C^0-Rigidity of $Ham(X,\omega)$ for symplectic rational surfaces

(joint with C-Y. Mak and W. Wu)

Seminário de Geometria em Lisboa

Marcelo S. Atallah

University of Sheffield

Let (M, ω) be closed with a fixed metric g. For $\varphi, \psi \in \mathrm{Diff}(M)$

$$d_{C^0}(\varphi, \psi) = \max_{x \in M} d(\varphi(x), \psi(x))$$

d_{C^0} induces C^0-topology on $\mathrm{Diff}(M)$.

Def: $\varphi \in \mathrm{Ham}$ if $\exists\, H : M \times S^1 \to \mathbb{R}$ s.t. $\varphi = \varphi_H^1$, where:

$$\begin{cases} \omega(X_H^t, -) = -dH_t \\ \frac{d}{dt}\varphi_H^t = X_H^t \circ \varphi_H^t \end{cases}$$

$$\mathrm{Ham} \overset{②}{\subseteq} \mathrm{Symp}_0 \overset{③}{\subseteq} \mathrm{Symp} \overset{①}{\subseteq} \mathrm{Diff}$$

Question: Are these subspaces closed in the $\mathcal{C}^0$ topology?

① Yes! (Gromov-Eliashberg)

② $\mathcal{C}^0$- Flux conjecture:

(i) True if $\Gamma = \Gamma^{top}$. (LMP '97, Buhovsky '13)

(ii) {spherically monotone or Lefschetz} and {$\chi(M) \neq 0$ or $Z(\pi_1(M))$ finite} (A.-Shelukhin, in progress).

③ Known only for surfaces.

Argument for surfaces:

(i) Homeo(M) is locally-path connected
(Edwards-Kirby, '71)

(ii) $\pi_0(\mathrm{Homeo}^+(\Sigma)) \cong \pi_0(\mathrm{Diff}^+(\Sigma))$.

(iii) $\mathrm{Symp}(\Sigma, \omega) \longrightarrow \mathrm{Diff}^+(\Sigma)$ induces

$$\pi_0(\mathrm{Symp}(\Sigma,\omega)) \cong \pi_0(\mathrm{Diff}^+(\Sigma))$$

Observation:

In general,

Torelli

$$f \in \overline{\mathrm{Symp}_0} \subseteq \mathrm{Symp} \implies f \in \mathrm{Symp}_h$$

In particular $\overline{\mathrm{Symp}_0} = \mathrm{Symp}_0$ when

$$\pi_0(\mathrm{Symp}_h) = \{1\}.$$

This is the case for:

(i) monotone $S^2 \times S^2$ (Gromov)

(ii) monotone $\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ $0 \leq n \leq 4$
(Gromov, Evans)

(iii) more examples (Li-Li-Wu)

Obs: $\mathrm{Symp}_0 = \mathrm{Ham}$ in these cases.

Setup: $X_n := (\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}, \omega)$

$H_2(X_n)$ admits a canonical basis $\{H, E_1, \ldots, E_n\}$, where:

(i) $E_i \cdot E_j = -\delta_{ij}$, $H \cdot E_i = 0$, $H^2 = 1$

(ii) $\omega(E_i) = \min\{\omega(E) \mid E^2 = -1, E_j \cdot E = 0, j \geq i+1\}$

Identifying $H_2 \overset{P.D}{\cong} H^2$ and rescalling

$$[\omega] = H - \sum_{i=1}^{n} a_i E_i, \quad a_i \geq a_{i+1}$$

$$= (1 \mid a_1, \ldots, a_n)$$

$$c_1 = (3 \mid 1, \ldots, 1)$$

Def: (X_n, ω) is a positive rational surface if $c_1 \cdot [\omega] > 0$.

Def: (Classification of ω)

ω is of type $\mathbb{D}$ if

$$[\omega] = (1 \mid \lambda, \underbrace{\tfrac{1-\lambda}{2}, \dots, \tfrac{1-\lambda}{2}}_{k}, a_{k+2}, \dots, a_n)$$

$a_{k+2} < (1-\lambda)/2$

and

$\lambda \in (1/3, 1), k \geq 4$ or $\lambda = 1/3, k = 4$.

ω is of type $\mathbb{E}$ if $\lambda = 1/3$, $k = 5, 6, 7$.

Otherwise, ω is of type $\mathbb{A}$.

Obs: When $n = k+1$ we call it of pure type.

Remark: ω is of type $\mathbb{E}$ if X_n is monotone $n=6,7,8$ (or small blow-ups of it)

Theorem (LLW '22)

Let (X_n, ω) be a positive rational surface. Then:

(i) ω of type $\mathbb{A}$:

$$\pi_0(\mathrm{Symp}_h(X_n)) = 1$$

(ii) ω of type $\mathbb{D}$, $\lambda \neq 1/3$:

$$\pi_0(\mathrm{Symp}_h(X_n)) = PB_k(S^2)/\mathbb{Z}_2$$

(iii) ω of type $\mathbb{D}$, $\lambda = 1/3$:

$$\pi_0(\mathrm{Symp}_h(X_n)) = PB_5(S^2)/\mathbb{Z}_2.$$

Obs: $\mathrm{Ham} = \overline{\mathrm{Ham}} \subseteq \mathrm{Symp}$ for type $\mathbb{A}$

Theorem (A.-Mak-Wu): Let (X, ω) be a positive rational surface of type $\mathbb{D}$. Then, $\mathrm{Ham}(X,\omega)$ is a connected comp. of $\mathrm{Symp}(X,\omega)$ in the $\mathcal{C}^0$-topology.

Corollary: $\mathrm{Ham}(X,\omega)$ is $\mathcal{C}^0$-closed in $\mathrm{Symp}(X,\omega)$.

Proof Overview:

It is enough to show that $\exists\ \varepsilon>0$ such that if $\psi \in Symp_h$ and

$$d_{C^0}(\psi, id) < \varepsilon \Rightarrow \psi \in Ham.$$

- **Ham is open:**

Let $\varphi \in Ham$ and $\psi \in Symp$ s.t.

$$d_{C^0}(\varphi, \psi) < \varepsilon$$

$\psi \in Symp_0$ for suff $\varepsilon>0$.

Then, by right invariance of d_{C^0}

$$d_{C^0}(id, \psi\varphi^{-1}) < \varepsilon \Rightarrow \psi \in Ham$$

Hence $B_\varepsilon^{C^0}(\varphi) \subseteq Ham$

- **Ham is closed:**

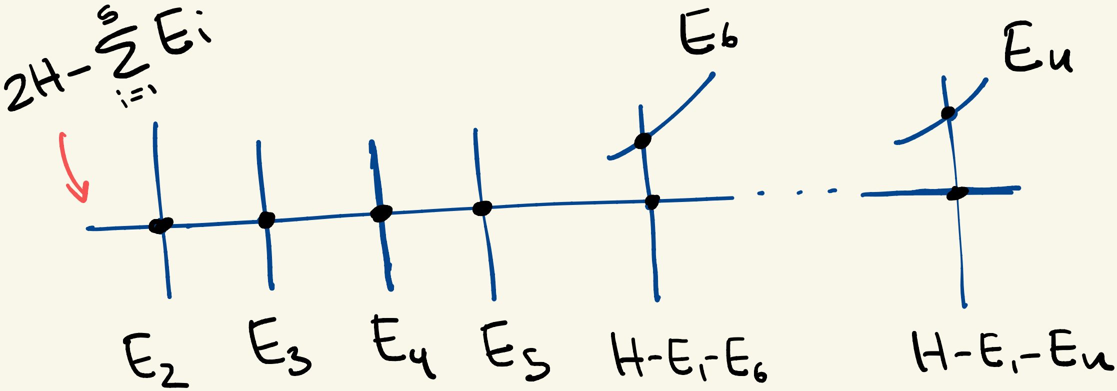
$$\varphi_n \xrightarrow{C^0} \psi \Rightarrow \varphi_n\psi^{-1} \xrightarrow{C^0} id$$

$$\Rightarrow \varphi_n\psi^{-1} \in Ham\ \forall\ n \geq N_0 \Rightarrow \psi \in Ham.$$

Step 1:

Obtain Criterion to determine when $\varphi \in \mathrm{Ham}(X, \omega)$.

Fix (X_n, ω), $n \geq 6$. There exists J-holomorphic config. Σ, $J \in \mathcal{J}_\omega$.



Let

$$\mathcal{C} = \{\text{Config with } \omega\text{-orthogonal int. } \& \text{ same homology.}\}$$

Note that $\mathrm{Symp}_h(X, \omega) \curvearrowright \mathcal{C}$ transitively.

We have the following fibration:

$$\begin{tikzcd}
\mathrm{Stab}'(\Sigma) \arrow[r] & \overset{\varphi|_\Sigma = \mathrm{id}_\Sigma}{\mathrm{Stab}^0(\Sigma)} \arrow[r] \arrow[d] & \overbrace{\mathrm{Stab}(\Sigma)}^{\varphi(\Sigma)=\Sigma} \arrow[r] \arrow[d] & \mathrm{Symp}_h(X,\omega) \arrow[d] \\
\mathrm{Symp}_c(X \setminus \Sigma) \arrow[u, phantom, "\cong" rotate=90] & \mathcal{G}(\Sigma) & \mathrm{Symp}(\Sigma) & \mathcal{C}
\end{tikzcd}$$

Proposition (LLW): When ω is of pure type $\mathbb{D}$ and

$$\lambda > \frac{n-3}{n-1}, \quad \lambda \in \mathbb{Q}.$$

Then,

$$\pi_0(\mathrm{Stab}^0(\Sigma)) \longrightarrow \pi_0(\mathrm{Stab}(\Sigma))$$

is trivial. Thus, if

$$\varphi \in \mathrm{Stab}^0(\Sigma) \Rightarrow \varphi \in \mathrm{Ham}(X,\omega)$$

.

Lemma: Let (X,ω) be a positive rational surface of pure type $\mathbb{D}$. If

$$\psi \in \mathrm{Stab}^{\circ}(\Sigma) \Rightarrow \psi \in \mathrm{Ham}(X,\omega)$$

Proof idea:

We have exact sequences:

$$\begin{tikzcd}
 & & PB_{n-1}(S^2)/\mathbb{Z}_2 \arrow[d, phantom, "\cong" rotate=90] \\
\pi_1(\mathcal{C}_1) \arrow[r, "A"] & \pi_0(\mathrm{Stab}(\Sigma_1)) \arrow[r, "B"] \arrow[d, phantom, "||"] & \pi_0(\mathrm{Symp}_h(X,\omega_1)) \\
\pi_0(\mathrm{Stab}^{\circ}(\Sigma_1)) \arrow[r, "A'"] & \pi_0(\mathrm{Stab}(\Sigma_1)) \arrow[r, "B'"] & \pi_0(\mathrm{Symp}(\Sigma_1)) \arrow[d, phantom, "\cong" rotate=90] \\
 & & PB_{n-1}(S^2)/\mathbb{Z}_2
\end{tikzcd}$$

Claim: $B' \circ A = 0 \Leftrightarrow B \circ A' = 0$

$$B' \circ A = 0 \Rightarrow \mathrm{im}\, A \subseteq \ker B' = \mathrm{im}\, A'$$

so we have a surjective homomorphism:

$$\pi_0(\mathrm{Stab}(\Sigma))/\mathrm{im}(A) \longrightarrow \pi_0(\mathrm{Stab}(\Sigma))/\mathrm{im}(A')$$

- hopfian property $\Rightarrow$ isomorphism.

Hence $\operatorname{im} A = \operatorname{im} A'$. Thus.

$$\operatorname{im}(B \circ A') = B(\operatorname{im} A') = B(\operatorname{im} A) = 0.$$

Converse is the same arg. $\square$.

Obs: Prop 1 $\Rightarrow$ $B_\lambda \circ A'_\lambda = 0$ $\lambda > \frac{n-3}{n-1}$ and $\lambda \in \mathbb{Q}$. Hence $B'_\lambda \circ A_\lambda = 0$.

Let $\lambda' < \lambda < 1$. We have:

$$B'_\lambda \circ A_\lambda : \pi_1(\mathcal{C}_\lambda) \xrightarrow{0} \pi_0(\mathrm{Symp}(\Sigma_\lambda))$$

Use syplectic inflation argument to show that

$$B'_{\lambda'} \circ A_{\lambda'} : \pi_1(\mathcal{C}_{\lambda'}) \xrightarrow{0} \pi_0(\mathrm{Symp}(\Sigma_{\lambda'}))$$

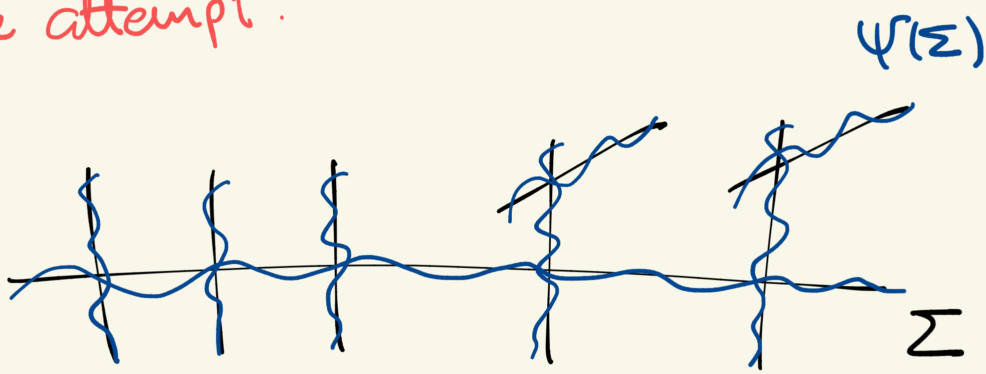
Step 2: For ψ C^0-small, isotope $\psi(\Sigma)$ back to Σ with C^0-control.

Lemma: There exists $\varepsilon > 0$ s.t. $\forall \psi \in \mathrm{Symp}_h$ satisfying:

$$d_{C^0}(\psi, id) < \varepsilon$$

there exist $\varphi \in \mathrm{Ham}$ s.t. $\varphi\psi \in \mathrm{Stab}^0(\Sigma)$.

Naive attempt.



Use J-isotopy between J and $\psi_* J$.
Yields $\varphi \in \mathrm{Ham}(X, \omega)$ s.t.

$$\varphi\psi \in \mathrm{Stab}(\Sigma)$$

If $\varphi\psi$ trivial in $\pi_0(\mathrm{Stab}(\Sigma))$ then can find ψ' s.t. $\varphi\psi' \in \mathrm{Stab}^\circ(\Sigma)$.

no control over $[\varphi\psi] \in \pi_0(\mathrm{Stab}(\Sigma))$.

Isotopy with C^0-control:

Let $F = H - E_1$ be the fiber class.

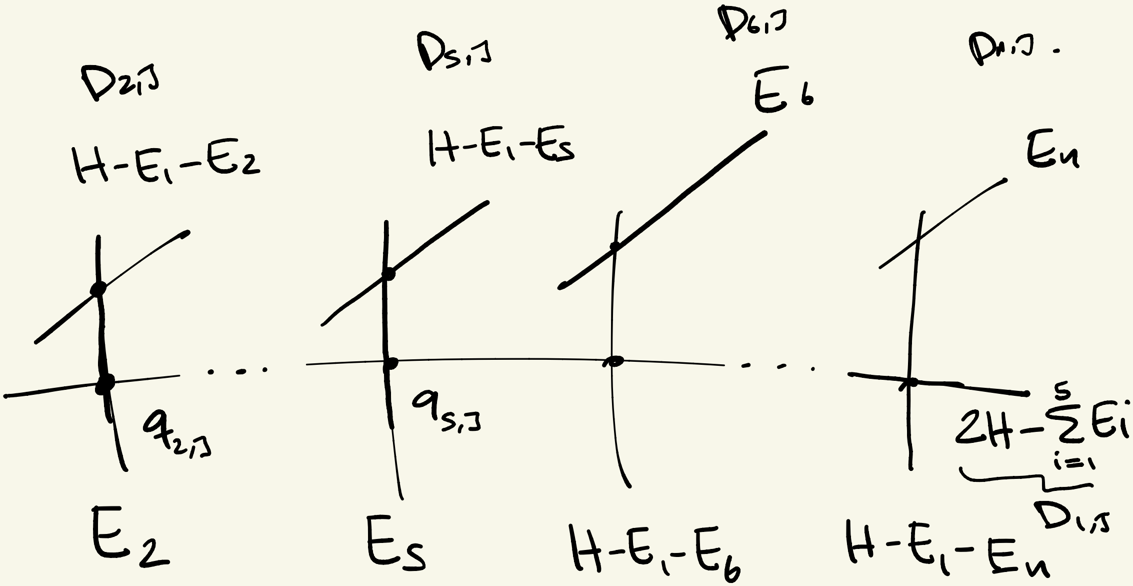
$\forall J \in \mathcal{J}^{reg}$ we have a map:

Gromov compactification

$$\chi_J : X \longrightarrow \mathcal{M}(F, J)$$

$\chi_J(x)$ is the unique J-curve u in class F (possibly nodal) s.t. $x \in \mathrm{im}(u)$.

Choose configuration $D \supsetneq \Sigma$, "compatible" with χ_J.



Obs:

$$\chi_J|_{D_{1,J}} : D_{1,J} \xrightarrow{\cong} \mathcal{M}(F,J)$$

so we have:

$$\pi_J : X \longrightarrow D_{1,J}$$

Where

$$D_{i,J} = \pi_J^{-1}(\{q_{i,J}\}) \quad i=2,\dots,n$$

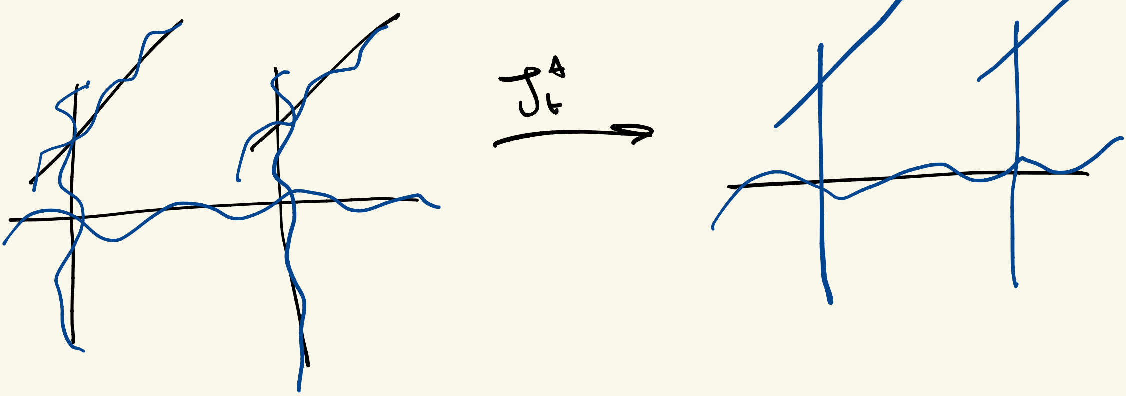
Upshot: while J is not close to $\varphi_* J$

π_J is C^0-close to $\pi_{\varphi_* J}$.

Construct J-isotopy in two steps.

Step A:

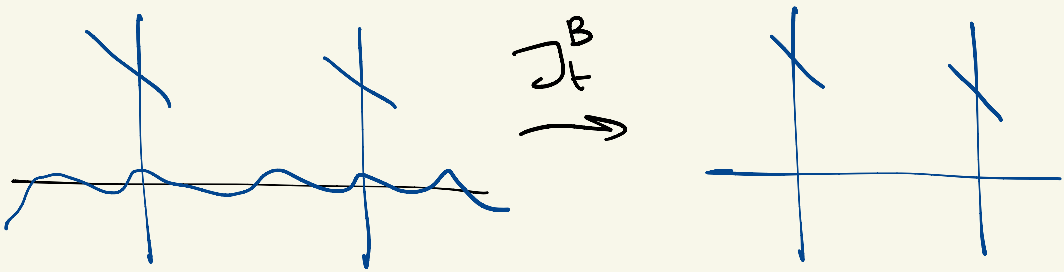
$$\varphi_* J \xrightarrow{J_t^A} J^A, \text{ s.t. } \begin{cases} J_1^A = J, & \text{near } D_{i\geq 2}. \\ J_t^A = \varphi_* J, & \text{far from } D_{i\geq 2} \end{cases}$$



important: by positivity of intersections we can keep D_{i,J_t} "close" to $D_{i,J}$.

Step B:

$$J^A \xrightarrow{J_t^B} J \quad \text{s.t.} \quad J_t^B = J \text{ near } D_{\geq 2}$$



Concatenating φ_A^t and φ_B^t yields φ^t s.t. $\varphi^1 \circ \psi(\Sigma) = \Sigma$.

$$\varphi^1 \circ \psi \in \mathrm{Stab}(\Sigma)$$

Looking at $\pi_J \circ \varphi^t \circ \psi$ it is possible to deduce, by the $\mathcal{C}^0$-control, that

$$[\varphi^1 \circ \psi] = 1 \in \pi_0(\mathrm{Stab}(\Sigma))$$

It is there fore possible to find

$\varphi'' \in \mathrm{Ham}$ s.t. $\varphi'' \circ \psi|_{\Sigma} = \mathrm{id}_{\Sigma}$

$$\Rightarrow \quad \psi \in \mathrm{Ham}$$

Thank you!!