A RELATIONSHIP BETWEEN GAUGE THEORIES WITH FINITE AND CONTINUOUS GAUGE GROUPS

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3-dimensional topological gauge theories.

DATA: - (compact) topological group G (gauge group)

- $\omega \in H^4(BG, \mathbb{R}/\mathbb{Z})$

Algebraic groups:

$G = \mathcal{G}(\mathbb{C})$ SU(2)

Chern-Simons/Witten-Reshetikhin-Turaev TQFT

$\updownarrow$?

$G = \mathcal{G}(\mathbb{F}_q)$ $SU(2, \mathbb{F}_q) \cong SL(2, \mathbb{F}_q)$

Dijkgraaf-Witten TQFT

Rmk: Partly motivated by Weil conjectures

$$\begin{tikzcd} & X(\mathbb{C}) \arrow[dd, leftrightsquigarrow] \\ \text{Algebraic variety } X(\mathbb{Q}) \arrow[ur] \arrow[dr] & \\ & X(\mathbb{F}_q) \end{tikzcd}$$

In the context of topological gauge theories

$$X(\mathbb{K}) = \mathrm{Hom}(\pi_1(M), G(\mathbb{K})) \text{ for a manifold } M$$

(representation variety)

In this talk we will be exploring relationship for the values of TQFTs

$$Z : \mathrm{Bord}_3 \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

This always stays $\mathbb{C}$ in the talk!

on closed 3-manifolds M only:

$$Z(M) \in \mathbb{C}$$

- $G = SU(2)$: Witten–Reshetikhin–Turaev invariant.

Let $M = S^3(L)$ — Dehn surgery on a framed link $L \subset S^3$ with N components.

Colored Jones polynomial $J_n(L;\zeta) \in \zeta^{\frac{a}{4}}\mathbb{Z}[\zeta,\zeta^{-1}]$

$n \in \mathbb{Z}^N_{\geq 1}$ $\qquad a \in \mathbb{Z}$

$$J_n\left(\begin{matrix}\text{0-framed}\\ \text{unknot}\end{matrix};\zeta\right) = [n]_\zeta := \frac{\zeta^{\frac{n}{2}} - \zeta^{-\frac{n}{2}}}{\zeta^{1/2} - \zeta^{-1/2}}$$

Take ζ to be an r-th primitive root of unity.

$$F(L;\zeta) := \sum_{n \in \{1,2,\ldots,r-1\}^N} J_n(L;\zeta) \prod_{i=1}^{N} [n_i]_\zeta$$

WRT invariant:

$$\tau_{SU(2)}(M;\zeta) := \frac{F(L;\zeta)}{F\left(\begin{smallmatrix}+1 \text{ framed} \\ \text{unknot}\end{smallmatrix}\right)^{b_+} F\left(\begin{smallmatrix}-1 \text{ framed} \\ \text{unknot}\end{smallmatrix}\right)^{b_-}} \in \mathbb{C}$$

$b_\pm := \#$ positive/negative eigenvalues of the linking matrix of L.

Rmk 1: Fits into the general RT construction of 3d TQFTs from a Modular Tensor Category (MTC)

— (semisimplified) category of finite-dimensional rep-s of the quantum group $\overline{U}_{\zeta}(sl_2)$

Rmk 2: For $\zeta = e^{\frac{2\pi i}{r}}$ provides a math definition of the partition function of the $SU(2)$ Chern-Simons gauge theory at level $(r-2) \in \mathbb{Z} = H^4(BSU(2), \mathbb{Z})$

Asymptotic expansion / quantum modularity conjecture:

[Witten '89, Freed-Gompf '91, Andersen '01, Zagier '10, Chen-Yang '15, Garoufalidis-Zagier '21, Wheeler '23, Andersen-Han-Li-Mistegård-Sauzin-Sun '25, Singh-P '25, ...]

$\xi = e^{\frac{2\pi i s}{r}}$, fix s, take $r \to \infty$ along coprimes with s, with fixed $(r \bmod s)$:

$$\tau_{SU(2)}(M;\xi) \simeq \sum_{\gamma \in \pi_0 \mathrm{Hom}(\pi_1(M), SL(2;\mathbb{C}))} e^{2\pi i \frac{r}{s} CS_\gamma} \, r^{\Delta_\gamma} \left(a_0^\gamma + \frac{a_1^\gamma}{r} + \ldots\right)$$

$SU(2)$

For $s=1$ only $SU(2)$ subset appears

$$CS_\gamma := \frac{1}{8\pi^2} \int_M \mathrm{Tr}\left(A_\gamma dA_\gamma + \frac{2}{3} A_\gamma^3\right)$$

$\leftarrow$ Chern-Simons functional of a representative flat connection 1-form $A_\gamma \in \Omega^1(M) \otimes \mathfrak{sl}_2(\mathbb{C})$

- Dijkgraaf – Witten invariant for G - finite group

$$\omega \in H^3_{alg}(G, U(1)) \cong H^3(BG, U(1)) \xrightarrow[\beta]{\cong} H^4(BG, \mathbb{Z})$$

("twist")

β: Bockstein for $1 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 1$

$$Z^{G,\omega}_{DW}(M) = \frac{1}{|G|} \sum_{\alpha \in \mathrm{Hom}(\pi_1(M), G)} \underbrace{(\alpha^* \omega)}_{\in H^3(M, U(1))}[M] \in \mathbb{C}$$

(up to conjugation, corresponds to a homotopy class of a map $M \to BG$)

Rmk: Fits into the general construction of Turaev-Viro TQFT with the input spherical fusion category Vec_G^ω, or into RT construction with the input MTC $D_\omega(G)$ (Twisted Drinfeld double of G)

($\Rightarrow$ Can be computed via triangulation / surgery)

TAKE $G = SU(2;\mathbb{F}_q)$

for a finite field $\mathbb{F}_q$ with $q = p^m$ elements where p is prime.

$$SU(2;\mathbb{F}_q) \subset SL(2;\mathbb{F}_{q^2}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\middle|\; \begin{matrix} A,B,C,D \in \mathbb{F}_q \\ AD-BC=1 \end{matrix} \right\}$$

$$\| $$

$$\left\{ \text{—}\,\text{//}\,\text{—} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$(\ )^*$ is an order 2 automorphism of $\mathbb{F}_{q^2}$: $x^* := x^q$,

i.e. the m-th power of the Frobenius : $x \longmapsto x^p$

$$\begin{array}{c} SL(2;\mathbb{F}_{q^2}) \\ \cup \end{array}$$

It turns out $SU(2;\mathbb{F}_q) \cong SL(2;\mathbb{F}_q)$.

For generic q $\quad H^3(SL(2;\mathbb{F}_q), U(1)) \cong \mathbb{Z}/(q^2-1)\mathbb{Z}$

$$G = SL(2;\mathbb{F}_q) \supset T' \cong \mathbb{Z}/(q+1)\mathbb{Z} \overset{\phi'}{\hookrightarrow} GL(1,\mathbb{C}) \cong \mathbb{C}^*$$

("codiagonal")

$$G \supset T \cong \mathbb{Z}/(q-1)\mathbb{Z} \overset{\phi}{\hookrightarrow} GL(1,\mathbb{C}) \cong \mathbb{C}^*$$

(diagonal)

Take a complex rep. $\rho := \mathrm{Ind}_T^G \phi \oplus \mathrm{Ind}_{T'}^G \phi' : G \to GL(2q^2,\mathbb{C})$

$c_2(\rho) := c_2$(associated bundle over BG) $\in H^4(BG,\mathbb{Z})$

$$\boxed{\omega := \beta^{-1} c_2(\rho) \in H^3(BG, U(1))}$$

$\beta : H^3(BG, U(1)) \xrightarrow{\cong} H^4(BG,\mathbb{Z})$

Bockstein

Conjecture [Nicosanti - P - Quenta '25]

For M with no hyperbolic components in Thurston's geometric decomposition & generic q

$(r \in \mathbb{Z})$

$$Z_{DW}^{SL(2,\mathbb{F}_q),\, \omega^{12r}}(M) \cdot |SL(2,\mathbb{F}_q)| = \sum_{\gamma \in \pi_0(\mathrm{Hom}(\pi_1(M), SL(2,\mathbb{C})))} e^{2\pi i \cdot 12 r \cdot CS_\gamma}\, T_\gamma(q)$$

for some

$$T_\gamma(q) \in \mathbb{Z}[q], \quad T_\gamma(q) = \Theta(q^{2\Delta_\gamma}) \quad q \gg 1$$

CS_γ & Δ_γ are the same as in WRT

The conjecture is verified by explicit calculation of $Z_{DW}^{SL(2,\mathbb{F}_q),\,\omega^{12}}(M)$ for various M, using the general formula of [Nosaka '23] for

$$\left(\beta^{-1}c_2(\rho)\right)^{12} \in H^3(BG, U(1)) \quad \text{for}$$

a finite group G and a rep. $\rho : G \to GL(N,\mathbb{C})$

<u>Example</u> $M = S^3/2I$ Poincaré homology sphere

$$Z_{DW}^{SL(2;\mathbb{F}_q),\,\omega^{12}} \cdot \underbrace{|SL(2,\mathbb{F}_q)|}_{q(q^2-1)} = e^{2\pi i \cdot 0} + e^{2\pi i \cdot 12 \cdot \frac{1}{120}}\, q(q^2-1)$$

$$+ e^{2\pi i \cdot 12 \cdot \frac{49}{120}}\, q(q^2-1)$$

CS values of the 3 $SU(2)$ flat connections on M

<u>Rmk</u> $U(1;\mathbb{F}_q) \cong \mathbb{Z}/(q+1)\mathbb{Z}$

vs

$U(1)$

works in a similar, but much simpler way (for all manifolds, non-conjecturally) (such stacks also appeared in TQFT context in [Gonzalez-Prieto – Hablicsek – Vogel])

Rmk Consider the character stack $\underline{X}(\mathbb{K}) := \frac{\mathrm{Hom}(\pi_1(M), SL(2,\mathbb{K}))}{SL(2,\mathbb{K})}$

For untwisted DW:

$$Z_{DW}^{SL(2,\mathbb{F}_q),\mathbb{1}}(M) = \#\underline{X}(\mathbb{F}_q) = \sum_i (-1)^i \operatorname{Tr}(Fr)^m \Big|_{H^i_c(X(\overline{\mathbb{F}}_p);\mathbb{Q}_\ell)}$$

$\frac{|\mathrm{Hom}(\pi_1(M), SL(2,\mathbb{F}_q))|}{|SL(2,\mathbb{F}_q)|}$ ⫽ $q = p^m$

Behrend – Grothendieck – Lefschetz

$$Z_{DW}^{SL(2,\mathbb{F}_q),\omega^m}(M) \overset{?}{=} \sum_i (-1)^i \operatorname{Tr}(Fr)^m \Big|_{H^i_c(X(\overline{\mathbb{F}}_p);\mathcal{L})}$$

Categorification of DW?

for some rank 1 sheaf $\mathcal{L}$ over $X(\overline{\mathbb{F}}_p)$ s.t. $Fr|_{\mathcal{L}_\alpha} = (\alpha^*\omega)[M]$, $\alpha: M \to BG$

Q: Does such $\mathcal{L}$ actually exist?

Q: What is the role of the corresponding zeta function in TQFT context?

$$\zeta_{DW}^{SL(2),\omega_+}(M;s) := \prod_{p \in \text{primes}} \exp \sum_{m=1}^{\infty} \frac{Z_{DW}^{SL(2,\mathbb{F}_{p^m}),\omega_{p^m}}(M)}{m} p^{-sm}$$

Thank you!