Fully augmented links, shadow links, and the T-V volume conjecture

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Outline

Knots and links

Invariants of Knots and Links

A **knot** is an embedding of a simple closed curve in a 3-manifold M, $S^1 \hookrightarrow M$. A **link** is an embedding of a disjoint union of a finite number of circles in M.

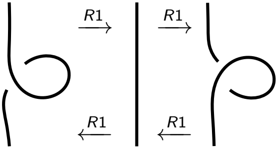
Ambient isotopy

Continuous deformation of the knot K in M in such a way that the knot never self-intersects.

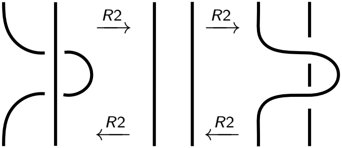
Invariant

$P(K)$ is an invariant of knots if $P(K)$ is defined for all knots and unchanged after applying ambient isotopy to K.
if $K \sim K'$, then $P(K) = P(K')$.

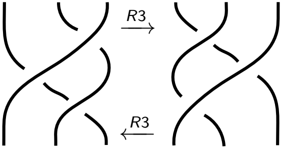
Knots and links



(a) Reidemeister 1 move.



(b) Reidemeister 2 move.



(c) Reidemeister 3 move.

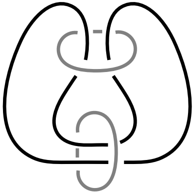
Figure: Reidemeister moves.

Fully Augmented links

Definition

A **flat fully augmented link** in S^3 is a link with a diagram consisting of two types of components:

1. Closed curves embedded in the plane of projection called **Knot strands**.
2. Simple unknots meeting the plane of projection orthogonally, bounding a disc that is punctured by the knot strands exactly twice called **crossing circles**.

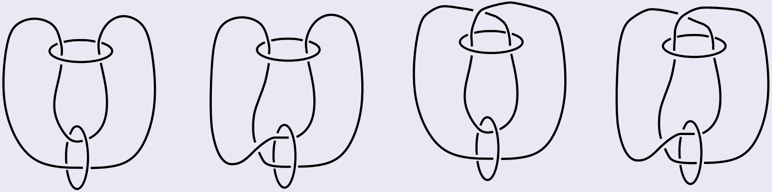


Fully Augmented links

Definition

A **fully augmented link** is obtained from a flat fully augmented link by inserting one or zero crossings into the knot strands in a neighborhood of a crossing circle.

Examples



Fully Augmented links

Fully augmented links

A **fully augmented link** can be obtained from a knot by inserting a crossing circle around all twist regions and removing an even number of crossings in the twist region until there are one or no crossings.

Twist region

Fully Augmented links

Fully augmented links

A **fully augmented link** can be obtained from a knot by inserting a crossing circle around all twist regions and removing an even number of crossings in the twist region until there are one or no crossings.

Twist region

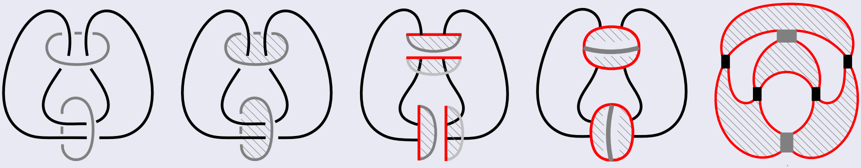
Hyperbolic fully augmented link

A fully augmented link with conditions:

1. Contains at least two crossing circles
2. Connected
3. Prime
4. None of its crossing circles are parallel

Fully Augmented links

Polyhedral decomposition by I. Agol and D. Thurston



Lemma(Purcell)

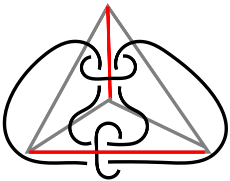
The polyhedral decomposition of the complement of a hyperbolic fully augmented link corresponds to a circle packing on S^2 whose nerve is a triangulation of S^2. The nerve satisfies the following two properties.

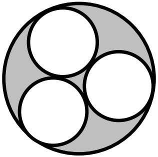
- Each edge of the nerve has distinct endpoints.
- No two vertices are joined by more than one edge.

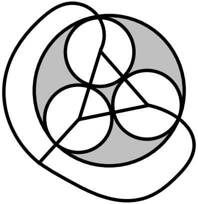
Fully augmented links

Lemma (Purcell)

Let γ be a triangulation of S^2 such that no two vertices are joined by more than one edge and each edge has distinct ends. Choose a collection of edges, called a **dimer**, such that each triangle of γ meets exactly one element in the dimer. Then there is a hyperbolic fully augmented link associated to this graph; it has γ as its nerve.







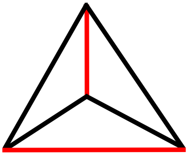
Fully augmented links

Octahedral fully augmented links

A fully augmented link with polyhedral decomposition obtained by gluing regular ideal octahedra is called octahedral.

Proposition (Purcell)

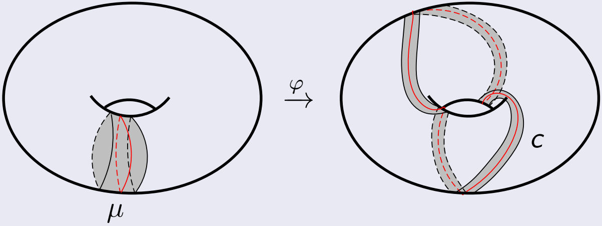
N is a nerve of a circle packing associated to an octahedral fully augmented link if and only if N is obtained by central subdivision of the complete graph on four vertices.



Links in orientable closed 3-manifolds

Dehn Surgery

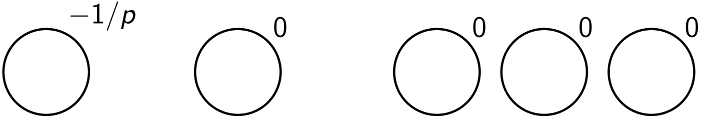
$$M_K = (S^3 - U(K)) \cup_\varphi (D^2 \times S^1)$$



Theorem (Lickorish-Wallace Theorem)

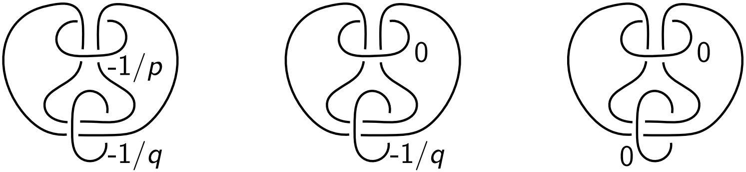
Every closed orientable 3-manifold can be obtained from S^3 *by performing integral surgery on a framed link.*

Links in orientable closed 3-manifolds



(a) S^3 (b) $S^2 \times S^1$ (c) $\#_3(S^2 \times S^1)$

Figure: Examples of 3-Manifolds



(a) $K \hookrightarrow S^3$ (b) $K \hookrightarrow S^2 \times S^1$ (c) $K \hookrightarrow \#_2(S^2 \times S^1)$

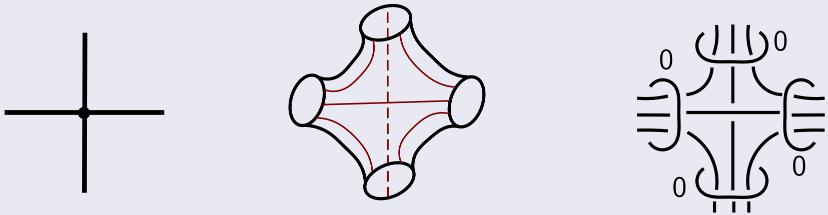
Figure: Examples of knots in 3-Manifolds

Shadow links

Fundamental shadow links

A **fundamental shadow link** is a link in the connected sums of copies of $S^1 \times S^2$ that are built out of **building blocks** which are a collection of 3-balls with four discs on its boundary and six arcs connecting the discs.

Building block

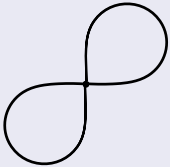


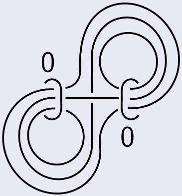
Shadow links

Fundamental shadow links

The building blocks are glued along the discs in such a way that the end points of each arc is glued to the endpoint of another arc yielding a handlebody (possibly non-orientable), with a link in its boundary from the arcs. The manifold is obtained from this handlebody by taking the orientable double.

Example





Octahedral fully augmented links and shadow links

Theorem (Wong, Yang 2020, I, Purcell, McQuire 2024)

Let L be an octahedral fully augmented link with c crossing circles. Then $S^3 - L$ is isometric to $\#^c S^1 \times S^2 - \tilde{L}$ for some fundamental shadow link $\tilde{L}$.

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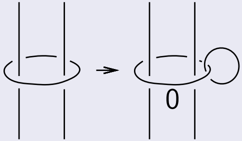
Wong & Yang move: from links in S^3 to links in $\#_n(S^2 \times S^1)$



Octahedral fully augmented links and shadow links

Lemma

Any fully augmented link of the Borromean family corresponds to a fundamental shadow link with associated graph consisting of a single vertex and two edges that are loops.

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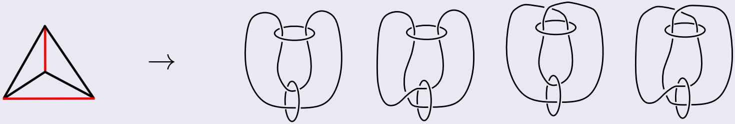
Borromean family



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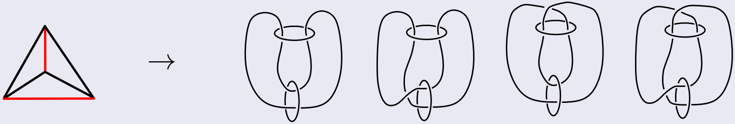
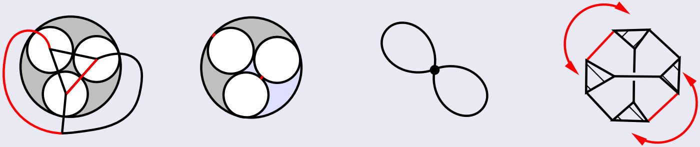
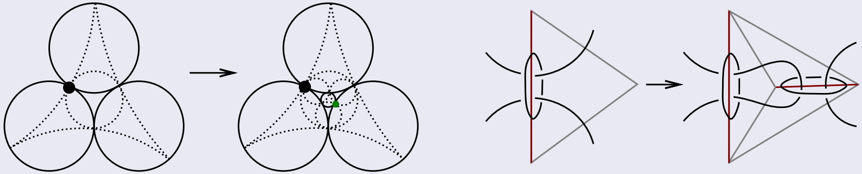


Fundamental shadow links

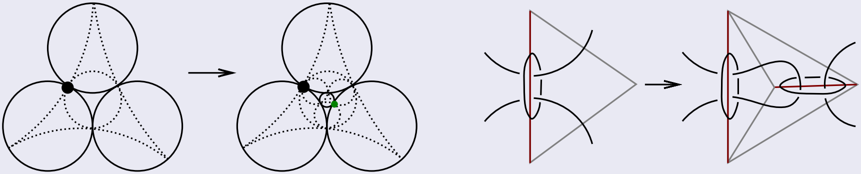


Octahedral fully augmented links and shadow links

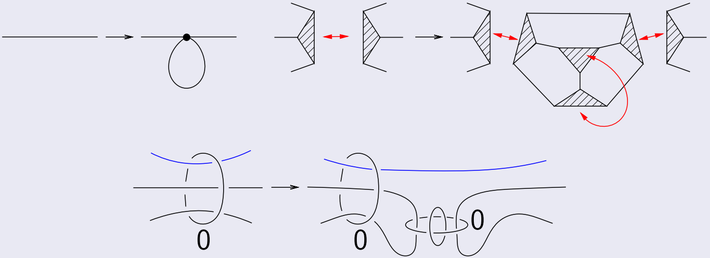
Octahedral fully augmented links from central subdivision



Octahedral fully augmented links from central subdivision



Shadow links from graph move

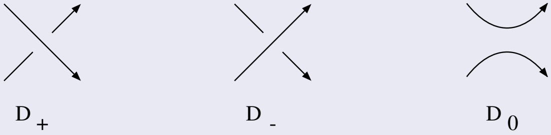


Invariants of knots and links

The Jones polynomial

The Jones polynomial is an invariant of oriented links defined by the initial condition and skein relation.

- $V_{\bigcirc}(t) = 1$
- $t^{-1}V_{D_+}(t) - tV_{D_-}(t) = (t^{1/2} - t^{-1/2})V_{D_0}(t)$.



The Kauffman bracket

Framed links

A **framed link** is an embedding of a disjoint union of annuli in M. When $M = S^3$, for each component of a framed link we can associate to it a framing number.

The unreduced Kauffman bracket

The **unreduced Kauffman bracket** is a version of the Kauffman bracket polynomial that is normalized to be 1 for the empty link $\varnothing$.

(i) $\langle \varnothing \rangle = 1$

(ii) $\langle \bigcirc \rangle = -(A^2 + A^{-2})$

(iii) $\langle \bigcirc \sqcup D \rangle = -(A^2 + A^{-2})\langle D \rangle$

(iv) $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle\,)\,(\, \rangle$

The Kauffman bracket and the Jones polynomial

Blackboard framing

A projection of a framed knot $\boldsymbol{K}$ is said to have **blackboard framing** if the normal vector for every point on the knot is perpendicular to the plane.

Framing number and writhe

The **writhe** of a projection of an oriented knot $\overrightarrow{K}$ is the sum of all of the signs of the crossings, $w(\overrightarrow{K}) = \sum_{c \in cr(K)} sgn(c)$, where

$sgn($  $) = -1,$ $sgn($  $) = 1.$

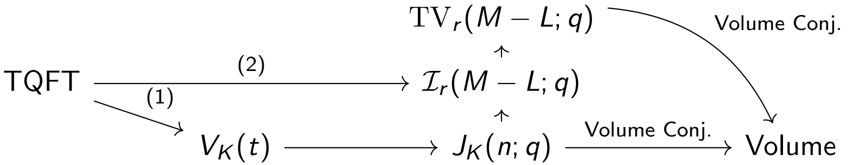
Relationship between the Kauffman bracket and Jones polynomial

Let K be a knot and $A = t^{-1/4}$, then

$$V_K(t) = (-A^3)^{w(K)}(-A^2 - A^{-2})^{-1}[K]$$

The Jones polynomial and TQFT

Physics | Jones polynomial | Quantum invariants | Hyperbolic geometry



1. Edward Witten made a connection to the Jones polynomial and TQFT by utilizing Tsuchiya-Kanie and Kohno's work that showed that Jones' representation of the braid group are ones that arise when one decomposes the correlation functions of 2-dimensional WZW model in conformal blocks.
2. Witten further speculated that there should exist a 3-manifold invariant. Reshetikhin and Turaev constructed this invariant by using a quantization of the Jones polynomial called the colored Jones polynomial.

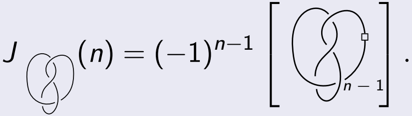
The colored Jones polynomial

Definition

Let $\boldsymbol{K}$ be a zero-framed knot, then the n^{th} **colored Jones polynomial** is defined as

$$J_{\boldsymbol{K}}(n) = (-1)^{n-1}[tr_{\boldsymbol{K}}(f_{n-1})].$$

Illustration

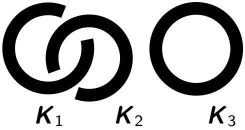


Definition

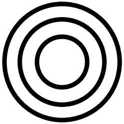
Let $\boldsymbol{L}$ be a framed link with k components in S^3, order the components such that $\boldsymbol{L} = \{\boldsymbol{K}_i\}_{i=1}^k$. Then the **multi-bracket** is a multilinear map,

$$\langle\langle\ ,\dots,\ \rangle\rangle_{\boldsymbol{L}} :$$
$$\mathcal{S}_{2,\infty}(\boldsymbol{K}_1 \times I; \mathbb{C}(A)) \times \cdots \times \mathcal{S}_{2,\infty}(\boldsymbol{K}_k \times I; \mathbb{C}(A)) \to \mathcal{S}_{2,\infty}(S^3; \mathbb{C}(A)).$$

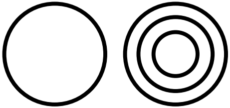
The colored Jones polynomial



(a) Link $\boldsymbol{L}$ with ordered components



(b) Decorated according to $\langle\langle 1, 1, z^3\rangle\rangle_{\boldsymbol{L}}$



(c) Decorated according to $\langle\langle A, A^{-1}z, z^3\rangle\rangle_{\boldsymbol{L}} = \langle\langle 1, z, z^3\rangle\rangle_{\boldsymbol{L}}$

Figure: Illustration of decorating links.

Definition

Let $\boldsymbol{L}$ be a k component link with $w(L) = 0$, then the $\boldsymbol{i}^{th}$ **colored Jones polynomial** is defined as

$$J_{\boldsymbol{L},\boldsymbol{i}}(A) = \langle\langle S_{i_1-1}(z), \cdots, S_{i_k-1}(z)\rangle\rangle_{\boldsymbol{L}},$$

where $\boldsymbol{i} = (i_1, \ldots, i_k)$. $S_0(z) = 1, S_1(z) = z$, $S_{n-1}(z) = S_n(z) - zS_{n-2}(z)$.

Quantum 3-manifold invariant

Witten-Reshetikhin-Turaev invariant

$$\mathcal{I}_r(M_{\boldsymbol{L}}) = \langle\langle \Omega, \ldots, \Omega \rangle\rangle_{\boldsymbol{L}} \langle\langle \Omega \rangle\rangle_{\infty}^{-b_+} \langle\langle \Omega \rangle\rangle_{\infty}^{-b_-}$$

is an invariant of $M_{\boldsymbol{L}}$, where $M_{\boldsymbol{L}}$ is a closed orientable 3-manifold obtained by integral surgery from $\boldsymbol{L} \in S^3$ where $\Omega = \sum_{n=0}^{r-2} \Delta_n S_n(x)$. $A = e^{\pi i/2r}$.

Turaev Viro 3-manifold invariant

$TV(M, q)$ is 3-manifold invariant defined using a triangulation of M.

Quantum $6j$-symbol

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$$

$$= (-1)^{(i+j+k+l+m+n)/2} \Delta(ijk)\Delta(imn)\Delta(ljn)\Delta(lmk) \sum_{z=\max\{T_1,T_2,T_3,T_4\}}^{\min\{Q_1,Q_2,Q_3\}} \mathcal{S}_z$$

Quantum invariants of 3-manifolds

6j-symbol

$$\Delta(ijk) = \left(\frac{[\frac{i+j-k}{2}]![\frac{i-j+k}{2}]![\frac{j+k-i}{2}]!}{[\frac{i+j+k}{2}+1]!} \right)^{1/2}$$

$$\mathcal{S}_z = \frac{(-1)^z[z+1]!}{[z-T_1]![z-T_2]![z-T_3]![z-T_4]![Q_1-z]![Q_2-z]![Q_3-z]!}$$

$$T_1 = \frac{i+j+k}{2}, \quad T_2 = \frac{j+l+n}{2}, \quad T_3 = \frac{i+m+n}{2}, \quad T_4 = \frac{k+l+m}{2}$$

$$Q_1 = \frac{i+j+l+m}{2}, \quad Q_2 = \frac{i+k+l+n}{2}, \quad Q_3 = \frac{j+k+m+n}{2}$$

and $[n] = (q^n - q^{-n})/(q - q^{-1})$.

Quantum 3-manifold invariants

Theorem (Walker, Turaev, Roberts)

For A a $2r^{th}$ root of unity, $TV(M_{\boldsymbol{L}}) = |\mathcal{I}(M_{\boldsymbol{L}})|^2$.

Volume Conjecture for TV-invariants (Chen, Yang 2018)

Let M be a hyperbolic, closed with cusp or compact with total geodesic boundary, then

$$\lim_{r\to\infty} \frac{2\pi}{r} \log |TV(M, e^{2\pi i/r})| = Vol(M),$$

as r varies along the odd natural numbers.

Quantum 3-manifold invariants

Theorem (Detcherry, Kalfagianni, Yang 2018)

Let L be a link in S^3 with n components. Let $r - 2m + 1 \geq 3$ be odd and A a primitive $2r^{th}$ root of unity with $q = A^2$. Then

$$TV(S^3 - L, q) = 2^{n-1}\eta^2 \sum_{1 \leq \boldsymbol{i} \leq m}^{r-1} |J_{L,\boldsymbol{i}}(A)|^2,$$

where $\eta = (A^2 - A^{-2})/\sqrt{-r}$ and $\boldsymbol{i} = (i_1, \ldots, i_n)$ and $1 \leq \boldsymbol{i} \leq m$ means $1 \leq i_k \leq m$ for each k.

Theorem (Belletti, Detcherry, Kalfagianni, Yang 2022)

The volume conjecture for TV-invariants hold for fundamental shadow links.

Quantum 3-manifold invariants

Theorem (Belletti, Detcherry, Kalfagianni, Yang)

For A a $2r^{th}$ root of unity, $\eta = \sqrt{2/r}\sin(\pi/r)$,

$$TV(M-K) = \sum_{i=1}^{r-1} |RT(M,K,i)|^2,$$

where $RT(M,K,i) = (\boldsymbol{L}_{\eta\Omega} \cup \boldsymbol{K}_{S_i})\langle\langle\eta\Omega\rangle\rangle_{\infty}^{\sigma(\boldsymbol{L})}\eta$ and $\sigma(\boldsymbol{L}) = b_+ - b_-$.

Lemma (Belletti, Detcherry, Kalfagianni, Yang 2022)

If the sign is chosen such that $\frac{r\pm1}{2}$ is even, then

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| \begin{vmatrix} \frac{r\pm1}{2} & \frac{r\pm1}{2} & \frac{r\pm1}{2} \\ \frac{r\pm1}{2} & \frac{r\pm1}{2} & \frac{r\pm1}{2} \end{vmatrix}_{q=e^{\frac{2\pi i}{r}}} \right| = v_8$$

Quantum 3-manifold invariants

Theorem (Wong, Yang 2020, I, Purcell, McQuire 2024)

The volume conjecture for TV-invariants holds for octahedral fully augmented links with no half twists.

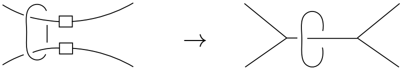
Proposition (I, Purcell, McQuire)

Let D be a diagram of an octahedral fully augmented link of n components with c crossings circles. Decorate D by $\boldsymbol{i} = (a_1, a_2 \ldots, a_c, i_1, i_2, \ldots, i_{n-c})$, where $a_1, \ldots, a_c$ colour the crossing circles and $i_1, \ldots, i_{n-c}$ colour the knot strands. Let $H \subseteq \{a_1, a_2, \ldots, a_c\}$ be the colours of the crossing circles adjacent to half-twists; $H = \varnothing$ if L has no half-twists. Then

$$\langle S_{a_1}(z), \ldots, S_{a_c}(z), S_{i_1}(z), \ldots, S_{i_{n-c}}(z)\rangle_D = \sum_{j_1, j_2, \ldots, j_c} N_{\boldsymbol{i}, j_1, j_2, \ldots, j_c}$$

Quantum 3-manifold invariants

Each j_l is a summation variable arising from merging the strands passing between the crossing circle coloured a_l



Proposition (I, Purcell, McQuire)

$N_{\boldsymbol{i},j_1,j_2,\ldots,j_c}$ is a product of:

(1) $\Delta_{j_l}\lambda_{j_l,a_l}$ for $1 \leq l \leq c$,

(2) $\gamma_{j_l}^{i_k,i_m}$ for each $l \in \{1,2,\ldots,c\}$ such that $a_l \in H$ and i_k, i_m are the colours of the knot strands passing through the crossing circle coloured a_l,

(3) $c-1$ quantum $6j$-symbols, each of the form $\begin{vmatrix} i_q & i_p & j_k \\ i_r & i_s & j_l \end{vmatrix}$, for some $q,p,r,s \in \{1,\ldots,n-c\}$, $k,l \in \{1,\ldots,c\}$, and every summation variable appears at least once in such a quantum $6j$-symbol.

Quantum 3-manifold invariants

Detcherry, Kalfagianni, and Yang proposed a question about the asymptotic behaviour of the coloured Jones polynomial. They asked whether $J_{L,m}(t)$ for $q^2 = t = e^{(2\pi i)/(m+\frac{1}{2})}$ grows exponentially in m with growth rate equal to the hyperbolic volume.

Theorem (I, Purcell, McQuire)

Let L be an octahedral fully augmented link with c crossing circles and no half-twists. Then as m varies over the even integers,

$$\lim_{m\to\infty} \frac{4\pi}{2m+1} \log \left| J_{L,(m,\ldots,m)}(t = e^{\frac{4\pi i}{2m+1}}) \right| = \mathrm{Vol}(S^3 \backslash L).$$

THANKS!