X!} \coloneqq \left(\gamma_* \mathcal{I}_{\widehat{X}!}\right)^{C_2} \xrightarrow{\gamma_*(\exp^{\mathrm{ar}})^{C_2}} \left(\gamma_* \mathcal{I}_{\widehat{X}}\right)^{C_2} \eqqcolon \mathcal{J}_X.$$

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$$C_2 \curvearrowright \mathcal{I}_{\widehat{X}!} \xrightarrow{\exp^{\mathrm{ar}}} \mathcal{I}_{\widehat{X}} \curvearrowleft C_2$$

and to obtain a morphism

$$\mathcal{J}_{X!} := \left(\gamma_* \mathcal{I}_{\widehat{X}!}\right)^{C_2} \xrightarrow{\gamma_*(\exp^{\mathrm{ar}})^{C_2}} \left(\gamma_* \mathcal{I}_{\widehat{X}}\right)^{C_2} =: \mathcal{J}_X.$$

Still denoting the previous morphism by $\exp^{\mathrm{ar}}$ and introducing the sheaf $J^{\mathrm{ar}} := \left(\gamma_* \widehat{\mathbf{Z}}_{\widehat{X}}\right)^{C_2} \hookrightarrow \mathcal{J}_{X!}$ we get the *real arithmetic exponential sequence* on X.

The real arithmetic exponential sequence

Theorem (M., 2025)

There is a short exact sequence of abelian sheaves on X

$$0 \longrightarrow J^{\mathrm{ar}} \longrightarrow \mathcal{J}_{X!} \xrightarrow{\exp^{\mathrm{ar}}} \mathcal{J}_X^+ \longrightarrow 0.$$

The universal property of $\bullet_!$ once more ensures the existence of a morphism between short exact sequences,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 2\pi J & \longrightarrow & \mathcal{O}_X & \xrightarrow{\exp^{\mathrm{an}}} & \mathcal{J}_X^+ & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle \exists!\widetilde{\exp^{\mathrm{an}}}} & & \big\| & & \\
0 & \longrightarrow & J^{\mathrm{ar}} & \longrightarrow & \mathcal{J}_{X!} & \xrightarrow[\exp^{\mathrm{ar}}]{} & \mathcal{J}_X^+ & \longrightarrow & 0.
\end{array}$$

Arithmetic reduction complex

As in the "analytic" case, one has an *arithmetic reduction morphism*

$$\rho^{\mathrm{ar}} : J^{\mathrm{ar}} \longrightarrow \{\pm 1\}_X \simeq \mathbf{Z}/2$$

obtained in a similar way

$$\begin{array}{c} J^{\mathrm{ar}} \overset{\frac{1}{2}}{\hookrightarrow\!\!\!\!\twoheadrightarrow} \tfrac{1}{2}J^{\mathrm{ar}} \overset{\pi}{\longrightarrow} \tfrac{1}{2}J^{\mathrm{ar}}/J^{\mathrm{ar}} \overset{\overline{\exp^{\mathrm{ar}}}}{\longrightarrow} j_!\mu_2 \hookrightarrow \mu_2 \simeq \mathbf{Z}/2 \\ \rho^{\mathrm{ar}} : J^{\mathrm{ar}} \longrightarrow \mu_2 \end{array}$$

and we define the *arithmetic reduction complex* $\mathbf{Z}(1)^{\mathrm{ar}}$ as the complex of abelian sheaves on X

$$\cdots \longrightarrow 0 \longrightarrow J^{\mathrm{ar}} \overset{\rho^{\mathrm{ar}}}{\longrightarrow} \mathbf{Z}/2 \longrightarrow 0 \longrightarrow \cdots$$

concentrated in degrees 0 and 1.

Arithmetic reduction complex

The morphism of complexes

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & J^{\mathrm{ar}} & \xrightarrow{\rho^{\mathrm{ar}}} & \mathbf{Z}/2 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \frac{1}{2}} & & \downarrow{\scriptstyle \iota} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{J}_{X!} & \xrightarrow[\exp^{\mathrm{ar}}]{} & \mathcal{J}_X & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

gives a quasi-isomorphism between $\mathbf{Z}(1)^{\mathrm{ar}}$ and $C(\exp^{\mathrm{ar}})[-1]$.

Arithmetic reduction complex

The morphism of complexes

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & J^{\mathrm{ar}} & \xrightarrow{\rho^{\mathrm{ar}}} & \mathbf{Z}/2 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow \tfrac{1}{2} & & \downarrow \iota & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{J}_{X!} & \xrightarrow[\exp^{\mathrm{ar}}]{} & \mathcal{J}_X & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

gives a quasi-isomorphism between $\mathbf{Z}(1)^{\mathrm{ar}}$ and $C(\exp^{\mathrm{ar}})[-1]$.

Corollary

Let X be a real space. There is a natural exact triangle in the real space, called the *arithmetic exponential triangle over* X,

$$\mathbf{Z}(1)^{\mathrm{ar}} \longrightarrow \mathcal{J}_{X!} \xrightarrow{\exp^{\mathrm{ar}}} \mathcal{J}_X \longrightarrow \mathbf{Z}(1)^{\mathrm{ar}}[1]$$

in the triangulated category $\mathcal{D}^b(X)$.

Higher complexes and prefinite completion

Unlike in the "analytic" setting, we'll not define the higher arithmetic complexes simply by taking tensor powers of $\mathbf{Z}(1)^{\mathrm{ar}}$. We need the notion of *prefinite completion*, a variant of the profinite completion which makes sense in a wide class of abelian categories.

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Definition

Let G be a group. If H is a normal subgroup of G we say that has *efinite index* in G whenever the quotient group ${}^{G}/_{H}$ has finite exponent. The *prefinite completion* of a group G is the inverse limit $\widehat{G}^{\mathrm{ef}} := \lim_{U \lhd_{\mathrm{ef}} G} {}^{G}/_{U}$ and there is a natural morphism $\eta : G \longrightarrow \widehat{G}^{\mathrm{ef}}$.

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Remark

There is a morphism $\epsilon : \widehat{G}^{\mathrm{ef}} \longrightarrow \widehat{G}$ given by the universal property of the profinite completion, which is an isomorphism for finitely generated abelian groups.

Higher complexes and prefinite completion

Definition

Let $\mathcal{A}$ be a Grothendieck abelian category. If A is an object of $\mathcal{A}$ the *prefinite completion* of A is the inverse limit $\widehat{A} = \lim\limits_{m|n} {}^{A}/_{nA}$ and A is *prefinite* when the natural morphism $\eta : A \longrightarrow \widehat{A}$ is an isomorphism.

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This gives us access to a notion of completed tensor product in the prefinite sense when $\mathcal{A} = \mathbf{Ab}(X)$ with X a topological space:

Definition

Let $\mathscr{F}, \mathscr{G}$ be abelian sheaves on a topological space X. The *completed tensor product* $\mathscr{F} \widehat{\otimes}^{\mathrm{ef}} \mathscr{G}$ is the sheaf associated to the presheaf that assigns to each open subset $\mathcal{U} \subseteq X$ the abelian group $\mathscr{F}(\mathcal{U}) \widehat{\otimes}^{\mathrm{ef}}_{\mathbf{Z}} \mathscr{G}(\mathcal{U})$.

Construction of the arithmetic variant

For $q \geq 2$, we construct the *higher arithmetic reduction complexes* as follows: $\mathsf{Z}(q)^{\mathrm{ar}} := \left(\mathsf{Z}(1)^{\mathrm{ar}}\right)^{\widehat{\otimes}^{\mathrm{ef}} q}$ (and we set $\mathsf{Z}(0)^{\mathrm{ar}} := \widehat{\mathsf{Z}}[0]$).

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Definition

The *bigraded arithmetic (integral) cohomology* of a real space X is the hypercohomology of these complexes of sheaves,

$$H^{p,q}_{\mathrm{ar}}(X) := H^{p,q}_{\mathrm{ar}}(X, \mathbf{Z}) := \mathbf{H}^{p}\left(X, \mathbf{Z}(q)^{\mathrm{ar}}\right).$$

There are natural morphisms induced in cohomology $H^{p,q}(X) \to H^{p,q}_{\mathrm{ar}}(X)$ by the morphisms of complexes $\mathbf{Z}(q) \to \mathbf{Z}(q)^{\mathrm{ar}}$ coming from the morphism of exact triangles

$$\begin{array}{ccccccc}
\mathbf{Z}(1) & \longrightarrow & \mathcal{O}_X & \xrightarrow{\exp^{\mathrm{an}}} & \mathcal{J}_X & \longrightarrow & \mathbf{Z}(1)[1] \\
\downarrow & & \downarrow{\scriptstyle \widetilde{\exp}^{\mathrm{an}}} & & \| & & \downarrow \\
\mathbf{Z}(1)^{\mathrm{ar}} & \longrightarrow & \mathcal{J}_{X!} & \xrightarrow[\exp^{\mathrm{ar}}]{} & \mathcal{J}_X & \longrightarrow & \mathbf{Z}(1)^{\mathrm{ar}}[1].
\end{array}$$

A comparison theorem

Theorem (M., 2025)

Let X be a real cell complex with finite skeleta. The bigraded arithmetic (integral) cohomology of X is obtained as the profinite completion of the bigraded (integral) cohomology of X: the cohomology groups $H^{p,q}_{\mathrm{ar}}(X)$ are profinite and the natural morphisms $H^{p,q}(X) \longrightarrow H^{p,q}_{\mathrm{ar}}(X)$ induce isomorphisms

$$\widehat{H^{p,q}(X)} \xrightarrow{\simeq} H^{p,q}_{\mathrm{ar}}(X)\,.$$

Ideas of the proof

We decompose the proof into several parts; all of them are encompassed in the following diagram,

$$\begin{array}{ccc}
H^{p,q}(X) & \longrightarrow & \mathbf{H}^{p}\left(X, \widehat{\mathbf{Z}(q)}\right) \\
\eta \big\downarrow & \overset{\exists!\phi}{\underset{\simeq}{\nearrow}} & \simeq \big\downarrow \mathbf{H}^{p}(\psi) \\
\widehat{H^{p,q}(X)} & \xrightarrow[\simeq]{} & H^{p,q}_{\mathrm{ar}}(X).
\end{array}$$

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Ideas of the proof

There are essentially three main steps:

- If the real space X is Hausdorff, locally connected, and has only finitely many connected components (*e.g.*, compact and locally connected), then $\mathbf{H}^*(\psi)$ is an isomorphism,
- If X is a real cell complex with finite skeleta, then the hypercohomology groups $\mathbf{H}^*\left(X, \widehat{\mathbf{Z}(*)}\right)$ are profinite,
- If X is a real cell complex with finite skeleta, then the morphism ϕ is an isomorphism.

The semi-algebraic side

Both versions of bigraded cohomology have their semi-algebraic counterparts, namely for algebraic varieties over any real closed field, so that all the previous results remain valid in this setting. This relies on the theory of locally semi-algebraic spaces developed by Delfs–Knebusch–Schwartz **[DK85]**, **[Sch89]** and on the work of Coste-Roy **[CR82]**, Delfs **[Del91]** on semi-algebraic cohomology (which takes the place of usual sheaf cohomology).

The semi-algebraic side

Theorem (M., 2025)

Let X be a real closed space over $\mathbf{R}$ (in particular, a real algebraic variety). For all integers p, q, there are canonical isomorphisms

$$\iota^* : H^{p,q}_{\mathrm{san}}(X) \xrightarrow{\simeq} H^{p,q}_{\mathrm{an}}(X_{\mathrm{top}}) \qquad ; \qquad \iota^* : H^{p,q}_{\mathrm{sar}}(X) \xrightarrow{\simeq} H^{p,q}_{\mathrm{ar}}(X_{\mathrm{top}}) \,.$$

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Theorem (M., 2025)

Let S/R be a real closed field extension and X a real closed space over R (in particular, an algebraic variety over R). For all integers p, q, there are canonical isomorphisms

$$\pi^* : H^{p,q}_{\mathrm{san}}(X) \xrightarrow{\simeq} H^{p,q}_{\mathrm{san}}(X_S) \quad ; \quad \pi^* : H^{p,q}_{\mathrm{sar}}(X) \xrightarrow{\simeq} H^{p,q}_{\mathrm{sar}}(X_S) .$$

Thanks for your attention!

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