Gravitational instantons and harmonic maps

Joint work with Song Sun.

- Gravitational instantons:

  Complete noncompact Ricci-flat 4-mflds (M,g) with $\int |Rm|^2 < \infty$.

- Axisymmetric harmonic maps:

  $\Phi: \mathbb{R}^3 \setminus \Gamma \to$ space of symmetric positive 2×2 matrices with $\det = 1$

  Γ ↑ z-axis $\qquad = \mathcal{H}^2$. hyperbolic space

  $\Gamma = z\text{-axis} \subseteq \mathbb{R}^3$

  Φ is invariant under rotation

  rotation $\qquad \Phi: \mathbb{H}^\circ \to \mathcal{H}^2 \qquad \mathbb{H} =$ right-half-plane $\quad \partial\mathbb{H} = \Gamma$

- Asymptotic geometry for gravitational instantons:

  ALE. $\qquad$ ALF. $\qquad$ AF. ....

  $\mathbb{R}^4/\Gamma \quad \Gamma \subseteq SO(4)$.

  AF_β : $\mathbb{R}^1 \times (\mathbb{R}^2 \times \mathbb{R}^1)/\mathbb{Z}$ $\qquad$ up to isometry. $\beta \in [0,1)$

  $\mathbb{Z} \mapsto 2\pi\beta$ on $\mathbb{R}^2$, $\mathbb{Z} \mapsto 1$ on $\mathbb{R}^1$

  Flat metric, hence Ricci-flat

  Cubic volume growth

  - If $\beta = \frac{q}{p}$ rational, the asymptotic cone is $\mathbb{R}^3/\mathbb{Z}_p$ $\qquad$ $\mathbb{Z}_p$ rotation by $2\pi\frac{q}{p}$ along z-axis

  - If β irrational, the asymptotic cone is $\mathbb{H}$. flat right-half-plane

Topology of the end

$\text{End} \simeq (\mathbb{R}^3 \setminus K) \times S^1$

Let $X_{2k+1} = (\mathbb{R}^2 \times S^2) \#_k \overline{\mathbb{CP}^2} \#_k \mathbb{CP}^2$, $X_{2k+2} = (\mathbb{R}^2 \times S^2) \#_{k+1} \overline{\mathbb{CP}^2} \#_k \mathbb{CP}^2$

Main theorem (Li-Sun 25). For $\forall \beta \in (0,1)$, $\forall X_n$, there exists an AF_β gravitational instanton on X_n.

Remark; • For $X_1 = \mathbb{R}^2 \times S^2$, the metric is Riemannian Kerr (If $\beta = 0$ then Schwarzschild)

• For $X_2 = (\mathbb{R}^2 \times S^2) \# \overline{\mathbb{CP}^2}$, the metric is Chen-Teo.

• For Kerr, Chen-Teo, they are conformally Kahler.

For $n \geq 3$, there is NO complex geometry.

Today; construction on X_2.

• Reduction for toric Ricci-flat.: Bach-Weyl. Mazur. Ernst, ...

Lott 20, Lucietti-Kunduri 21.

Setting: (M, g) Ricci-flat. with free isometric T^2-action

integrable. (surface-orthogonal).

Integrability; distribution orthogonal to T^2-action

is integrable

Step 1; Pick $\partial_{\phi_1}, \partial_{\phi_2}$ that generate T^2-action.

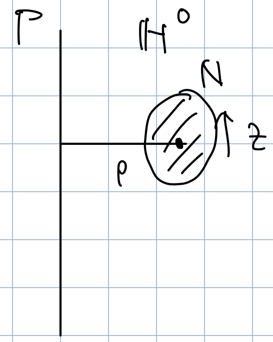
Set $G_{ij} = g(\partial_{\phi_i}, \partial_{\phi_j})$, $\rho := \sqrt{\det G} > 0$.

Denote the quotient 2d metric as (N, h).

Step 2; Ricci-flat $\underset{\not\Leftarrow}{\Rightarrow}$ ρ is a harmonic function on (N, h)

$\Rightarrow$ Find a harmonic conjugate z. $dz = *_h d\rho$

$$h = e^{2\nu}(d\rho^2 + dz^2)$$



$(\rho, z) : N \to \mathbb{H}^0$

Integrable $\Rightarrow$ $g = e^{2\nu}(d\rho^2 + dz^2) + G$

Step 3: Set $\Phi = \frac{1}{\rho} G$, $\det \Phi = 1$

$\Phi : N \subseteq \mathbb{H}^0 \to$ space of positive symmetric 2×2 matrices

with $\det \Phi = 1$

$= \mathcal{H}^2$.

$g = e^{2\nu}(d\rho^2 + dz^2) + \rho \Phi$, is Ricci-flat $\Leftrightarrow$ Φ is harmonic, ν satisfies an ODE $\begin{cases} \nu_\rho = \Phi_\rho, \Phi_z \\ \nu_z = \Phi_\rho . \Phi_z \end{cases}$

Once have Φ, we can determine ν

- Tameness condition for harmonic maps

Need harmonic map $\Phi : \mathbb{H}^0 \to \mathcal{H}^2$, to behave like model maps near $\partial \mathbb{H}$. and infinity so that the Ricci-flat metric closes up to 4-mflds

There are two families of model maps:

- For any vector $v \in \mathbb{R}^2$, there is the model

$$\Phi_v := P_v^T \begin{pmatrix} \rho & \\ & \frac{1}{\rho} \end{pmatrix} P_v. \quad \rightsquigarrow \quad \underset{2}{\mathbb{R}^3} \times \underset{2}{S^1}$$

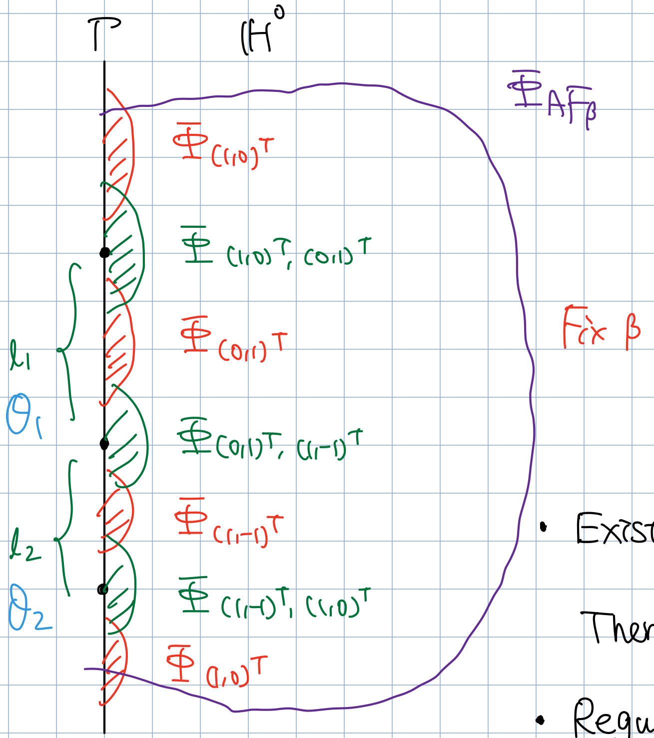
$P_v \in SO(2)$, the matrix rotates the direction of v to $(1,0)^T$

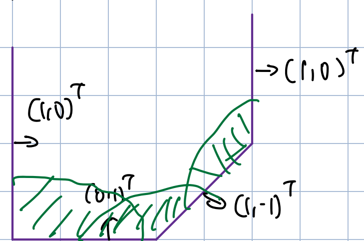
$\underset{2}{\mathbb{R}^2} \times \underset{2}{\mathbb{R}^2}$

- For any vector $v_1, v_2 \in \mathbb{R}^2$,

$P_{v_1, v_2} \in SL(2)$ $\quad \Phi_{v_1, v_2}$. $\quad \Phi_{(1,0)^T, (0,1)^T} = \frac{1}{\rho}\begin{pmatrix} r-z & \\ & r+z \end{pmatrix}$ $\quad r = \sqrt{\rho^2 + z^2}$

To get Ricci-flat metric on $X_2 = (\mathbb{R}^2 \times S^2) \# \overline{\mathbb{CP}^2}$





$\Phi_1 \neq \Phi_2$

$\Phi : \mathbb{H}^0 \to \mathcal{H}^2$

<u>Definition</u> (Tameness): Harmonic map $\Phi : \mathbb{H}^0 \to \mathcal{H}^2$ is called tame if:

$$\mathrm{dist}_{\mathcal{H}^2}(\Phi, \Phi_{\text{model}}) < C$$

$$\mathrm{dist}_{\mathcal{H}^2}(\Phi, \Phi_{AF_\beta}) \to 0 \quad \text{as } r \to \infty$$

- Existence & uniqueness (Weinstein. Li-Tian).

  There $\exists\,!$ tamed harmonic map (Fix β ℓ_1, ℓ_2)

- Regularity (Weinstein. Li-Tian. Nguyen).

  For the tamed harmonic map Φ, the Ricci-flat metric (X_2, g). at most has conical singularities along S^2.

  (Edge cone singularity)

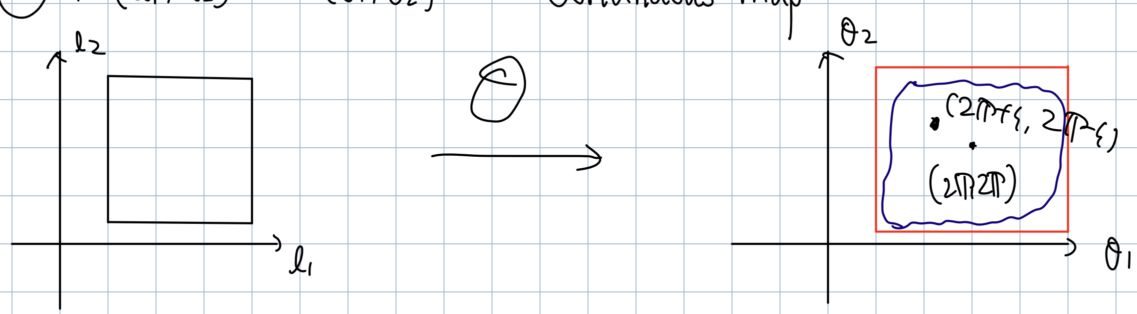
Each finite interval corresponds to a totally geodesic S^2 in X_2

Problem: Can we adjust (ℓ_1, ℓ_2) s.t. $(\theta_1, \theta_2) = (2\pi, 2\pi)$?

Difficulty: No explicit knowledge of (θ_1, θ_2)

Key idea: Can understand (θ_1, θ_2) if (ℓ_1, ℓ_2) is very degenerate!

$\Theta : (\ell_1, \ell_2) \to (\theta_1, \theta_2)$ continuous map



$Q = [\varepsilon, N]^2$. $\varepsilon \ll 1$. $N \gg 1$

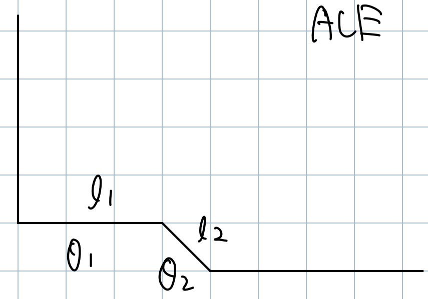
∂Q : (l_1, l_2) very degenerate

We proved $\Theta|_{\partial Q}$ is homotopic to a standard map of degree one

Homotopic does not touch $(2\pi, 2\pi)$

ALE : One less parameters than angles

ALE



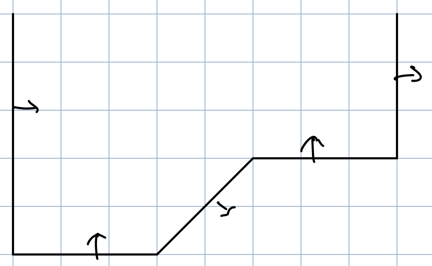
$$\frac{l_1}{l_2} = \frac{l_1'}{l_2'} \Rightarrow \theta_1 = \theta_1', \ \theta_2 = \theta_2'$$

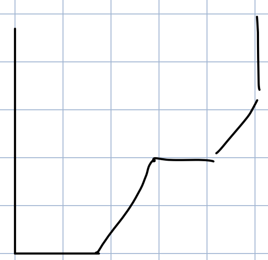
ALF :

$(\mathbb{R}^2 \times S^2) \# \overline{\mathbb{CP}}^2 \# \mathbb{CP}^2$

$\|$

$X_3 \cdots X_4 \cdots\cdots$ $\leftarrow$ $(\mathbb{R}^2 \times S^2) \#_2 \overline{\mathbb{CP}}^2 \# \mathbb{CP}^2$





$\cdots\cdots$

X_n $n \geq 3$