Knotted bi-disk embeddings

based on joint work in progress with:

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Lisbon Geometry Seminar — 18.11.2025

Outline:

I) Motivation: topology of symplectic embedding spaces

II) Results

III) Toric and almost toric geometry

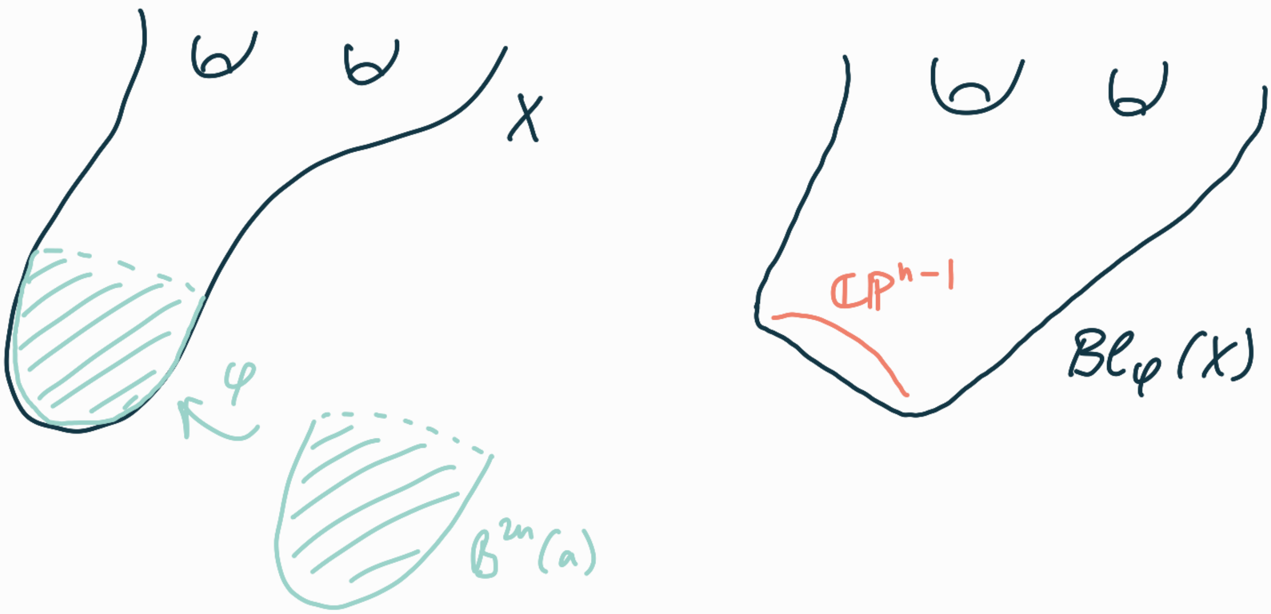
IV) Proofs

I. Motivation

Symplectic blow-up :

1) Fix symplectic ball-embedding
$$\varphi : B^{2n}(a) \hookrightarrow X, \qquad \varphi^*\omega = \omega_0 .$$
2) Consider $X \setminus \varphi(\mathring{B}^{2n}(a))$ and collapse the "characteristic foliation" of its boundary.

In the chart φ, this is the Hopf action
$$\partial B^{2n}(a) = S^{2n-1}/S^1$$



Question : "How much" does the blow-up space depend on the symplectic embedding φ ?

Note: The resulting exceptional divisor $\mathbb{CP}^{n-1} \subset Bl_\varphi(X)$ has symplectic volume $\sim a^{n-1}$

$\rightsquigarrow Bl_\varphi(X)$ will depend on a!

McDuff '96: If $\dim X = 4$ and not of simple SW-type (e.g. rational surfaces) then
$$Bl_\varphi(X) \cong Bl_{\varphi'}(X)$$
if $a = a'$.

Equivalently: $\exists \varphi_t : B^4(a) \hookrightarrow X$ s.th. $\varphi_0 = \varphi$, $\varphi_1 = \varphi'$.

$\rightsquigarrow$ Isotopy deforming ω into ω'.

Def: (Space of symplectic ball embeddings)

$$\boxed{\mathcal{E}mb(B^{2n}(a), X) := \{ \varphi : B^{2n}(a) \hookrightarrow X \mid \varphi^*\omega = \omega_0 \}}$$

For X of dim 4 and rational, connected:
$$\# \mathcal{E}mb = 1$$
In higher dimensions, nothing is known.

(blow-ups may be highly not unique!)

Ball embeddings are of independent interest, e.g.

Gromov width $\rightsquigarrow$ capacity.

II. Results

What about other domains ?

Def: Let $C^4(a) = \{(z_1, z_2) \in \mathbb{C}^2 \mid \pi|z_1|^2, \pi|z_2|^2 \leq a\}$ be the cube or bi-disk or polydisk.

Thm A: (B. – Mikhalkin – Schlenk)

Let $X = \mathbb{CP}^2(1), B^4(1)$, then:

$$\# \mathcal{E}mb(C^4(a), X) \geq n+1 \quad \text{for} \quad a \in [1/3, a_n)$$

where $a_n := \dfrac{g_{n-1} g_n}{g_{n-1}^2 + g_n^2}$, $\underbrace{g_n = (1,1,2,5,13,34,\dots)}_{\text{Fibonacci with odd index.}}$

Remarks:

1) $\# \mathcal{E}mb(C^4(1/3), X) = \infty$!

2) It was previously known that:

$$\# \mathcal{E}mb(C^4(1/3), B^4(1)) \geq 2$$

independently: Gutt–Usher & Dimitroglou-Rizell.

3) We have similar results for $X = S^2 \times S^2, C^4$.

4) Can upgrade the result to a weaker equivalence relation: $\varphi_0 \sim_{im} \varphi_1 \iff \exists \phi \in Symp$ s.th. $\phi(im\, \varphi_0) = im\, \varphi_1$

Quantitative questions:

Recall: Gromov width:

$$w_B(X) := \sup \{ a>0 \mid \exists\, B^4(a) \hookrightarrow X \}$$

Can define Cube width:

$$w_C(X) := \sup \{ a>0 \mid \exists\, C^4(a) \hookrightarrow X \}$$

Known: $w_C(B^4(1)) = 1/2 \rightsquigarrow w_C(\mathbb{CP}^2(1)) = 1/2$

Ekeland-Hofer, neck-stretching

Note: If Emb is connected

$\rightsquigarrow$ get unique width $(B^4 \hookrightarrow \mathbb{CP}^2, B^4)$

If Emb is not connected

$\rightsquigarrow$ get width for every component.

Let $\varphi_0, \varphi_1, \varphi_2, \ldots \in \mathrm{Emb}(C^4(1/3), \mathbb{CP}^2)$ generating the components in Thm A.

Def: $w(\varphi_n) := \sup \left\{ a>0 \;\middle|\; \begin{array}{l} \exists\, \varphi : C^4(a) \hookrightarrow \mathbb{CP}^2 \\ \varphi|_{C^4(1/3)} \sim \varphi_n \end{array} \right\}$

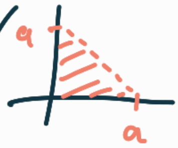
Thm B: (B.-M.-S.)

$$w(\varphi_n) = \frac{g_{n-1}\, g_n}{g_{n-1}^2 + g_n^2}.$$

III. Toric and almost toric geometry

$\mu_0 : \mathbb{C}^2 \to \mathbb{R}^2_{\geq 0}$, $\mu_0(z_1, z_2) := (\pi |z_1|^2, \pi |z_2|^2)$

standard moment map.

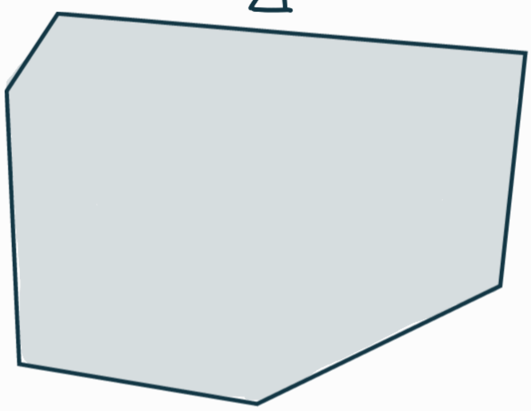
$B^4(a) := \mu_0^{-1}\Big($  $\Big)$, $C^4(a) := \mu_0^{-1}\Big($  $\Big)$

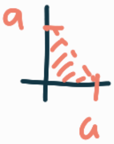
"toric domains"

Delzant '88 : moment map images downstairs determine symplectic geometry upstairs.

Let $\mu : X^4 \to \Delta$ toric

Δ



 $\subset \Delta \rightsquigarrow B^4(a) \hookrightarrow X$

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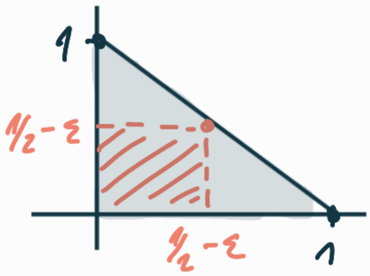
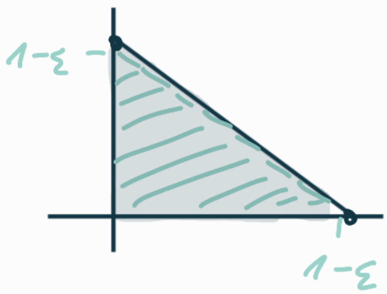
Examples: $\mathbb{CP}^2(1)$:

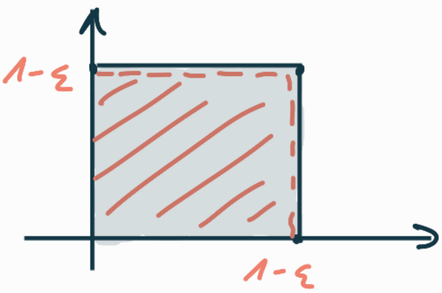
$C^4(1/2 - \varepsilon)$ $B^4(1-\varepsilon)$

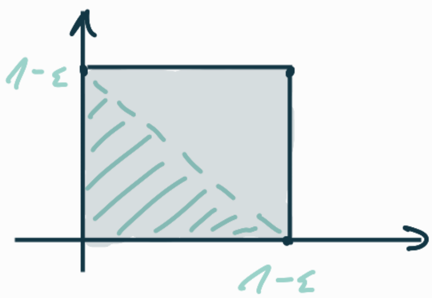




$S^2 \times S^2$

$C^4(1-\varepsilon)$ $B^4(1-\varepsilon)$



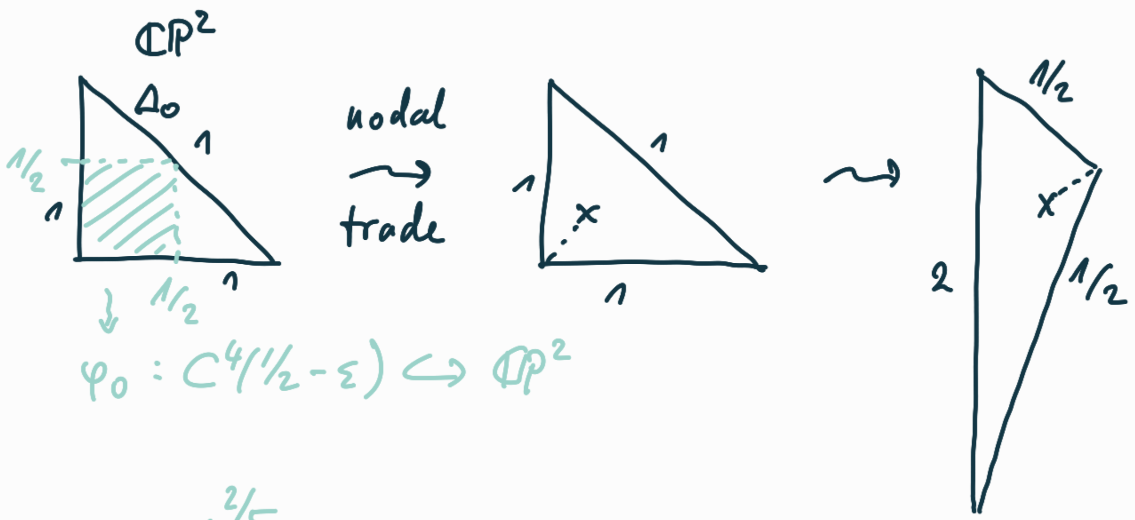


"Experimental observation":

often, these diagrams are sharp, i.e. no larger embeddings exist.

(This is somewhat surprising...)

<u>Idea</u>: Produce new embeddings by almost toric mutations on Δ.



$\varphi_0 : C^4(1/2 - \varepsilon) \hookrightarrow \mathbb{C}P^2$

$\rightsquigarrow \varphi_1 : C^4(2/5 - \varepsilon) \hookrightarrow \mathbb{C}P^2$

$\varphi_2 : C^4(10/29 - \varepsilon) \hookrightarrow \mathbb{C}P^2$

$\overset{\text{mutate}}{\rightsquigarrow}$ ...

<u>Symington (& Zung)</u>: Almost toric base digrams (downstairs) still determine the symplectic manifold (upstairs).

Mutate in alternation 2 vertices (keeping one Delzant) produces $\varphi_n : C^4\left(\frac{g_{n-1} g_n}{g_{n-1}^2 + g_n^2} - \varepsilon\right) \hookrightarrow \mathbb{C}P^2$.

IV. Proof

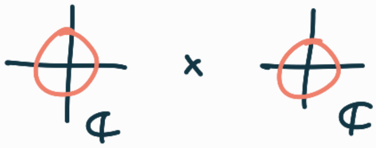
$\varphi_0, \varphi_1, \varphi_2, \ldots \in \mathrm{Emb}(C^4(a), \mathbb{CP}^2)$
constructed using ATFs.

How to distinguish them $\rightsquigarrow$ proof of Thm A:

<u>Def</u>: Let $\varphi : C^4(a) \hookrightarrow \mathbb{CP}^2$ for $a \geq 1/3$,

$$L_\varphi = \varphi(T(1/3, 1/3))$$

called torus associated to φ.

where $T(1/3, 1/3) =$  Lagrangian product torus.

Observations:

1) L_φ is monotone: $\frac{1}{6} m_{L_\varphi} = [\omega]_{L_\varphi}$

Maslov area.

2) Let $\varphi_0, \varphi_1 : C^4(a) \hookrightarrow \mathbb{CP}^2$, $a > 1/3$

$$\varphi_0 \sim \varphi_1 \implies L_{\varphi_0} \hat{=} L_{\varphi_1}$$

(i.e. $\exists \phi \in \mathrm{Symp}\,\mathbb{CP}^2$ s.th. $\phi(L_{\varphi_0}) = L_{\varphi_1}$)

but: by construction $L_{\varphi_n} =$ "Vianna tori".

<u>Vianna '16</u>: $L_{\varphi_n} \hat{=} L_{\varphi_{n'}} \implies n = n'$.

$\implies$ Thm A for $X = \mathbb{CP}^2$.

<u>proof of Thm B</u>: $w(\varphi_n) \geq \frac{g_{n-1}\, g_n}{g_{n-1}^2 + g_n^2}$

follows from ATFs.

Why can't we do any better?

Return to Vianna tori L_{φ_n}:

<u>Def</u>: Let $L \subset X$ monotone Lagrangian (torus).
Its disk-count polytope is:

$$\Delta(L) := \left\{ \alpha \in H^1(L;\mathbb{R}) \;\middle|\; \begin{array}{l} \langle \alpha, \partial\beta \rangle \leq \int_\beta \omega \\ \#_\beta(L) \neq 0 \end{array} \right\}$$

mod 2 count of J-holomorphic Maslov two disks with bdry on L and one marked bdry pt. in the class $\beta \in H_2(X,L)$.

This is a symplectic invariant, i.e.

$$L \underset{\phi}{\cong} L' \implies \Delta(L) = \phi^* \Delta(L').$$

(essentially the dual of the Newton polytope of the disk-counting potential of L)

<u>Thm</u>: (Vianna '16, Galkin-Mikhalkin '22)

$$\Delta(L_{\varphi_n}) = \Delta_n \qquad (\text{under } \mathbb{R}^2 \cong H^1(T^2;\mathbb{R}))$$

$\implies$ distinguishes L_{φ_n}.

Thm: (Shelukhin – Tonkonog – Vianna '24)

Let $L \subset X$ monotone Lag. torus and $\{L_t\}_{t\in[0,1]}$ a Lagrangian star isotopy of L, i.e.

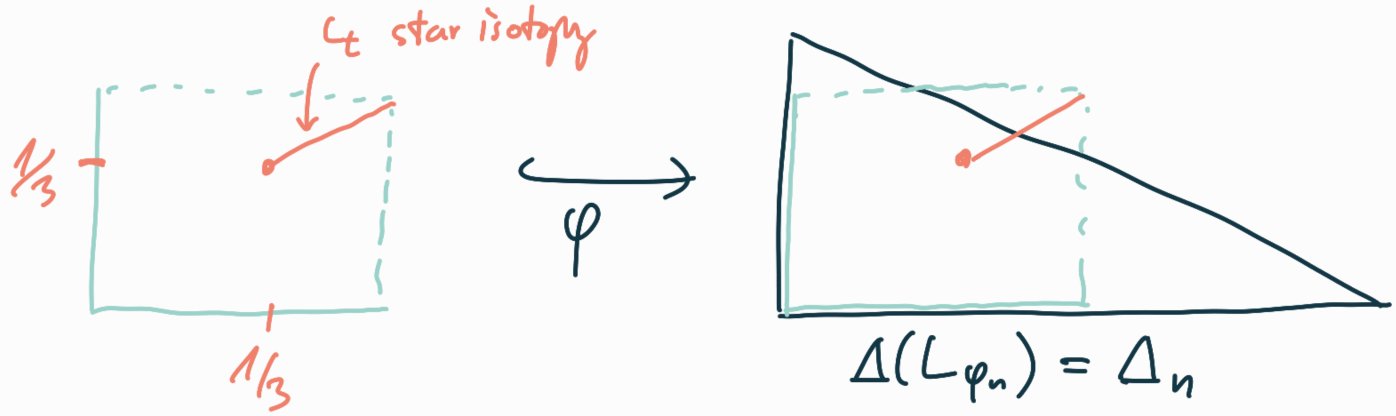
- L_t Lagrangian w/ $L_0 = L$,
- $\mathrm{Flux}\{L_t\}_{t\le t_0} = t_0\,\alpha$ for some $\alpha \in H^1(L;\mathbb{R})$

Then $\mathrm{Flux}\{L_t\}_{t\in[0,1]} \in \Delta(L)$.

Suppose now $\exists\, \varphi : C^4(a) \hookrightarrow \mathbb{CP}^2$

with
- $\varphi|_{C^4(1/3)} = \varphi_n$
- $a > \dfrac{g_{n-1}\,g_n}{g_{n-1}^2 + g_n^2}$

$\Rightarrow \quad L_\varphi \overset{\text{Ham}}{\cong} L_{\varphi_n}$



Contradicts S.-T.-V.

$\Longrightarrow$ Thm. B for $\mathbb{CP}^2$ & B^4.

Thank You!