The Number of Periodic Points of Surface Symplectomorphisms

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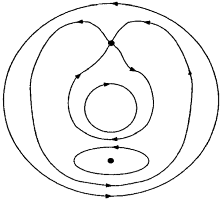
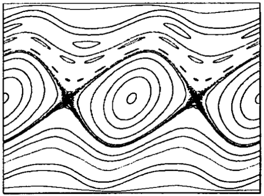
Theorem (Poincaré's last geometric theorem - Poincaré-Birkhoff, 1920)

An area preserving twist map of the annulus must have at least two different fixed points.

- typical twist map on the annulus





- this result can be viewed as the first theorem in global symplectic geometry

Symplectic Manifolds and Symplectic Diffeomorphisms

- symplectic manifold: (M^{2n}, ω), ω is a 2-form
  - nondegenerate: $\omega^n = \underbrace{\omega \wedge \ldots \wedge \omega}_{n \text{ times}} \neq 0$ (or, equiv., $\omega(u, v) = 0$ for all $v \in T_pM \Rightarrow u = 0$)
  - closed: $d\omega = 0$

- examples:
  - standard: $\mathbb{R}^{2n} = \underbrace{\mathbb{R}^n}_{x=(x_1,\ldots,x_n)} \times \underbrace{\mathbb{R}^n}_{y=(y_1,\ldots,y_n)}$, $\omega_0 = dy \wedge dx \, (= \sum_{i=1}^n dy_i \wedge dx_i)$
  - 2-sphere: $(S^2, \text{area form})$
  - complex projective space: $(\mathbb{C}P^n, \omega_{FS})$
  - $2n$-torus: $\mathbb{T}^{2n} = \underbrace{S^1 \times \ldots \times S^1}_{2n \text{ times}} = \underbrace{\mathbb{R}/\mathbb{Z} \times \ldots \times \mathbb{R}/\mathbb{Z}}_{2n \text{ times}} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$
  - surface with genus $g \geq 2$: $(\Sigma_{g\geq 2}, \text{ area form})$

dim $M = 2$

ϕ is a symplectomorphism $(\phi^*\omega = \omega) \Leftrightarrow$ ϕ is an area preserving diffeomorphism

dim $M > 2$

ϕ is a symplectomorphism $(\phi^*\omega = \omega) \Rightarrow$ ϕ is a volume preserving diffeomorphism

$\not\Leftarrow$ (Gromov's non-squeezing theorem '85)

Example

- $H_t \colon M \to \mathbb{R},\ t \in \mathbb{R}/\mathbb{Z};$ $\qquad \omega(X_{H_t}, \cdot) = -dH_t$

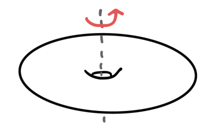
  $\frac{d}{dt}\varphi_H^t = X_H \circ \varphi_H^t$ $\qquad$ Hamiltonian flow
- $\varphi_H := \varphi_H^1$ $\qquad$ Hamiltonian diffeomorphism (time-1 map of φ_H^t)

  $\varphi_H^*\omega = \omega$ $\qquad$ φ_H preserves the symplectic structure

Example (not all symplectomorphisms are Hamiltonian diffeomorphisms)

$\mathbb{T}^2$ $\quad$ horizontal translation by θ

- area preserving
- not a Hamiltonian diffeomorphism



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  $\mathrm{Per}(\varphi_H) = \bigcup_{k \in \mathbb{Z}} \mathrm{Fix}(\varphi_H^k) \leftrightarrow \{$periodic orbits of $X_H\}$ (with integer period)

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Example

$M = \mathbb{C}P^n$, $\#\mathrm{Fix}(\varphi_H) \geq n + 1$

- recall the horizontal translation on the 2-torus: there are no fixed points
- contributions: Eliashberg ('79), Conley-Zehnder ('83),..., Floer ('89),...

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For a "large class" of symplectic manifolds (e.g. surfaces w/$g \neq 0$), $\#\operatorname{Per}(\varphi_H) = \infty$.

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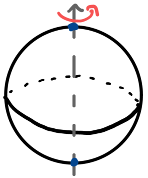

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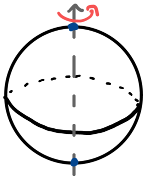

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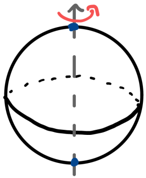

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Theorem (Collier, Kerman, Reiniger, Turmunkh, Zimmer, '12)

If $\varphi \in Ham(S^2)$, then $\#Per(\varphi) = 2$ or ∞. If $Per(\varphi) = \{x, y\}$ then x, y are non-degenerate and

$$\Delta(x) + \Delta(y) \equiv 0 \quad mod\, 4$$

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$\#\mathrm{Fix}(\varphi_H) >$ "A.C." $\Rightarrow \#\mathrm{Per}(\varphi_H) = \infty$

- e.g. $M = \mathbb{C}P^n$, A.C. $= n + 1$;
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Theorem (Ginzburg-Gürel, '14)

For a class of symplectic manifolds (which includes $\mathbb{C}P^n$),
φ_H has a (contractible) hyperbolic fixed point $\Rightarrow \#\mathrm{Per}(\varphi_H) = \infty$

Symplectic Diffeomorphisms

- $\phi\colon M \to M$ symplectomorphism; ϕ^t isotopy s.t. $\phi^0 = \mathrm{id}, \phi^1 = \phi$; $\quad \frac{d}{dt}\phi^t = X_t \circ \phi^t$

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Theorem (variants of Ginzburg-Gürel results; B.'13, B.'15)

For a class of symplectic manifolds (+ conditions on $Flux(\phi)$),
$\phi \in Symp(M,\omega)$ with a hyperbolic fixed point $\Rightarrow \#Per(\phi) = \infty$

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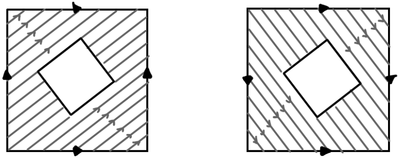
Example (B.'18)

Construction of a symplectic flow on a surface $\Sigma_{g\geq 2}$ with exactly $2g-2$ hyperbolic fixed points and no other periodic orbits.



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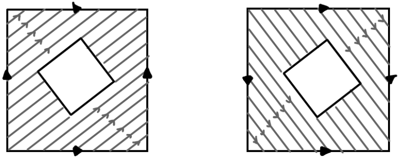

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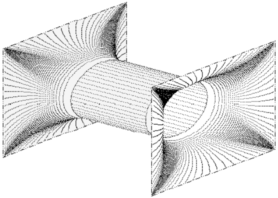

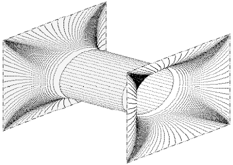

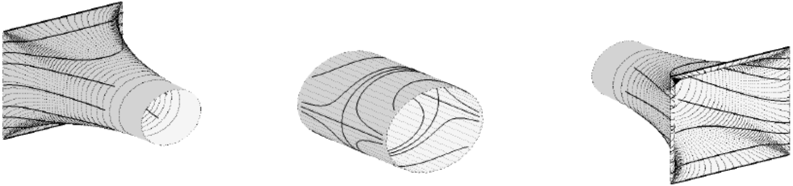

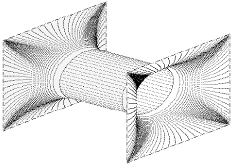

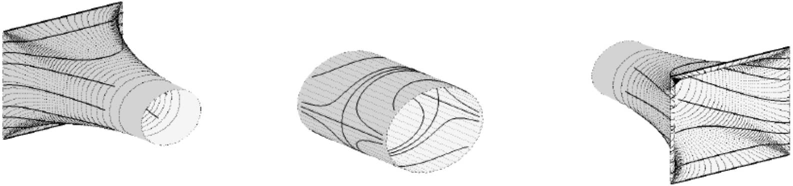

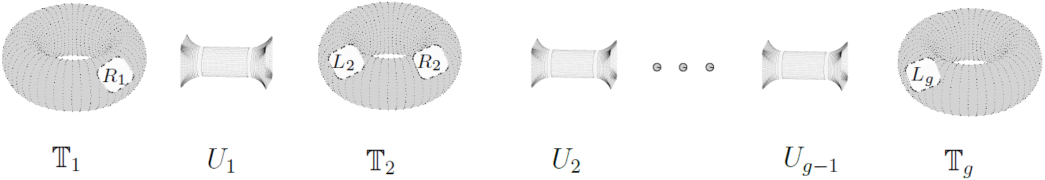
R_1
L_2
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$\mathbb{T}_1$
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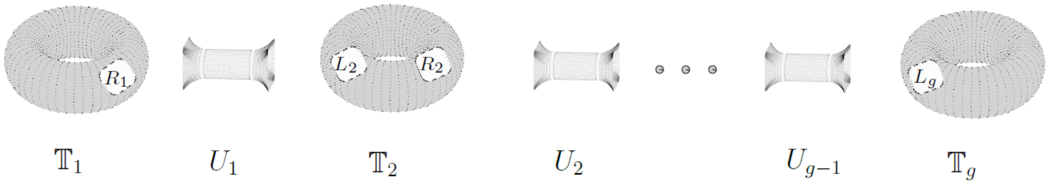

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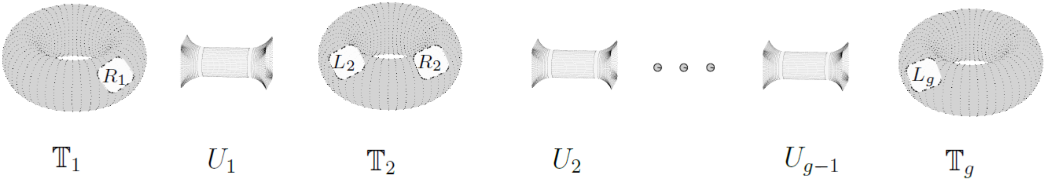

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Theorem (B.'18)

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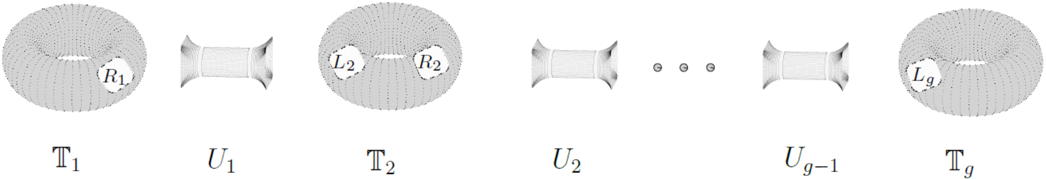

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Theorem (Prasad '25, Guiheneuf - Le Calvez - Passeggi '23)

$\#Per(\phi) = \infty$ if $Flux(\phi) \in H^1(\Sigma; \mathbb{Q})$

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Question 2

Is there a quantitative threshold for $\#\mathrm{Fix}(\phi)$ which forces $\#\mathrm{Per}(\phi) = \infty$?

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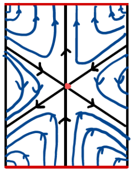

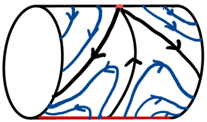

$H \sim x(x + y)(x - y)$

Question 1

$\exists \phi \in \mathrm{Symp}_0(\Sigma, \omega)$ with $\#\mathrm{Per}(\phi) < 2g - 2$?

Example (Atallah - B. - Ferreira, '24)

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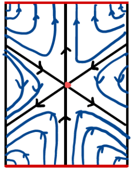

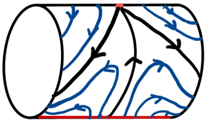

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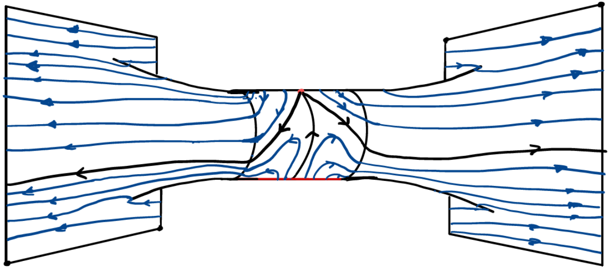


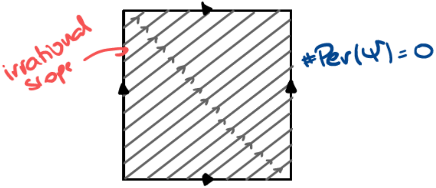
Corollary

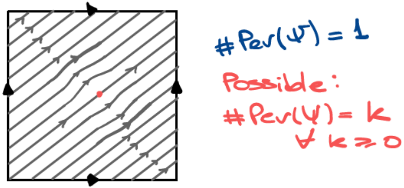
$\exists$ *a symplectic flow with exactly* $k \geq 1$ *fixed points and no other periodic orbits*

Corollary

∃ a symplectic flow with exactly $k \geq 1$ fixed points and no other periodic orbits

- 2-torus





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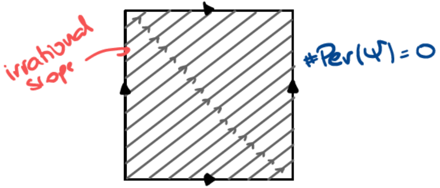
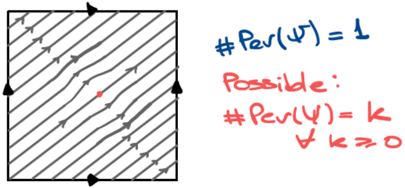
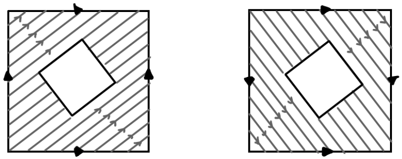
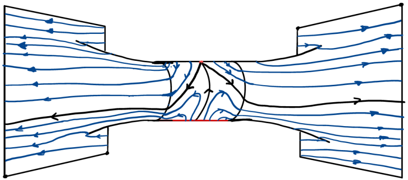




- g-surface, $g \geq 2$





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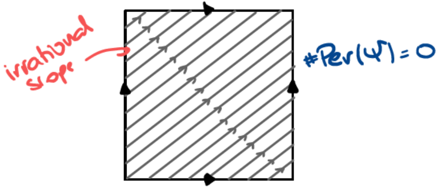

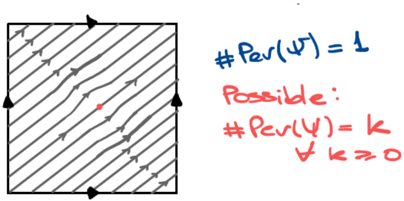

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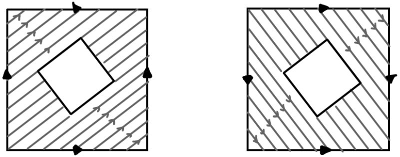
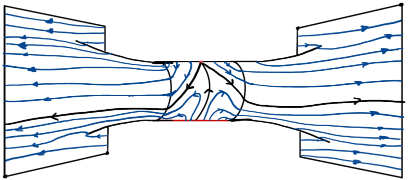




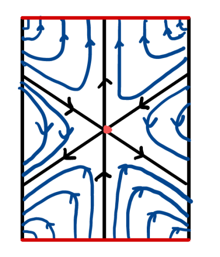
- Remark: $L(\phi, x) = 0$

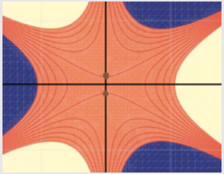
Corollary

$\exists$ a symplectic flow with exactly $k \geq 1$ fixed points and no other periodic orbits

Remark

- by changing H by $H + \varepsilon x$, we obtain a symplectic flow with exactly $2g - 2$ fixed points (each with index -1).





- for every partition

$$\sum_{i=1}^{k} a_i = 2g - 2, \quad 1 \leq a_1 \leq \ldots \leq a_k$$

there is a symplectic flow with k fixed points $x_1, \ldots, x_k$ with $L(\phi, x_k) = -a_i$. (Done for $g = 2, 3$)

Question 2

Is there a quantitative threshold for $\#\mathrm{Fix}(\phi)$ which forces $\#\mathrm{Per}(\phi) = \infty$?

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- Remark: For surfaces, $\dim HF^{\mathrm{loc}}(\phi, x) = |L(\phi, x)|$. If ϕ is non-degenerate then L.H.S. $= \#\mathrm{Fix}(\phi)$.

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$\sum_{x \in \mathrm{Fix}(\varphi_H^t)} \dim HF^{\mathrm{loc}}(\varphi_H, x) > \dim H_*(M) \Rightarrow \varphi_H$ has simple p-p.o. for each suff. large prime p.

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Theorem (variant of the Arnold Conjecture (generic case); Lê-Ono '95)

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Theorem (Atallah - B. - Ferreira '24)

$$\sum_{x \in Fix(\phi^t)} \dim HF^{loc}(\phi, x) > 2g - 2 \Rightarrow \phi \textit{ has simple (contractible) p-p.o. for each suff. large prime p.}$$

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General Statement (Conjecture)

$$\sum_{x \in \mathrm{Fix}(\phi^t)} \dim HF^{\mathrm{loc}}(\phi, x) > \dim HN_*(M, Flux(\phi^t)) \Rightarrow \phi \text{ has simple } p\text{-p.o. for each suff. large prime } p.$$

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Theorem A

Assume $\#\text{Fix}(\phi) < \infty$. If there is $x \in \text{Fix}(\phi)$ such that $\Delta(x) \neq 0$ and $HF^{\text{loc}}(\phi, x) \neq 0$ then there is a simple p-periodic point for suff. large prime p.

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- either there is $x \in \mathrm{Fix}(\phi)$ such that $\Delta(x) \neq 0$ and $HF^{\mathrm{loc}}(\phi, x) \neq 0$
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- x^p contributes non-trivially to HFN_μ with $\mu \notin$ supp HFN_*. $\square$

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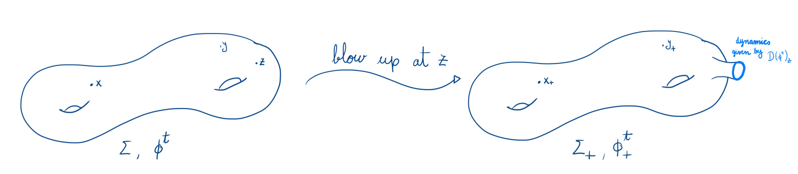

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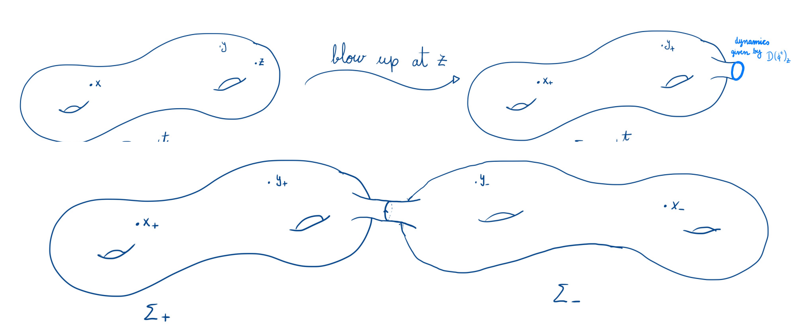

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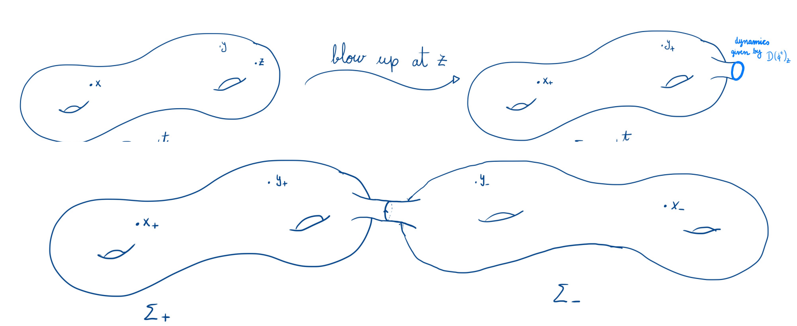

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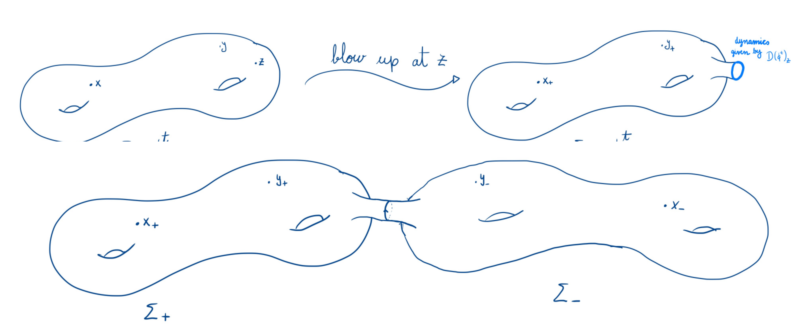

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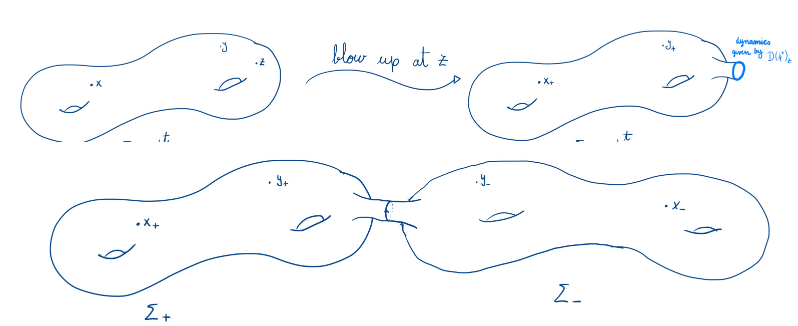

- Remark: $\Delta(z) \in 2\mathbb{Z} \Rightarrow$ there are only possibly new non-contractible p.o.
- cycles representing a non-trivial class of $HF^{\mathrm{loc}}(x_\pm)$ remain non-trivial in HFN. $\square$

proof

Theorem B

If $(\star) \sum_{x \in \mathrm{Fix}(\phi^t)} \dim HF^{\mathrm{loc}}(\phi, x) > 2g - 2$ then

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- Suppose that $\Delta(x) = 0$, for all $x \in \mathrm{Fix}(\phi)$ with $HF^{\mathrm{loc}}(x) \neq 0$
- Hence supp $HF^{\mathrm{loc}}(x) = \{-1, 0, 1\}$ for all $x \in \mathrm{Fix}(\phi)$

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If $(\star)$ $\sum_{x \in \mathrm{Fix}(\phi^t)} \dim HF^{\mathrm{loc}}(\phi, x) > 2g - 2$ then

- either there is $x \in \mathrm{Fix}(\phi)$ such that $\Delta(x) \neq 0$ and $HF^{\mathrm{loc}}(\phi, x) \neq 0$
- or there is a symplectic degenerate extremum (i.e. $x \in \mathrm{Fix}(\phi)$ with $HF_1^{\mathrm{loc}}(\phi, x) \neq 0$ or $HF_{-1}^{\mathrm{loc}}(\phi, x) \neq 0$)

- Suppose that $\Delta(x) = 0$, for all $x \in \mathrm{Fix}(\phi)$ with $HF^{\mathrm{loc}}(x) \neq 0$
- Hence supp $HF^{\mathrm{loc}}(x) = \{-1, 0, 1\}$ for all $x \in \mathrm{Fix}(\phi)$

- $CFN = \oplus_{x \in \mathrm{Fix}(\phi)} HF^{\mathrm{loc}}(x) \overset{(\star)}{\Longrightarrow}$ non-trivial differential

$\Longrightarrow$ there is a k s.t. CFN_k and CFN_{k+1} are non-trivial

$\Longrightarrow$ there is an SDE $\square$

Thank you!