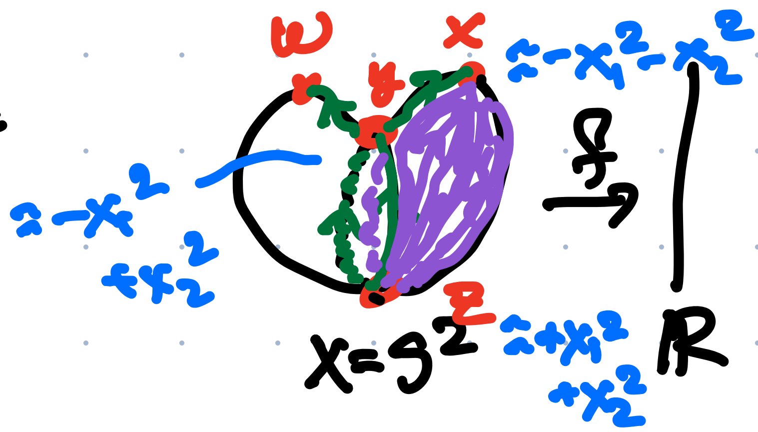
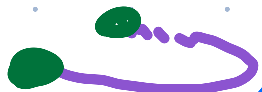
Broken

(jt w/ J. Lurie)

Fix X cmpct C^∞ mfld
$f: X \to \mathbb{R}$ C^∞
h Riemann metric





$$\boxed{\partial x = \sum_{|y| = |x| - 1} \#\{\nabla \text{ trajs } y \to x\}\, y}$$

$$H_* \cong \mathbb{Z}/2\mathbb{Z}_2 \oplus \mathbb{Z}/2\mathbb{Z}_0 \cong H_*(S^2; \mathbb{Z}/2\mathbb{Z})$$

Thm $H_*(\text{this cplx}) \cong H_*(X; \mathbb{Z}/2\mathbb{Z})$

Q: Why $\partial^2 = 0$?

Sketch: $\partial^2 x = \partial(\partial x)$
$$= \partial\Big(\sum_{|y| = |x| - 1} \#\{y \to x\}\, y\Big)$$

Claim:

Blue # = # pts in bdy of cmpct 1mfld (w/ ∂).

$$= \sum_y \#\{y\to x\} \sum_{|z|=|y|-1} \#\{z\to y\}\, z$$

$$= \sum_z \Big(\underbrace{\sum_y \{y\to x\}\cdot\{z\to y\}}_{=0\ \forall z?} \Big)\, z$$

$$= 0. \quad //$$

Lesson: It pays to consider "broken" trajectories.

Today: Apply lesson when $X = pt$, and study the moduli of ∇ trajs on X.

$$\underbrace{M}_{\text{moduli of } \nabla \text{ trajs}} := \{\gamma : \mathbb{R}\to X \mid \dot\gamma(t) = \nabla f(\gamma(t))\}/\mathbb{R}$$

$$\underset{X=pt}{=} pt/\mathbb{R} \quad \text{as a } \underline{\text{stack}}.$$

$$= B\mathbb{R}$$

moduli of once-broken trajs $:= \left\{\gamma : \mathbb{R}_{t_1}\sqcup\mathbb{R}_{t_2}\to X \;\middle|\; \begin{array}{l}\dot\gamma = \nabla f\circ\gamma \\ \lim\limits_{t_1\to+\infty}\gamma(t_1) = \lim\limits_{t_2\to-\infty}\gamma(t_2)\end{array}\right\}/\mathbb{R}^2$

$$= B\mathbb{R}^2 \quad \text{as a stack.}$$

Moduli of k-broken trajs $\simeq B\mathbb{R}^{k+1}$ $\left(= pt/\mathbb{R}^{k+1}\right)$

"<u>Defn</u>" Broken $\simeq \bigcup_{k\geq 0} B\mathbb{R}^{k+1}$

<u>Why Care?</u>

(1) Note $\exists$ Broken $\times$ Broken $\xrightarrow{m}$ Broken

Call a sheaf F <u>factorizable</u> when F is equipped w/ "compatibility data w/ m"

$$m^*F \simeq F \boxtimes F$$ (up to 1st order)

<u>Thm</u> (Lurie-T.) Fix $\mathcal{C}^{\otimes}$ cmptly gen. monoidal ∞-cat.

(Sets, $\times$)
(Spaces, $\times$)
(Chain$_R$, $\otimes_R$)
(Spectra, $\wedge$)
(Vect$_k$, $\otimes_k$)

Factorizable

$$Shv^{\otimes}(\text{Broken}; \mathcal{C}) \simeq \text{AssAlg}^{nu}(\mathcal{C}^{\otimes})$$

non-unital.

AssAlg = E_1 Alg

(2) Using Broken, we can make geometric (field-theory-inspired) invariants linear over (or valued in) many examples of $\mathcal{C}^{\otimes}$.

Floer cochains (for Lagrangians)

$\mathcal{C}$ = Spectra.

<u>Ex</u>: It's expected that many Floer-theoretic invariants (currently defined over $\mathbb{Z}$, $\mathbb{Z}/2$) have lifts to spectra (defined over $\mathbb{S}$, MU, ...) Broken gives a roadmap to create such lifts (Ex: Lagrangian Floer cohomology
Fukaya categories
Instanton Floer homology
SW-style invariants (Manolescu)
....)

Floer
Cohen-Jones-Segal
Abouzaid-Blumberg
T. Large
T. Kragh

(3) $\exists$ many generalizations/analogues of Broken for other dimensions + other algebraic structures.

Broken } ∂-tgs

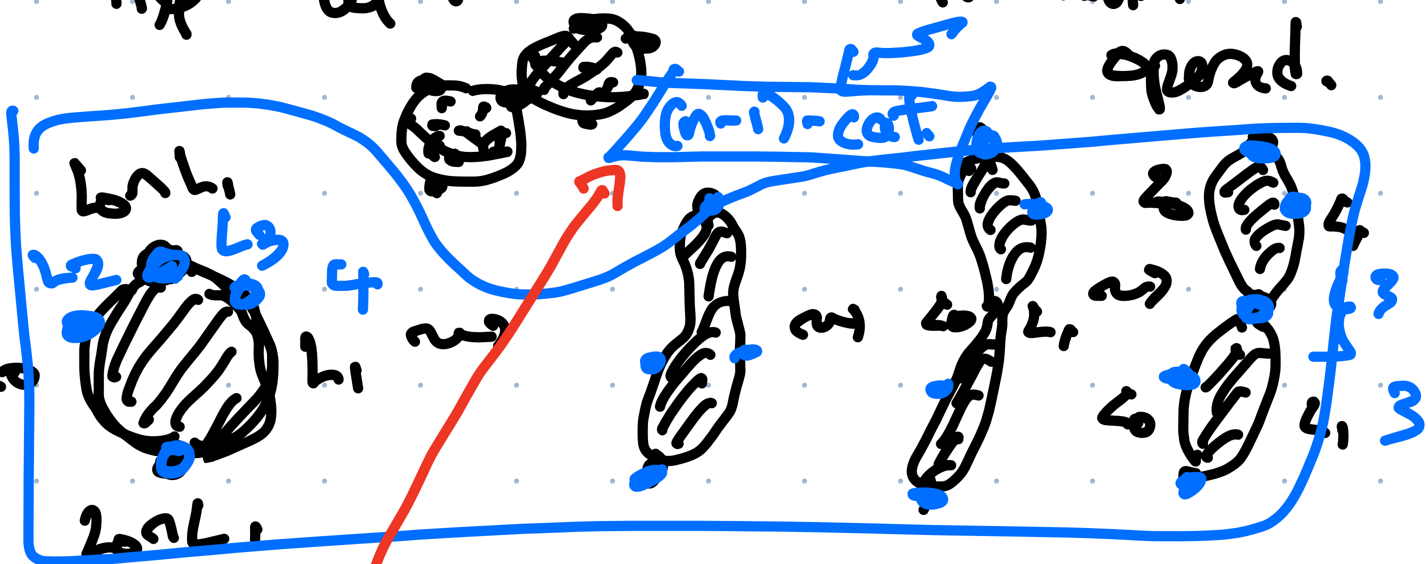
Ass Algs

def thy $\rightsquigarrow$ something linear (in $\mathcal{C}$)

$\rightsquigarrow HH^{*}$ (lin thy)

Broken Disks w/ marked pts | holom disks in pt = X | Planar Operads ⇝ A_∞ cat'ye

Broken^n Hyp | hyperbolic balls of dim $n \geq 2$, w/ marked pts | Operads over the $\mathbb{E}_{n-1}$ operad.



(What you assign to a pt in some n-dim field thy.)

Q: Are these examples of moduli (of maps out of hyperbolic n-balls) that compactify in "broken" ways?

<u>One candidate</u> (n=3): Fix $X^{7\text{dim}}$ G_2 mfld

Guess: $\exists$ a 2-cat. where obj. are 4dim coassociative (Lagrangians) submflds of X $\boxed{\text{3-dim}}$

And instantons are associative submfds, holom disks,
w/ ∂ on coassociatives.

Ex $X = CY3 \times S^1$

$\mathrm{Fuk}(CY3) \underset{?!?}{\simeq} HH_*(2\text{-cat})$

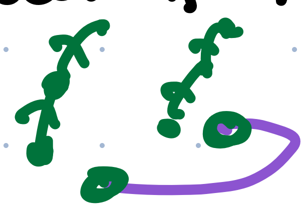
Ex $X = \Lambda^2_{ASD}(\Sigma_{\text{symp } 4\text{-mfld}})$

$\mathrm{Fuk}(\Sigma) \underset{\text{full}}{\subset} 2\text{-cat}(X)$.

<u>What is the deformation theory?</u>

Fix $M \xrightarrow{p} \mathrm{Broken}$ in factorizable way.

(Ex: M = usual moduli of ∇ trajs)



$\underline{\Bbbk}$ on M is fact.

$p_! \Bbbk \in \mathrm{Shv}^{\otimes}(\mathrm{Broken})$
$\simeq \mathrm{AssAlg}$.

Fix an orientation on M compatible w/ fact.

Str, get an M-C element in $p^! \Bbbk$.

Idea: $p^! \Bbbk \circlearrowleft \oplus \mathbb{Z}/2\mathbb{Z}$

and M-C element deforms (w/ 0 diff.) to something w/ a differential

$dx = x^2$, $|x| = 1$.

$\Updownarrow$

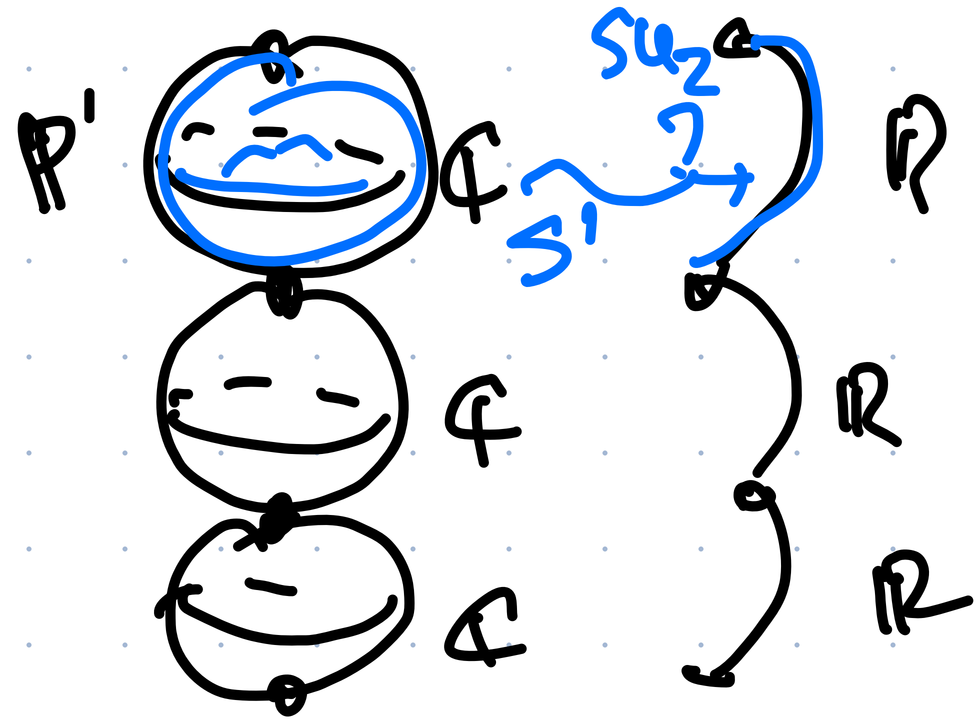
$x^2 \sim 0$

$\mathrm{Sh}^{\mathrm{sing}}_{\mathrm{fr}}(\mathbb{P}_{\mathrm{red}}\,\mathbb{P}^1) \simeq \mathrm{AssAlg}^{\mathrm{nu}}$ S^1

$\mathbb{R} \times G$

$\mathbb{C}P^1$

$\mathbb{H}P^1$



k

$\boxed{\text{Bnden} := B(R \cup B(R^2 \cup B(R^3 \cup \cdots}$

F

$V \leftarrow V \otimes V \quad V \otimes V \otimes V$

$V \in \text{AssAlg}$

Koszul Duality

Rep(R)