Computations in 3-categories of Topological Order

What is topological order?

anomaly free
(n+1) D
Topological phases

Low energy states

fully extended
(n+1) D TQFTs $\xrightarrow[\text{Ev}(\cdot)]{\text{Cobordism Hypothesis } \sim}$ fusion (n-1) categories

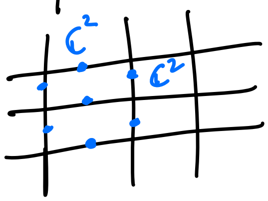
TVBW

String-net models (implement TVBW TQFT)

Fusion 1-category: Linear monoidal 1-category with

- finitely many simple objects $\quad \text{End}(-) \cong \mathbb{C}$
- every object is finite $\oplus$ of simples
- 1 is simple $\qquad \text{Rep}(G) \quad \text{Vec}[G]$
- all objects have duals $\qquad \text{Irr}(\text{Vec}[G]) = G$

Example ($n=2$): Kitaev's toric code

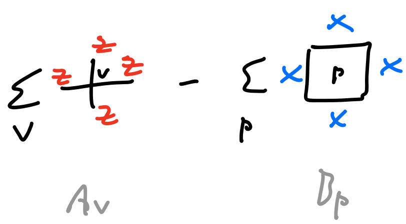


$$X = \begin{array}{cc} & \begin{matrix} 1 & g \end{matrix} \\ \begin{matrix} 1 \\ g \end{matrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$XZ = -ZX$$

$$H = -\sum_v A_v - \sum_p B_p$$



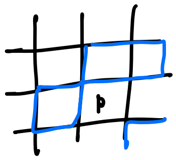
$$\mathcal{H} = \bigotimes_{\ell} \mathcal{H}_{\ell}$$

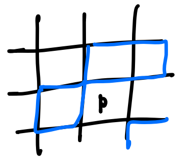
The fusion category of edge labels: $\mathcal{X} \cong \text{Vec}[\mathbb{Z}/2]$.

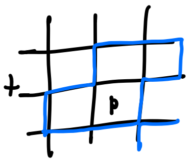
$$\text{Irr}(\mathcal{X}) = \{1, g\}, \quad 1 \otimes 1 \cong 1 \cong g \otimes g, \quad 1 \otimes g \cong g \otimes 1 \cong 1.$$

$F^{g}_{ggg} = \pm 1$ (let's pick $+1$)

Ground states:

Of $-\sum_v A_v$: 

Of $-\sum_v A_v - B_p$: 

+ 

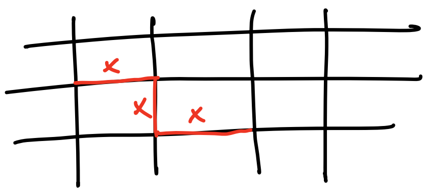
Of H: $\mathbb{Z}/2$-graded cohomology of manifold: how many blue loops go around each element of π_1?

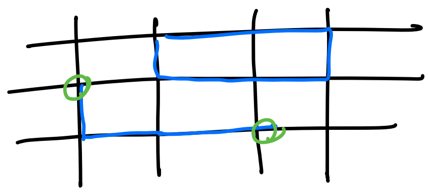
H is a frustration-free commuting projector Hamiltonian:

Commuting projector: $\frac{1-A_v}{2}$, $\frac{1-B_p}{2}$ are commuting projectors

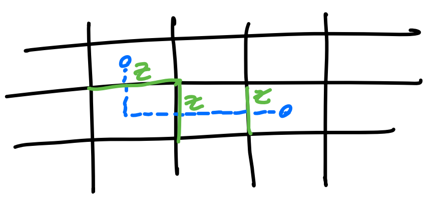
Frustration-free: there are simultaneous lowest-energy eigenstates for all terms.

Anyonic excitations:



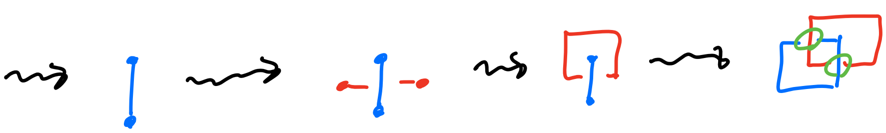


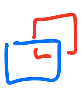
Type e-excitations

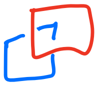


creates type m-excitations

zx or xz at crossings



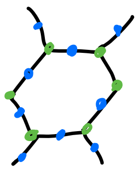
$= (-1)$  $= (-1)$  $= (-1)$.

Since in  each unlinked loop preserves ground states.

Overall: 1, e, m, ε

ground state both in parallel

General string net model: Given fusion category $\mathcal{X}$,

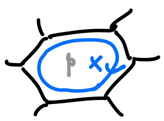


$$\mathcal{H}_e = \mathbb{C}\,\mathrm{Irr}(\mathcal{X})$$

$$\mathcal{H}_v = \bigoplus_{x,y,z} \mathcal{X}(x \to yz)$$

$$H = -\sum_v A_v - \sum_p B_p$$

A_v: checks that edge labels agree with vertex

$$B_p = \sum_x \frac{d_x}{D_{\mathcal{X}}}$$ 

Two Questions:

- What is the notion of equivalence for fusion categories which agrees with equivalence of TVBW TQFT?
- where do the anyonic excitations come from?

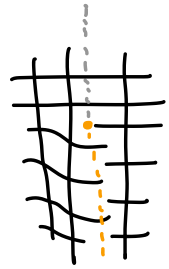
Topological defects:

(n+1) D top defects ← string-net models Kitaev Kong 12

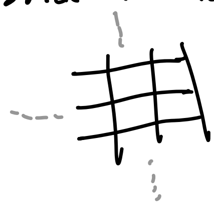
(n+1) D defect TQFT → (n+1) cat of top. order

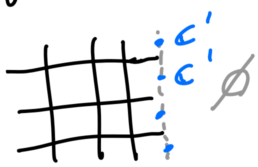
Carqueville Meusburger
Runkel Schaumann et all

KK12



Smooth and rough boundaries:





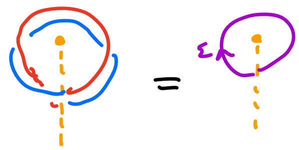
Module categories of edge labels:

$\mathrm{Vec}[\mathbb{Z}/2] \curvearrowright \mathrm{Vec}[\mathbb{Z}/2] \curvearrowleft \mathrm{Vec}$

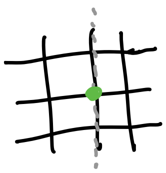
$\mathrm{Vec}[\mathbb{Z}/2] \curvearrowright \mathrm{Vec} \curvearrowleft \mathrm{Vec}$

Boundary Hamiltonian:

$D_p =$  , e and m swap



Excitations are defects between trivial boundaries:



Edge labels:

$\mathrm{Vec}(\mathbb{Z}/2) \curvearrowright \mathrm{Vec}[\mathbb{Z}/2] \curvearrowleft \mathrm{Vec}[\mathbb{Z}/2]$

$\otimes$ forgets grading

$$\mathrm{End}_{x\text{-}x}(x) = \mathrm{End}(\mathrm{Id}_x) =: Z(x)$$

A modular tensor category

3-categories of (2+1D) topological order:

Category	FusCat	String net $\longrightarrow$	2D bulk lattice models
Objects	multifusion 1-cats		1D domain walls
1-morphisms	bimodule cats		0D junctions
2-morphisms	bimodule functors		Local ops commuting with Hamiltonian
3-morphisms	bimodule nat. transf's		

Side note:

This perspective extends to [all-but-once = twice]-extended TQFT's:

Twice extended (2+1)D TQFT's $\Longleftrightarrow$ (2+1)D boundaries for invertible (3+1)D TQFTs

Invertible (3+1)D TQFT's $\Longleftrightarrow$ $\{\mathcal{A}$, $\mathcal{A}$ a MTC / with. eq.

Folklore, Kong + Zheng, Johnson-Freyd, etc.

Anomalous picture: $\mathrm{FusCat}^{\mathcal{A}}$,

$$\begin{tikzcd} & Z(\mathcal{X}) \arrow[d] \\ \mathcal{A} \arrow[ur, dashed] \arrow[r] & \mathcal{X} \end{tikzcd}
\qquad
\begin{tikzcd} \mathcal{X} \arrow[r] & \mathcal{M} & \mathcal{Y} \arrow[l] \\ \mathcal{A} \arrow[u] \arrow[urr] \arrow[ur, phantom, "\circlearrowright"] \end{tikzcd}
\quad \overset{Z^{\mathcal{A}}}{\rightsquigarrow} \quad
\begin{tikzcd} Z(\mathcal{X}) \arrow[r] & \mathrm{End}_{\mathcal{X}-\mathcal{Y}}(\mathcal{M}) \\ & Z(\mathcal{Y}) \arrow[u] \\ \mathcal{A} \arrow[uu] \arrow[ur] \arrow[u, phantom, "\circlearrowright"] \end{tikzcd}$$

Category	$\mathrm{FusCat}^{\mathcal{A}}$		
Objects	$\mathcal{A}$-enriched multifusion 1-cats	String net $\longrightarrow$	2D bulk lattice models on boundary of Walker–Wang
1-morphisms	$\mathcal{A}$-enriched bimodule cats		1D domain walls on boundary ...
2-morphisms	$\mathcal{A}$-balanced bimodule functors		0D junctions on boundary ...
3-morphisms	bimodule nat. transf's		Local ops commuting with Hamiltonian

All later technology in this talk is really in $\mathrm{FusCat}^{\mathcal{A}}$, but we stick to the case $\mathcal{A} = \mathrm{Vec}$ for a first glance.

[Burnell H Jones Penneys __], [Green H Penneys Pordel __, Sanford]

ongoing work.

Computational Tools

Wanted: • practical tools for computations in FusCat
(of composition of 1-morphisms, identifying simple 2-morphisms, vertical and horizontal compositions of 2-morphisms)

• that don't rely on choices of representatives for Morita class

• that have a natural lattice model / string operator interpretation.

we build up category level by category level.

0-morphisms:

Thm (Müger) $Z(\cdot)$ is a complete Morita invariant.
($Z(\mathcal{X}) = \mathrm{End}_{\mathcal{X}-\mathcal{X}}(\mathrm{Id}_{\mathcal{X}})$ is clearly a Morita invariant.)
Completeness $\Leftarrow\Rightarrow$ no invertible (2+1)D TQFTs

$$\mathcal{X} \underset{\text{Morita}}{\simeq} \mathcal{Y} \iff Z(\mathcal{X}) \underset{\text{br-}\otimes}{\cong} Z(\mathcal{Y})$$

1-morphisms:

Thm (Kong, Zheng) The center functor is fully faithful on 0 and 1 morphisms.
The center functor: $\mathcal{X} \mapsto Z(\mathcal{X})$

$$[\mathcal{X} \overset{\mathcal{M}}{\nrightarrow} \mathcal{Y}] \mapsto Z(\mathcal{X}) \xrightarrow{\mathrm{End}_{\mathcal{X}-\mathcal{Y}}(\mathcal{M})} Z(\mathcal{Y})$$

A Witt equivalence $\mathcal{C} \overset{\mathcal{W}}{\nrightarrow} \mathcal{D}$ is a multifusion category $\mathcal{W}$ (MTC's)
equipped with a braided monoidal equivalence $\mathcal{C} \boxtimes \overline{\mathcal{D}} \cong Z(\mathcal{W})$.
$Z(\mathrm{End}_{\mathcal{X}-\mathcal{Y}}(\mathcal{M})) \cong Z(\mathcal{X}) \boxtimes \overline{Z(\mathcal{Y})}$, so $\mathrm{End}_{\mathcal{X}-\mathcal{Y}}(\mathcal{M})$ is Witt eq.

Bad news: Z is not a functor once we include 2 morphisms.
Egs) Let $\mathcal{X} \cong \mathcal{Y} \cong \mathrm{Vec}$, $\mathcal{M} \cong \boxplus_n \mathrm{Vec}$.

$$\mathrm{End}_{\mathcal{X}-\mathcal{Y}}(\mathcal{M}) \cong M_n(\mathrm{Vec}), \quad Z(M_n(\mathrm{Vec})) \cong \mathrm{Vec}.$$

So $\mathcal{M}$ is decomposable, $\mathrm{End}_{\mathcal{X}-\mathcal{Y}}(\mathcal{M})$ is indecomposable.

Takeaway: we have to think a little harder to get a Morita-invariant way to work with 2- and 3-morphisms.

Classification of 1-morphisms: DMNO, The Witt Group of BFC's

Given a bimodule $\mathcal{X} \overset{\mathcal{M}}{\nrightarrow} \mathcal{Y}$, there are algebras

$$A := \mathcal{X}_{\mathcal{M}} \curvearrowleft \mathcal{Y} \in Z(\mathcal{X})^{\mathrm{loc}}_{A} \qquad \mathcal{X} \curvearrowright^{\mathcal{M}} \mathcal{Y} \in Z(\mathcal{Y})^{\mathrm{loc}}_{B} =: B,$$

and a Morita equivalence $\mathcal{N} : \mathcal{X}_A \overset{\sim}{\nrightarrow} \mathcal{Y}_B$

These pics: 1 dim down.

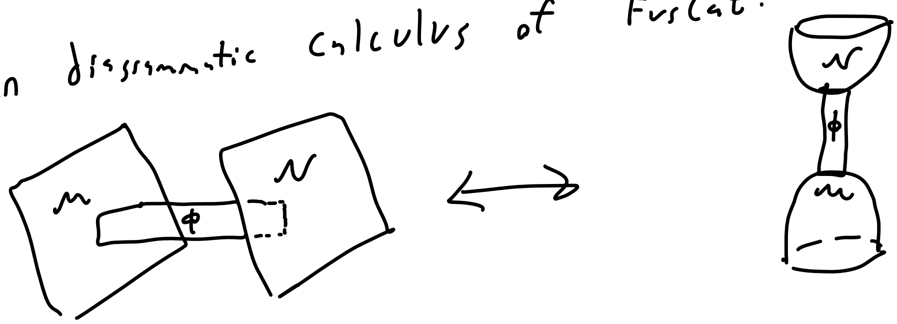
such that

$$\mathcal{X} \underset{\mathcal{M}}{|} \mathcal{Y} \cong \mathcal{X} \underset{\mathcal{X}_A}{|} \mathcal{X}_A \underset{\mathcal{N}}{|} \mathcal{Y}_B \underset{\mathcal{Y}_B}{|} \mathcal{Y} =: \mathcal{X} \underset{\mathcal{M}(A,\mathcal{N},B)}{|} \mathcal{Y}$$

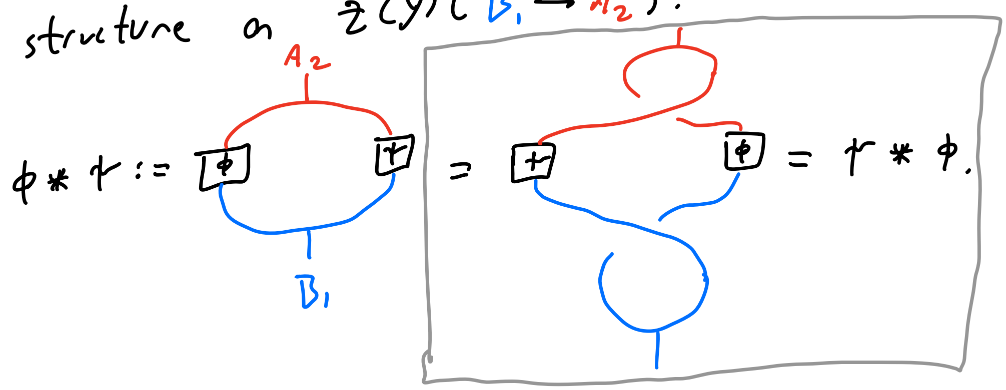
Composition of 1-morphisms: $\dagger \ast \rightsquigarrow (\dagger \mathcal{Y} \ast)$

$$\mathrm{End}(\mathrm{Id}_{\mathcal{M}(A_1,B_1,\mathcal{N}_1)} \boxtimes \mathcal{M}(A_2,B_2,\mathcal{N}_2)) \cong Z(\mathcal{Y})(B_1 \to A_2).$$

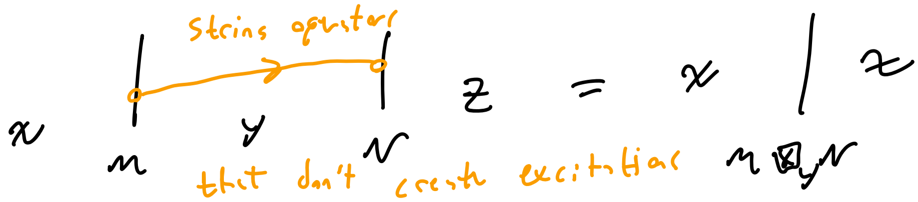
In diagrammatic calculus of FusCat:



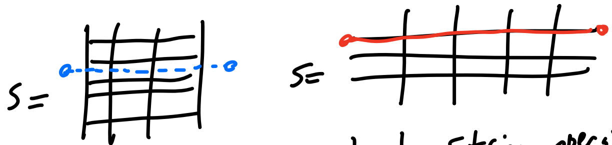
Algebra structure on $Z(\mathcal{Y})(B_1 \to A_2)$:



Lattice:



Example: toric code

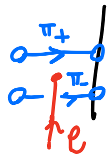


2D degeneracy associated with condensed string operators.

Local operators: $\mathbb{C}\{1, s\}$, $s^2 = 1$.

Diagonalize: $\pi_+ = \frac{1+s}{2}$, $\pi_- = \frac{1-s}{2}$

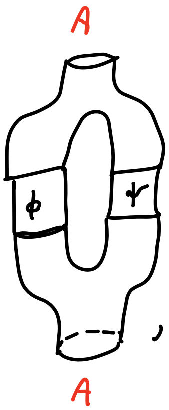
Confined anyons become defects between sectors.



Burnell H Penneys Jones

General story: $\mathcal{C}(A \to A)$ is a commutative C^*-algebra

$\phi * \psi =$



minimal idempotents $\leftrightarrow$ indecomposable summands of 1-morphism from composition,

Prop) All point defects between different summands arise from confined anyons.

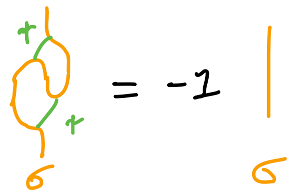
Example: $\mathrm{vec}[\mathbb{Z}/2] \subseteq \mathrm{Ising}$

$\mathrm{Irr}(\mathrm{Ising}) = \{\underbrace{1, \psi}_{\mathrm{vec}[\mathbb{Z}/2]}, \sigma\}$, $\quad \psi \otimes \psi \cong 1$, $\quad \sigma \otimes \sigma \cong 1 \oplus \psi$

$\dim(\psi) = 1 \qquad \dim(\sigma) = \sqrt{2}$

$Z(\mathrm{Ising}) \cong \mathrm{Ising} \boxtimes \overline{\mathrm{Ising}}$

$$F^{\sigma}_{\sigma,\sigma,\sigma} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



$$F^{\sigma}_{\psi,\sigma,\psi} = F^{\psi}_{\sigma,\psi,\sigma} = [-1]$$

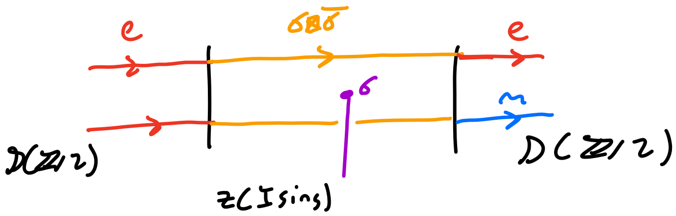
Use Ising as $\mathrm{Vec}[\mathbb{Z}/2]$ - $\mathrm{Vec}[\mathbb{Z}/2]$ bimodule.

UMTC story: $Z(\mathrm{Ising}) \cong \mathrm{Ising} \boxtimes \overline{\mathrm{Ising}}$,

$A = 1 \oplus (\psi \boxtimes \overline{\psi})$, $Z(\mathrm{Ising})_A^{loc} \cong Z(\mathrm{Vec}[\mathbb{Z}/2])$,

$w := {}_A Z(\mathrm{Ising})_A \cong \mathrm{Id} \boxplus \mathbb{T}$

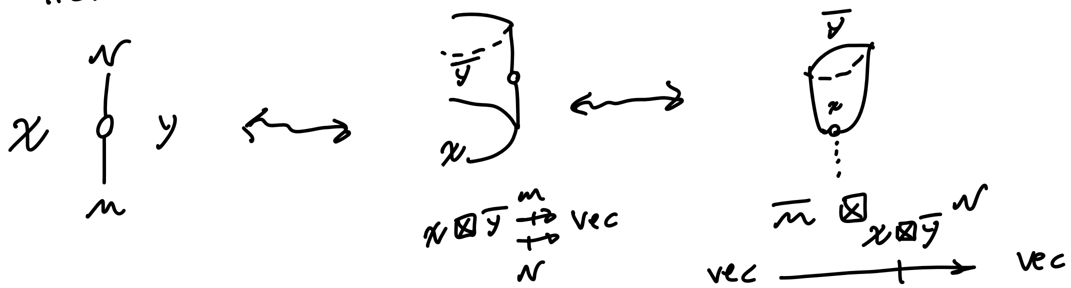
where $\mathbb{T}$ swaps e and m "electromagnetic duality"



σ becomes the "twist defect" between trivial and nontrivial boundaries.

Classification of 2-morphisms:

"Folding Tricks": use duality to deform the area near a defect.



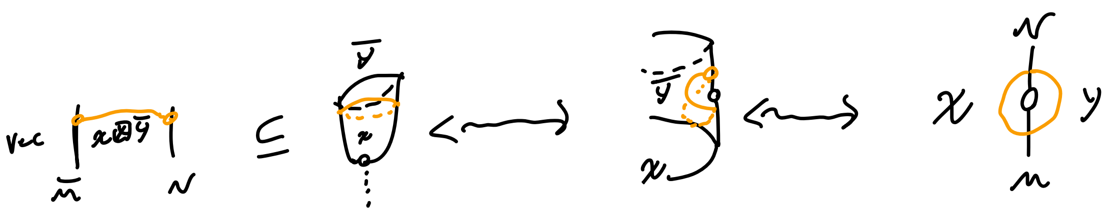
$$\mathrm{Fun}_{x-y}(\mathcal{M} \to \mathcal{N}) \cong \mathrm{Fun}_{x \boxtimes \bar{y}}(\mathcal{M} \to \mathcal{N}) \cong \mathrm{Fun}(\mathrm{vec} \to \bar{\mathcal{M}} \boxtimes_{x \boxtimes \bar{y}} \mathcal{N})$$

vec - vec bimodules are just linear categories, and

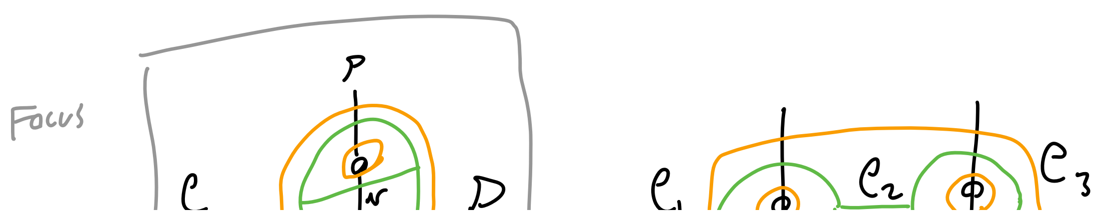
$$\mathrm{Fun}(\mathrm{vec} \to -) \cong - .$$

On the other hand, if $\mathcal{C}$ is tensored over vec, $\mathcal{C} \cong \boxplus_{c \in \mathrm{Irr}(\mathcal{C})} \mathrm{vec} \otimes c$

Conclusion: Classifying point defects $\cong$ Classifying summands of certain 1-morphisms



Vertical and horizontal compositions:



Orange subalgebras: commutative, central.

an amusing string diagram game

Full green algebra may not be.

If M, N, P are $(\mathcal{C}\boxtimes\overline{\mathcal{D}})_{L_1, L_2, L_3}$ respectively,

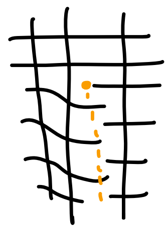
$$\mathcal{C}(1\to L_1L_2), \quad \mathcal{C}(1\to L_2L_3), \quad \mathcal{C}(1\to L_1L_3)$$

$$\supseteq \qquad \cap| \qquad \subseteq$$

$$\mathcal{C}(1\to L_1L_2L_3)$$

$\mathcal{C}\boxtimes\overline{\mathcal{D}}$

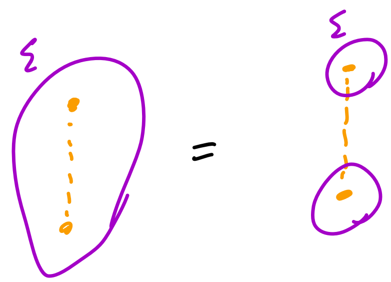
Example:

Earlier we saw  could be encircled in two ways:

- nothing
- 

So there are 2 species of defects, which differ by fusion with e or m, can absorb a fermion.

 $= e \oplus m$ or $1 \oplus \varepsilon$, depending on whether the two species align.



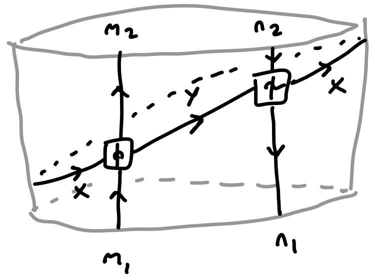
Comparison to Skein Theory Approach: [Kitaev Kong 12]

How are calculations in FusCat normally done?

Given $x \underset{\mathcal{N}}{\overset{\mathcal{M}}{\rightrightarrows}} y$, we construct a C^*-Hopf algebroid

$$\mathrm{Tube}(\mathcal{M}\to\mathcal{N}) \cong \bigoplus_{\substack{x,y,m_1,m_2,n_1,n_2 \\ \in \mathrm{Irr}(\cdot)}} \mathcal{M}(x m_1 \to m_2 y) \otimes \mathcal{N}(n_2 y \to x n_1)$$

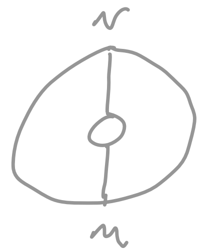
Puncture containing point defect



Think of $\phi \otimes \psi$ as

multiplication: stack tubes.

Rest of lattice

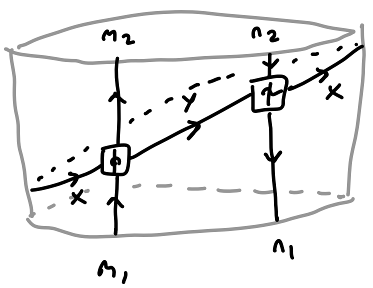


(If $x \cong y \cong \mathcal{M} \cong \mathcal{N}$, Morita eq to usual $\mathrm{Tube}(x)$)

Hopf-algebroid: $\mathrm{Tube}(\mathcal{M}\to\mathcal{M})$ is a Hopf algebra

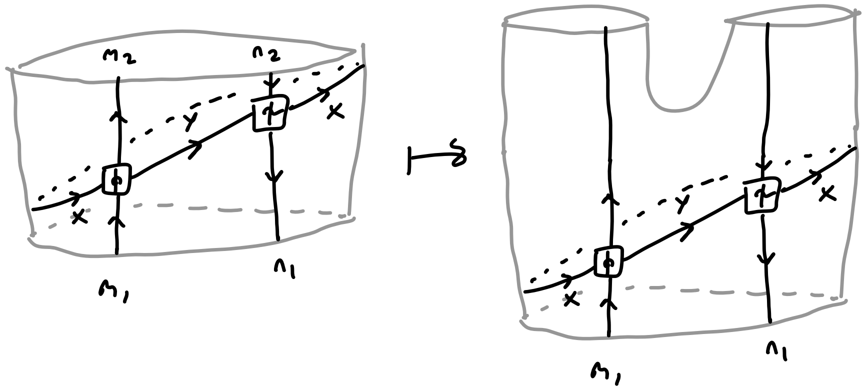
Comultiplication $\mathrm{Tube}(\mathcal{M}_1 \to \mathcal{M}_3) \to \mathrm{Tube}(\mathcal{M}_1 \to \mathcal{M}_2) \otimes \mathrm{Tube}(\mathcal{M}_2 \to \mathcal{M}_3)$

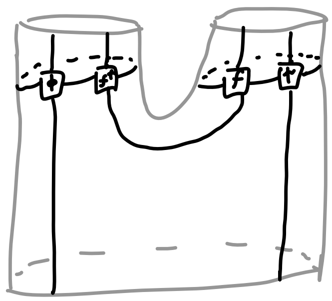
is corner of comult. of $\mathrm{Tube}\left(\bigoplus_{j=1}^{3} \mathcal{M}_j \to \bigoplus_{k=1}^{3} \mathcal{M}_k\right)$.



$$\longmapsto \sum_{p_1,p_2,f} \sqrt{\frac{d_{p_1} d_{p_2}}{d_x d_y}} \left(\phi \otimes f^\dagger\right) \otimes \left(f \otimes \psi\right)$$

in ONB

Think: 

$$= \sum_{p_1,p_2,f} \sqrt{\frac{d_{p_1} d_{p_2}}{d_x\, d_y}}$$ 

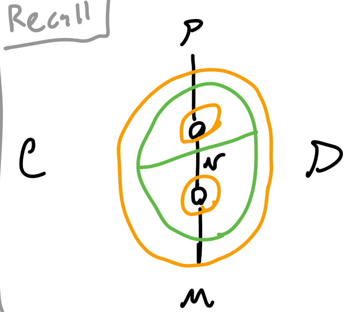
Remark: $\mathcal{C}(1\to L_1L_2L_3)$ is non-Abelian in general.
If there is ever fusion multiplicity,

$$\mathrm{Im}[\mathcal{C}(1\to L_1L_2)\otimes \mathcal{C}(1\to L_2L_3)\otimes \mathcal{C}(1\to L_1L_3)] \subsetneq \mathcal{C}(1\to L_1L_2L_3)$$

So this is not a comultiplication, but a bimodule that acts

Like $H \underset{\Delta}{\curvearrowright} H\otimes H \curvearrowleft H\otimes H.$ "Hopfish algebra" [Tang, Weinstein, Zhu 06]

Recall



$\mathcal{C}(1\to L_1L_2)$ $\quad$ $\mathcal{C}(1\to L_2L_3)$ $\quad$ $\mathcal{C}(1\to L_1L_3)$

$\cong$ $\quad$ $\cap$ $\quad$ $\cong$

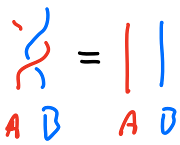
$\mathcal{C}(1\to L_1L_2L_3)$

Some Interesting Defect Lattices in (2+1)D [Burnell H 26?]

Goal: Draw a grid of topological defects such that

- G.S.D. scales with length (not area),
- excitations exhibit geometrically-restricted mobility ("lineons")

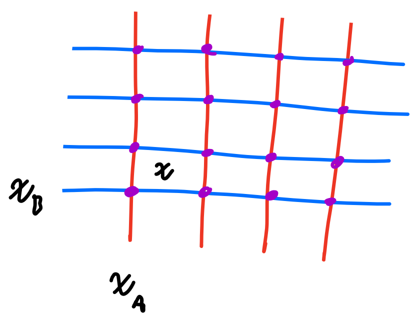
Inputs:
- UMTC $\mathcal{C}$
- étale algebras $A, B \in \mathcal{C}$:



"commuting condensates"

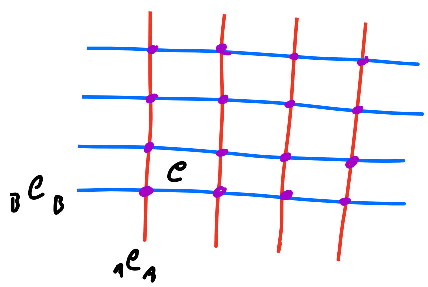
If $Z(\mathcal{X}) \simeq \mathcal{C}$,

The network:



Labels in $\mathrm{FusCat}^{\mathcal{B}}$

$\bullet = \mathcal{X}_B \boxtimes_{\mathcal{X}} \mathcal{X}_A \simeq \mathcal{X}_{BA} \simeq \mathcal{X}_{AD} \simeq \mathcal{X}_A \boxtimes_{\mathcal{X}} \mathcal{X}_B$



Labels by categories of excitations

$\bullet = \cdots$
more complicated to state.

If $\mathcal{C}(A \to B) \simeq \mathcal{C}(1 \to 1) \simeq \mathbb{C}$,
no common anyons

$\pi_\bullet \leq$

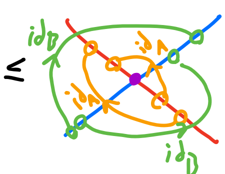


Image of this projection:

$\pi_\bullet$ = lift of $\pi_{1_{\mathcal{C}^{loc}_{AB}}}$

Example (generalizes many features of $D(S_3)$):

Let $D_n = \langle \underset{\text{Rotation}}{r}, \underset{\text{reflection}}{f} \mid r^n, f^2, (rf)^2 \rangle$ Dihedral, $|D_n| = 2n$

Then if $b \mid a \mid n$, we have $\underset{\text{normal}}{\langle r^a \rangle_{D_n}} \subseteq \underset{\text{not normal}}{\langle r^b, f \rangle_{D_n}} \subseteq \langle r, f \rangle = D_n$

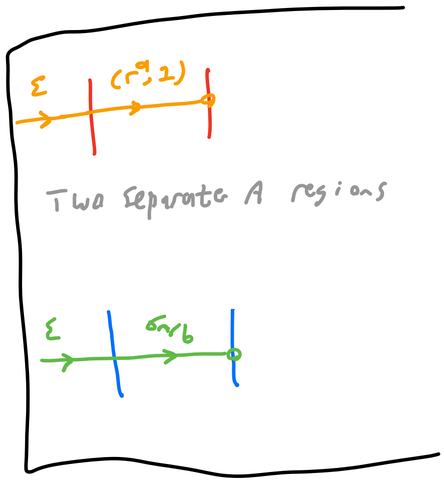
$$K \le H \le D_n$$

$A \cong \mathbb{C}[K] \cong \underbrace{\bigoplus_{k \in K} k}_{\text{in Vec}[D_n]}$, $B \cong \mathbb{C}^{G/H} \in \mathrm{Rep}(D_n) \subseteq D(D_n)$

A and B commute because if $\rho \in \mathrm{Rep}(G)$, ρ contains an H-invariant vector, then ρ is trivial on $\bigcap_{g \in G} gHg^{-1} \supseteq K$.

$H/K \cong \langle r^b, f \mid r^a, f^2, (rf)^2 \rangle \cong D_a$, so $D(D_n)^{loc}_{AB} \cong D(D_a)$.

But: $${}_Ae_A \;\; {}_Ae_A \cong {}_Ae_A \boxplus \left(\boxplus_{k=1}^{\overset{\frac{n-a}{2a}}{\overset{\|}{\lfloor \frac{n}{2a} \rfloor}}} {}_{\tilde{A}}e_{\tilde{A}} \right)$$



where $\tilde{A} \cong \mathbb{C}[K] \otimes \underbrace{\mathbb{C}^{D_n/\langle r \rangle}}_{1 \oplus \varepsilon}$

Abelian boson which detects f

$${}_Be_B \;\; {}_Be_B \cong {}_Be_B \boxplus \left(\boxplus_{k=1}^{\lfloor \frac{b}{2} \rfloor} {}_{\tilde{B}}e_{\tilde{B}} \right) \qquad \tilde{B} \cong \mathbb{C}^{D_n / \overbrace{\bigcap_{g \in D_n} gHg^{-1}}^{=\langle r^b \rangle}}$$

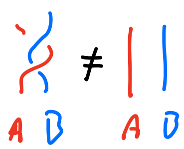
And $\mathcal{C}^{\mathrm{loc}}_{\tilde{A}\tilde{B}} \cong \mathcal{C}^{\mathrm{loc}}_{A\tilde{B}} \cong D(\mathbb{Z}/\frac{a}{b})$ toric code.

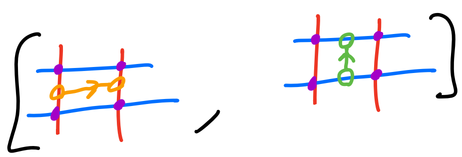
Stacking domain walls can confine f-lines, condense ε bosons.

Prop) Given an $\ell_A \times \ell_B$ defect lattice, untwisted periodic boundary conditions (torus), by stabilizer code analysis, ground state degeneracy is $\left(\frac{n}{a}\right)^{\ell_A} \cdot \left(\frac{n}{b}\right)^{\ell_B} \cdot$ [topological GSD from $DCD_{a/b}$]]

So: choosing commuting condensates $\Rightarrow$ degeneracy $\sim 2^{\ell_A + \ell_B}$:
"subextensive", not constant but not scaling with area.

If A and B did not commute,



Then  $\neq 0$, so we do not get this behavior.

(Could be "extensive" degeneracy $\sim 2^{\ell_A \ell_B}$, could be constant degeneracy because choice of  permits big simplification)