# An introduction to the KPZ universality class (Lecture 1)

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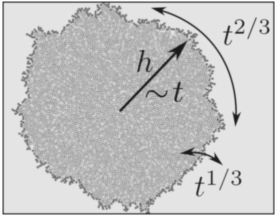
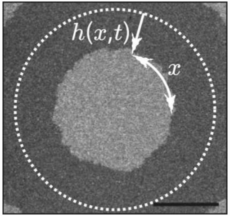
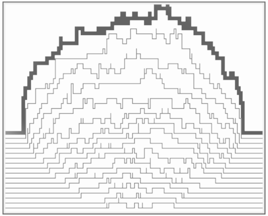
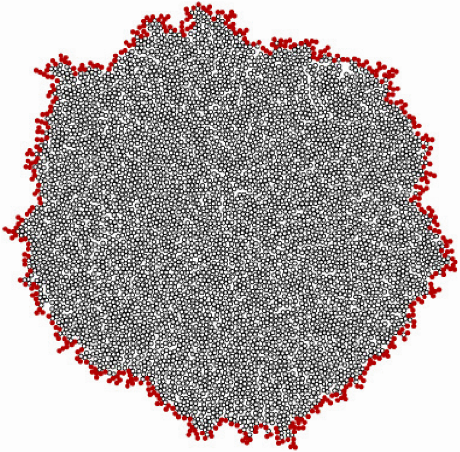


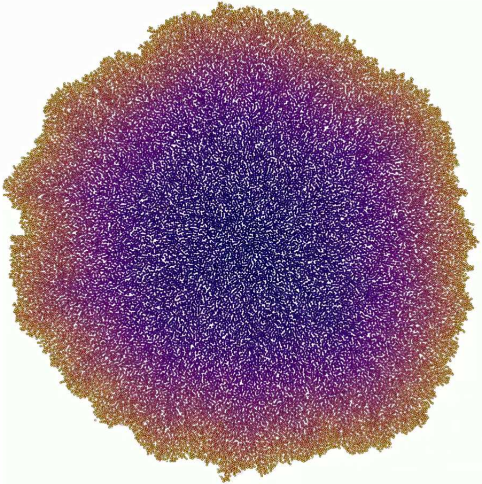



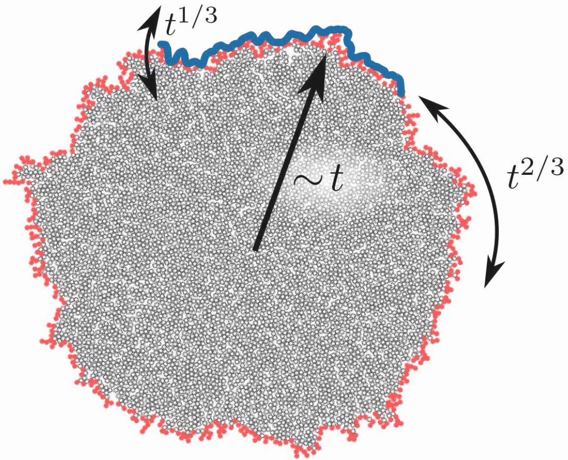
# One dimensional random growth

Markov processes with state space height fns. $h : (\mathbb{R} \text{ or } \mathbb{Z}) \longrightarrow \mathbb{R}$



Eden growth model



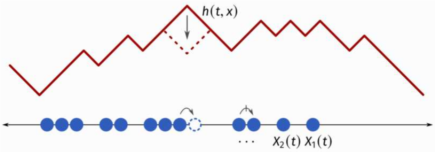



Fluctuations of order $t^{1/3}$

Spatial correlations at scale $t^{2/3}$

# Other examples

Particle systems, e.g. asymmetric exclusion processes



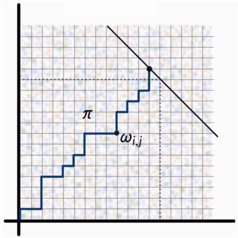

Bacterial colony

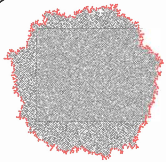
Paper burning

Stochastic reaction diffusion equations

Directed random polymers

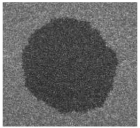
(paths weighted by their journey through a random background)



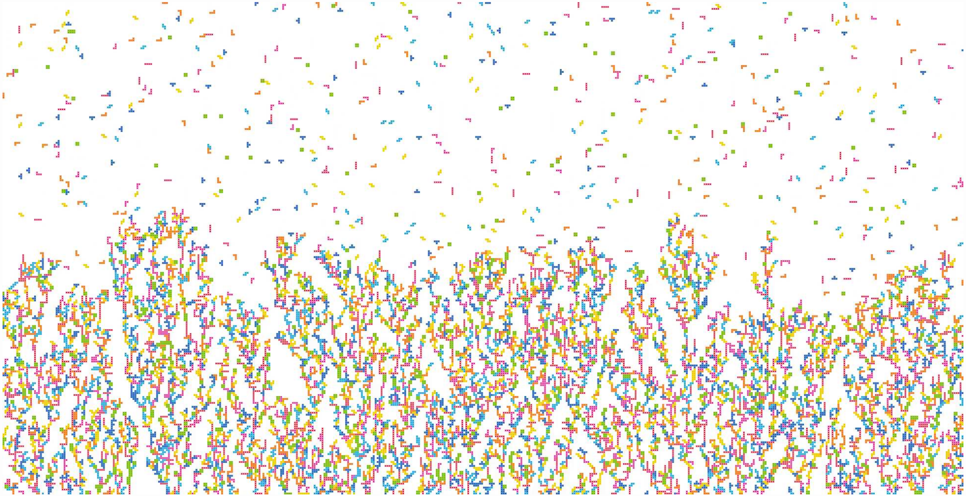
Eden model

Coffee stains



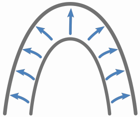
Liquid crystal turbulence

Unstable phase invades stable phase



# Kardar-Parisi-Zhang (KPZ) Equation ('86)

$$\partial_t h = \underbrace{(\partial_x h)^2}_{\text{lateral growth}} + \underbrace{\partial_x^2 h}_{\text{smoothing}} + \underbrace{\xi}_{\text{space-time white noise}}$$



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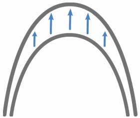

Without the lateral growth mechanism, fluctuations are Gaussian, of size $t^{1/4}$

(Edwards-Wilkinson universality class / additive SHE)

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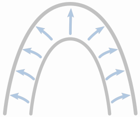

Physics prediction: [Résibois-Pomeau '75, Forster-Nelson-Stephen '77, KPZ '86]

Fluctuations should be of order $t^{1/3}$

Non-trivial correlations at spatial scale $t^{2/3}$

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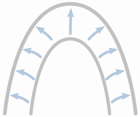

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**KPZ 1:2:3 scaling:** $h(t,x) \longmapsto t^{-1/3}\big(h(t, t^{2/3}x) - ct\big), \qquad t \longrightarrow \infty$

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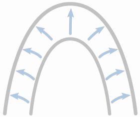

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Fluctuations should be of order $t^{1/3}$

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**KPZ 1:2:3 scaling:** $h(t,x) \longmapsto \varepsilon^{1/2}\left(h(c_1\varepsilon^{-3/2}t, c_2\varepsilon^{-1}x) - c_3\varepsilon^{-3/2}t\right), \qquad \varepsilon \to 0$

## Where does this scaling come from?

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi$$

Introduce the scaled solution $h_\varepsilon(t,x) = \varepsilon^b h(\varepsilon^{-z}t, \varepsilon^{-1}x)$.

Since

$$\partial_x h(t,x) = \varepsilon^{1-b}\partial_x h_\varepsilon(\varepsilon^z t, \varepsilon x), \qquad \partial_x^2 h(t,x) = \varepsilon^{2-b}\partial_x^2 h_\varepsilon(\varepsilon^z t, \varepsilon x),$$

$$\partial_t h(t,x) = \varepsilon^{z-b} h_\varepsilon(\varepsilon^z t, \varepsilon x), \qquad \xi(t,x) \overset{(d)}{=} \varepsilon^{\frac{z}{2}+\frac{1}{2}}\xi(\varepsilon^z t, \varepsilon x),$$

the KPZ equation for $h_\varepsilon$ reads

$$\partial_t h_\varepsilon = \varepsilon^{2-z}\partial_x^2 h_\varepsilon + \varepsilon^{2-z-b}(\partial_x h_\varepsilon)^2 + \varepsilon^{b-\frac{z}{2}+\frac{1}{2}}\xi$$

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Brownian motion should be invariant (modulo global height) $\rightsquigarrow$ b=1/2

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$$\rightsquigarrow \quad \partial_t h_\varepsilon = \varepsilon^{2-z}\partial_x^2 h_\varepsilon + \varepsilon^{\frac{3}{2}-z}(\partial_x h_\varepsilon)^2 + \varepsilon^{1-\frac{z}{2}}\xi \quad \rightsquigarrow \quad z = 3/2$$

So the good scaling is $h_\varepsilon(t,x) = \varepsilon^{1/2} h(\varepsilon^{-3/2}t, \varepsilon^{-1}x) - c_\varepsilon \varepsilon^{-3/2} t$

# KPZ universality conjecture

Conjecture: For all models in the KPZ universality class, the *height function* $h(t,x)$ satisfies (with model-dependent $c_1, c_2, C_\varepsilon$)

$$\varepsilon^{1/2}\left[h(c_1\varepsilon^{-3/2}t, c_2\varepsilon^{-1}x) - C_\varepsilon t\right] \longrightarrow \mathfrak{h}(t,x)$$

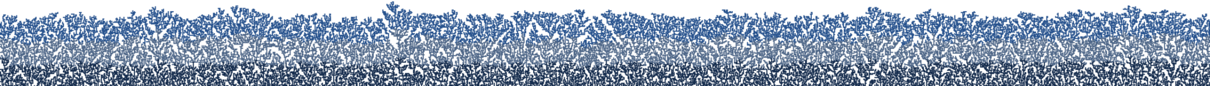
for some universal limit $\mathfrak{h}(t,x)$ (as a process in $t$ and $x$)

The distribution of $\mathfrak{h}(t,x)$ is universal, depending only on the initial data

$\mathfrak{h}(t,x)$ is known as the KPZ fixed point (≠ KPZ eqn.!)

It is invariant in distribution under KPZ 1:2:3 rescaling:

$$\mathfrak{h}(t,x;\mathfrak{h}_0(x)) \overset{(d)}{=} \alpha\,\mathfrak{h}(\alpha^{-3}t, \alpha^{-2}x; \alpha\mathfrak{h}_0(\alpha^{-2}x))$$



# KPZ universality conjecture/definition

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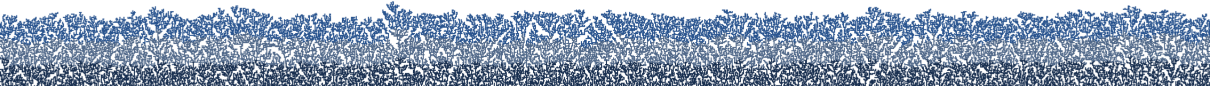
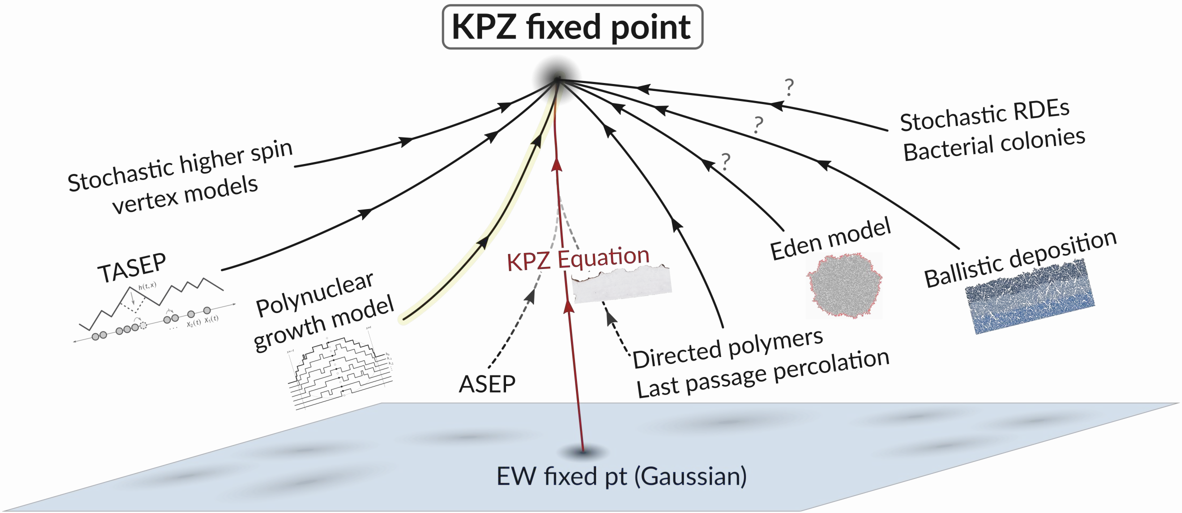
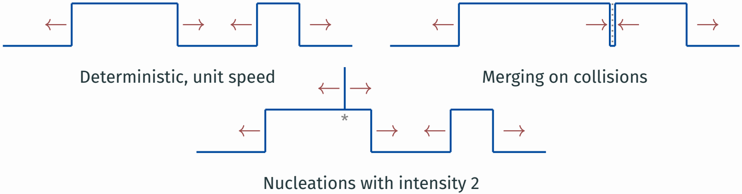


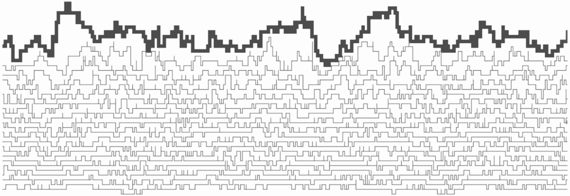

KPZ fixed point
Stochastic higher spin vertex models
TASEP
Polynuclear growth model
KPZ Equation
ASEP
Directed polymers
Last passage percolation
Eden model
Ballistic deposition
Stochastic RDEs
Bacterial colonies
?
?
?
EW fixed pt (Gaussian)

# The polynuclear growth model (PNG)

Height function $h : \mathbb{R} \longrightarrow \mathbb{Z} \cup \{-\infty\}$ evolves as follows:




[Gates-Westcott '95] model of crystal growth



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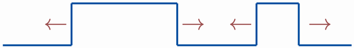
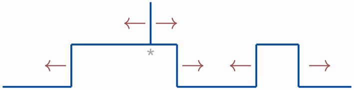


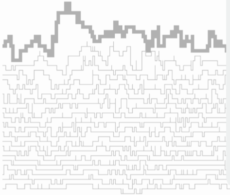
Deterministic, unit speed

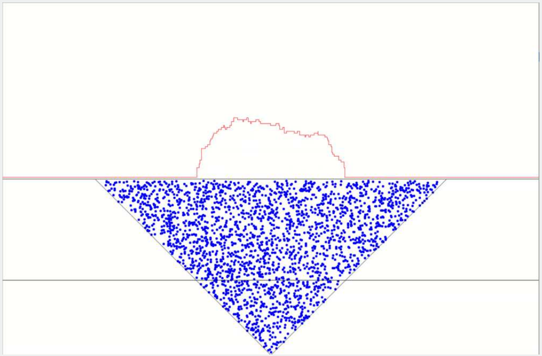
Merging on collisions



Nucleations with intensity 2

[Gates-Westcott '95] model of crystal growth





[Simulation by P. Ferrari]

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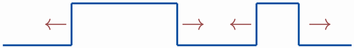

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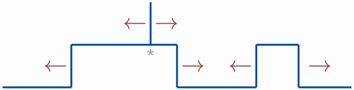
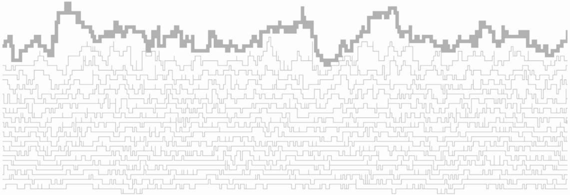


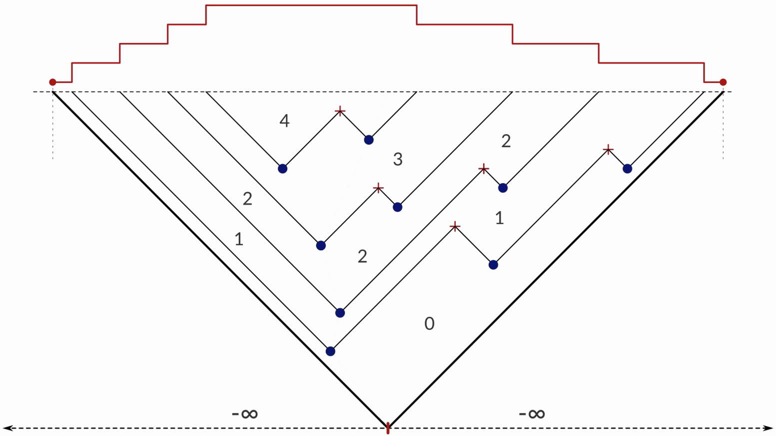
Nucleations with intensity 2

Independent Poisson up and down jumps, rate 1, are invariant (modulo global height)

[Gates-Westcott '95] model of crystal growth



# Droplet/narrow wedge initial condition



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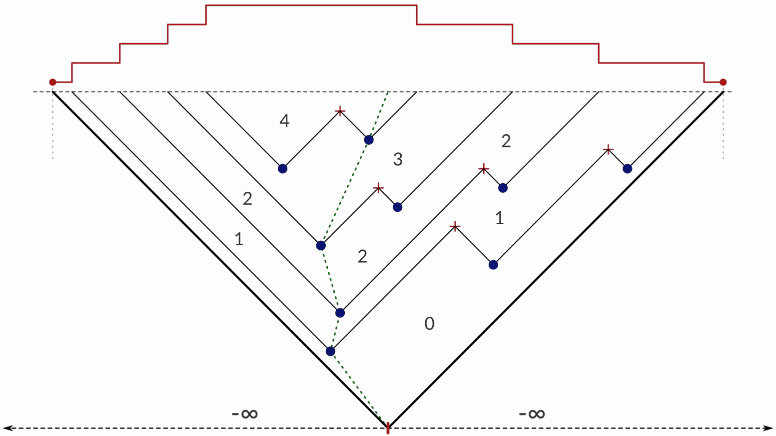


Equivalent to Poissonian last passage percolation

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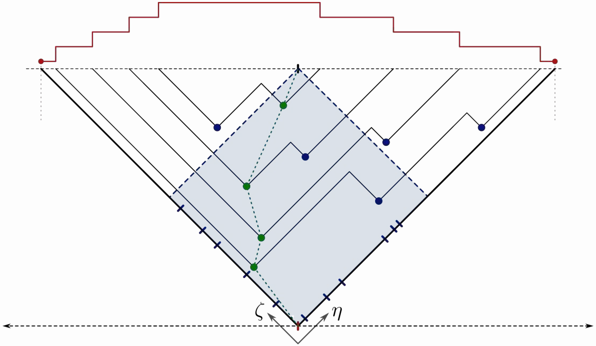


$\eta$-$\zeta$ order defines random permutation of $N \sim \text{Poisson}[2t^2]$

$h(t, 0)$ is the length $\ell_N$ of longest increasing subsequence

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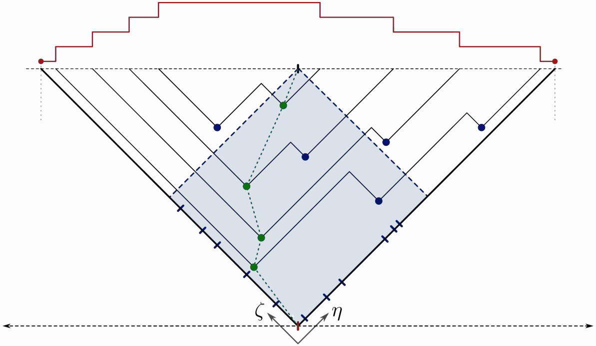


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[Hammersley '72] $\frac{\ell_N}{\sqrt{N}} \longrightarrow c$ (Ulam's problem '61)

Pf. by Kingman's subadditive ergodic theorem

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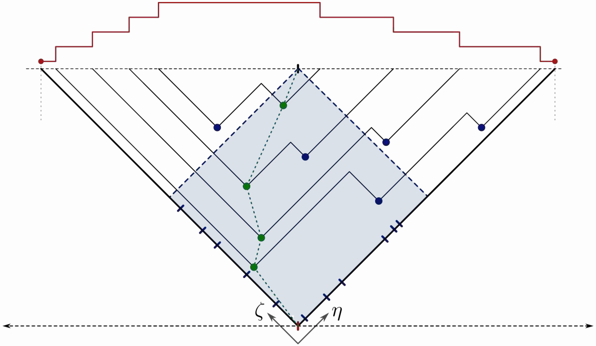


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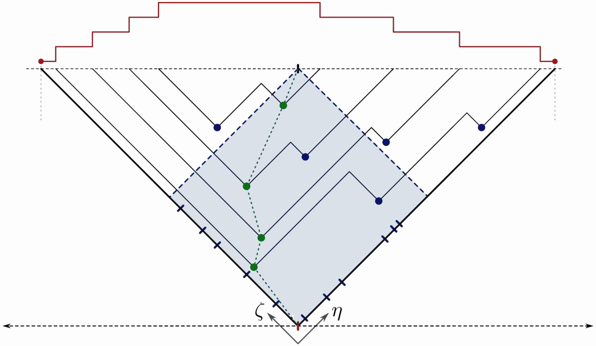
[Logan-Shepp, Vershik-Kerov '77] $c = 2$

Pf. based on RSK (alg. combinat.), later by hydrodynamics of Hammersley process [Aldous-Diaconis '95]

# Droplet/narrow wedge initial condition

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$h(t,0)$ is the length $\ell_N$ of longest increasing subsequence

[Baik-Deift-Johansson '99] $N^{-1/6}(\ell_N - 2\sqrt{N}) \longrightarrow$ Tracy-Widom GUE

Pf. based again on RSK + asymptotics of Toeplitz determinants

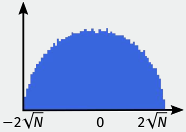
## Gaussian Unitary Ensemble

An $N \times N$ GUE matrix is a random Hermitian matrix

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,N} \\ * & A_{2,2} & \cdots & A_{1,N} \\ & & \ddots & \vdots \\ * & * & & \\ * & * & \cdots & A_{N,N} \end{bmatrix} \quad \text{with} \quad \begin{cases} A_{i,j} = \mathcal{N}(0, \frac{1}{2}) + \mathrm{i}\mathcal{N}(0, \frac{1}{2}) & \text{if } i > j, \\ A_{i,i} = \mathcal{N}(0, 2). \end{cases}$$

The *largest eigenvalue* $\lambda_{\mathrm{GUE}}(N)$ satisfies [Tracy-Widom '94]

$$N^{1/6}\left(\lambda_{\mathrm{GUE}}(N) - 2\sqrt{N}\right) \xrightarrow[N\to\infty]{(\mathrm{d})} \text{Tracy-Widom GUE}$$

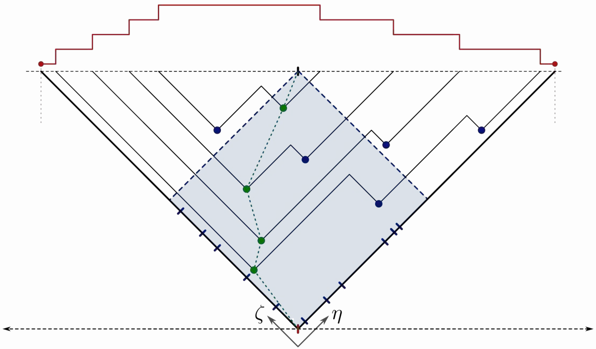



$$F_{\mathrm{GUE}}(r) = \exp\left(-\int_r^\infty (r-s)q(s)^2\,\mathrm{d}s\right), \quad q'' = rq + 2q^3 \text{ with } q(x) \sim \mathrm{Ai}(x) \text{ as } x \to \infty$$

Painlevé II

Pf. based again on RSK + asymptotics of Toeplitz determinants

# Droplet/narrow wedge initial condition

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$h(t,0)$ is the length $\ell_N$ of longest increasing subsequence

[Baik-Deift-Johansson '99] $N^{-1/6}(\ell_N - 2\sqrt{N}) \longrightarrow$ Tracy-Widom GUE

So $t^{-1/3}(h(t,0) - 2t) \longrightarrow$ Tracy-Widom GUE

# PNG with special initial data was solved in the 00's

Narrow wedge/droplet: $t^{-1/3}(h(t,t^{2/3}x)-2t) \longrightarrow \mathcal{A}_2(x)-x^2$

[Prähofer-Spohn '02, Johansson '03]

Airy$_2$ process, TW-GUE marginals

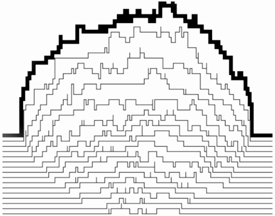
Flat ($h_0 \equiv 0$): $t^{-1/3}(h(t,t^{2/3}x)-2t) \longrightarrow \mathcal{A}_1(x)$

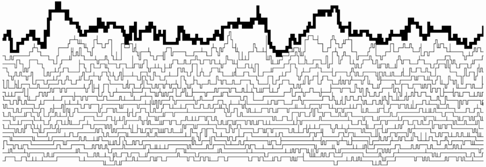
[Borodin-Ferrari-Sasamoto '08]

Airy$_1$ process, TW-GOE marginals

(symmetrized random permutations [Baik-Rains '01])

(connected also to Painlevé II [Tracy-Widom '96])





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Narrow wedge/droplet: $t^{-1/3}(h(t, t^{2/3}x) - 2t) \longrightarrow \mathscr{A}_2(x) - x^2$

[Prähofer-Spohn '02, Johansson '03]

Airy$_2$ process, TW-GUE marginals

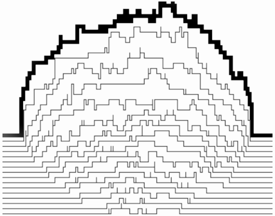
Flat ($h_0 \equiv 0$): $t^{-1/3}(h(t, t^{2/3}x) - 2t) \longrightarrow \mathscr{A}_1(x)$

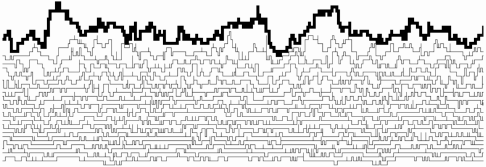
[Borodin-Ferrari-Sasamoto '08]

Airy$_1$ process, TW-GOE marginals

(symmetrized random permutations [Baik-Rains '01])

(connected also to Painlevé II [Tracy-Widom '96])





General initial data?

# The KPZ fixed point

Let $h(t,x)$ be the height function associated to PNG.

**Thm:**

Suppose $\mathfrak{h}_0(t) = \lim_{\varepsilon\to 0} \varepsilon^{1/2} h(0, 2\varepsilon^{-1}x)$ exists suitably. Then the distribnl. limit

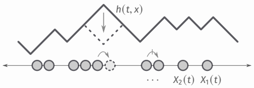
$$\mathfrak{h}(t,x) = \lim_{\varepsilon\to 0} \varepsilon^{1/2}\Big[h(\varepsilon^{-3/2}t, \varepsilon^{-1}x) - 2\varepsilon^{-3/2}t\Big]$$

exists, and it defines a Markov process with explicit trans. probabs. of the form

$$\mathbb{P}_{\mathfrak{h}_0}\big(\mathfrak{h}(t,x_i) \le r_i,\ i=1,\dots n\big) = \det\big(I - K^{\mathfrak{h}_0}_{t,x}\big)_{L^2(\mathbb{R}_{>r_1})\oplus\cdots\oplus L^2(\mathbb{R}_{>r_n})}.$$

$\mathfrak{h}(t,x)$ is the KPZ fixed point, first constr. as limit of TASEP [Matetski-Quastel-R '21]

Limit follows from explicit formulas [Matetski-Quastel-R '17-'22 ]
or from a variational description [Dauvergne-Virág '18]




*Integrable probability*

## Some basic properties

- Invariant under KPZ 1:2:3 rescaling: $\alpha\mathfrak{h}(\alpha^{-3}t, \alpha^{-2}x; \alpha\mathfrak{h}_0(\alpha^{-2}x)) \overset{(d)}{=} \mathfrak{h}(t, x; \mathfrak{h}_0)$
- $\mathfrak{h}(t, x)$ is Hölder $\frac{1}{2}-$ in $x$
- $\mathfrak{h}(t, x)$ is Hölder $\frac{1}{3}-$ in $t$
- $\mathfrak{h}(t, x)$ is locally Brownian in $x$ (even locally abs. cont. w.r.t. Brownian bridges)
- Brownian motion is invariant for $\mathfrak{h}(t, x)$ modulo global height
- Many symmetries
- Recovers Airy processes at fixed times for special initial data

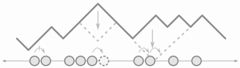
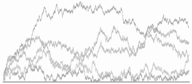
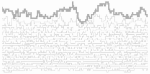
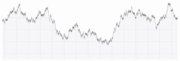
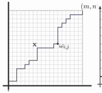
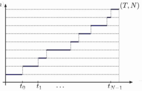
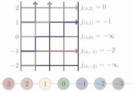
## Important progress on universality during the past few years

KPZ fixed point limit known for:

TASEP, PushASEP, one-sided RBMs, several discr. time variants of TASEP/PushTASEP, polynuclear growth model [Matetski, R, Quastel, Nica, Arai] | Brownian, geometric, exponential and Poisson LPP [Dauvergne, Ortmann, Virag] | Brownian semi-discrete polymer, KPZ equation [Virág] | ASEP, stochastic six vertex model [Aggarwal, Corwin, Hegde] ..........

## Also on the geometric and probabilistic structure of the KPZ fixed point

Regularity, geodesics, construction, couplings, fractal properties ..........

## Coming next

- Explicit solution of PNG
- Sketch of scaling limit
- Integrable PDEs for PNG and the KPZ fixed point and connection to RMT