The superstring measure on the moduli spaces of genus-zero super Riemann surfaces (SRSs)

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1. Introduction: the context

(Super)string theory $\qquad Z = \sum_{g \geq 0} \hbar^g \int_{\mathcal{M}_g} d\mu_g$ partition function

$$A_{g,m,n}(V) = \int_{\mathcal{M}_{g,m,n}} V \, d\mu_{g,m,n}$$ scattering amplitudes

m = # of NS punctures
n = # of Ramond punctures

Physicists developed computationally elaborate methods to compute those in low genus g, e.g. D'Hoker–Phong (2002–08). They used ad hoc methods and instead of integrating over $\mathcal{M}_{g,m,n}$, assuming they can use fiberwise integration along a nonexistent projection: $(\mathcal{M}_{g,m,n}) \to (\mathcal{M}_{g,m,n})_{red}$

a classical moduli space

In reality, $(\mathcal{M}_{g,m,n})_{red} \hookrightarrow \mathcal{M}_{g,m,n}$, without a holomorphic retraction $\mathcal{M} \to \mathcal{M}_{red}$.

Witten protested!!! Donagi-Witten: the moduli space is not projected, i.e., no holomorphic retraction $\mathcal{M}_{g,n,0} \to (\mathcal{M}_{g,n,0})_{red}$ for $g \gg n$. Witten: get back to holomorphic/AG methods used in the 80s (Friedan-Shenker, Belavin-Knizhnik, Beilinson-Manin, A. Schwarz, ...)

Seemed to be computationally unrealistic: physicists didn't understand them well enough.

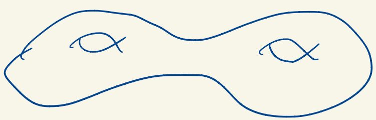
Cacciatori-Grushevsky-V project: develop explicit formulas and computations of scattering amplitudes, starting from low genus $g = 0, 1, \ldots$, and using first principles, i.e., analytic geometry.

2. SRSs & Their Moduli

Classical Riemann surfaces

X (compact) cx. mnfld of dim 1

loc. ringed space, loc. modeled on $(U, \mathcal{O}_{an}|_U)$



$g = 2$

$U \subset \mathbb{C}$ open

$\mathcal{O}_{an}|_U$ sheaf of holom. fncts on U

Super Riemann surfaces (SRSs)

X compact cx. supermnfld of dim $1|1$

locally $(U, \mathcal{O}|_U[\xi])$

$U \subset \mathbb{C}$ open

supercomm. ring ($\Rightarrow \xi^2 = 0$)

$\mathcal{O}(U)[\xi] = \mathcal{O}(U) \oplus \mathcal{O}(U)\xi$

+ "maximally nonintegrable" odd distribution $\mathcal{D}^{0|1} \subset T_X^{1|1}$

(locally $D = \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial z}$, $D^2 = \frac{\partial}{\partial z}$)

$[D,D] = DD + DD = 2\partial/\partial z$

$[\,,\,] : \mathcal{D} \otimes \mathcal{D} \xrightarrow{\sim} T_X / \mathcal{D}$ (think of $\mathcal{D}$ as spinors)

Serre duality $H^i(X, \mathcal{F}) \cong H^{1-i}(X, \omega \otimes \mathcal{F}^*)$

$\omega_X = \Omega^1_X = \{f(z)\,dz\}$ dualizing sheaf

$T_X = \omega_X^*$

$H^i(X, \mathcal{F}) \simeq H^{1-i}(X, \omega \otimes \mathcal{F}^*)$

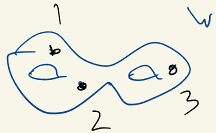
$\omega_X = \mathrm{Ber}(\Omega^1_X) \cong \mathcal{D}^{-1} = \mathcal{D}^*$

rank $\Omega^1_X = 1|1$ (Ber = sdet)

$\Omega^1_X = \{f(z,\xi)\,dz + g(z,\xi)\,d\xi\}$

moduli space $\mathcal{M}_{g,n} = \{ X,$ compact R. surface of genus g with n punctures $\} / \simeq$

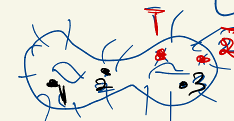
$g = 2$



$\in \mathcal{M}_{2,3}$

$\dim_{\mathbb{C}} = 3g - 3 + n$

$\mathcal{M}_{g,m,n} = \{ X$ cmpct SRS of gen g with m NS and and n Ramond pncts $\} / \simeq$



$\in \mathcal{M}_{2,3,2}$

$\dim_{\mathbb{C}} = 3g - 3 + m + n \mid 2g - 2 + m + \frac{n}{2}$

univ. curve $\mathcal{C}_{g,n} \xrightarrow{\pi} \mathcal{M}_{g,n}$, fiber over $[X]$ is X

universal SRS: $\mathcal{C}_{g,m,n} \xrightarrow{\pi} \mathcal{M}_{g,m,n}$

$\boxed{\text{Set } \mathcal{M} := \mathcal{M}_{g,0} \text{ or } \mathcal{M}_{g,0,0}}$

tangent bundle $\mathcal{T}_{\mathcal{M}}$ on $\mathcal{M}$

$$0 \to \mathcal{T}_{\mathcal{C}/\mathcal{M}} \to \mathcal{T}_{\mathcal{C}} \to \pi^* \mathcal{T}_{\mathcal{M}} \to 0$$

Kodaira-Spencer map: LES of higher direct images

$$\cdots \to \mathcal{T}_{\mathcal{M}} \xrightarrow{\sim} R^1 \pi_* \mathcal{T}_{\mathcal{C}/\mathcal{M}} \to \cdots$$

$R^1 \pi_* \mathcal{T}_{\mathcal{C}/\mathcal{M}}$ v. bundle over $\mathcal{M}$ with fiber $H^1(X, \mathcal{T}_X)$

$\mathcal{T}_{\mathcal{M}_{g,m,n}}$ on $\mathcal{M}_{g,m,n}$

$$0 \to \mathcal{T}^{sc}_{\mathcal{C}/\mathcal{M}} \to \mathcal{T}^{sc}_{\mathcal{C}} \to \pi^* \mathcal{T}_{\mathcal{M}} \to 0$$

super Kodaira-Spencer map

$$\cdots \to \mathcal{T}_{\mathcal{M}} \xrightarrow{\sim} R^1 \pi_* \mathcal{T}^{sc}_{\mathcal{C}/\mathcal{M}} \to \cdots$$

Fact: $\mathcal{T}^{sc}_{\mathcal{C}/\mathcal{M}} \simeq \mathcal{D}^{\otimes 2}$

$$\mathcal{T}_{\mathcal{M}} \xrightarrow{\sim} R^1 \pi_* (\mathcal{D}^{\otimes 2})$$

Dualizing sheaf on $\mathcal{M}$:

$$\omega_{\mathcal{M}} = \det T^*_{\mathcal{M}} \overset{KS}{=} (\det R^1\pi_* T_{\mathcal{C}/\mathcal{M}})^* \underset{\text{Serre}}{=} \det \pi_* \omega^{\otimes 2}_{\mathcal{C}/\mathcal{M}} := \lambda_2$$

sheaf of holom. volume forms on $\mathcal{M}$

det. line bundle

$$T^*_{\mathcal{C}/\mathcal{M}} = \Omega^1_{\mathcal{C}/\mathcal{M}} = \omega_{\mathcal{C}/\mathcal{M}}$$

$$\lambda_k := \det R^{\cdot}\pi_* \omega^{\otimes k}_{\mathcal{C}/\mathcal{M}}$$

$$\omega_{\mathcal{M}} = \operatorname{Ber} T^*_{\mathcal{M}} \overset{KS/\text{Serre}}{=} \operatorname{Ber} \pi_* \omega^{\otimes 3}_{\mathcal{C}/\mathcal{M}} =: \lambda_{3/2}$$

Berezinian line bdle

$$\lambda_{k/2} := \det R^{\cdot}\pi_* \omega^{\otimes k}_{\mathcal{C}/\mathcal{M}}$$

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measure on $\mathcal{M}$:

$\varphi \wedge \bar\varphi$, $\varphi \in \Gamma(\mathcal{M}, \lambda_2) \rightsquigarrow \int_{\mathcal{M}} \varphi \wedge \bar\varphi$ makes sense

$\varphi \wedge \bar\varphi$, $\varphi \in \Gamma(\mathcal{M}, \lambda_{3/2})$

Mumford isom-m $\quad \lambda_2 \simeq \lambda_1^{\otimes 13}$

not canonical

$\rightsquigarrow \mu \in \Gamma(\lambda_2 \otimes \lambda_1^{-13})$, defined up to constant

Mumford form

string measure on $\mathcal{M}$: $\quad \dfrac{\mu \wedge \bar\mu}{(\det \bar Q)^{13}}$, where

$$Q : H^0(X, \omega_X) \otimes H^0(X, \omega_X) \to \mathbb{C}; \quad \det Q : \lambda_1 \otimes \bar\lambda_1 \to \mathcal{O}_{\mathcal{M}}$$

$$Q(\alpha, \beta) := \int_X \alpha \wedge \bar\beta$$

super Mumford isom-m

$$\lambda_{3/2} = \lambda_{1/2}^{\otimes 5}$$

canonical

super Mumford form $\mu \in \Gamma(\lambda_{3/2} \otimes \lambda_{1/2}^{-5})$

Thm! ('88, Deligne, V)

superstring measure on $\mathcal{M}$

$$\frac{\mu \wedge \bar\mu}{(\operatorname{Ber} \bar Q)^5}$$

($\bar Q$ is a mystery)

String partition function

$$Z_g = \int_{\mathcal{M}_g} \frac{\mu \wedge \bar{\mu}}{(\det \bar{Q})^{13}}$$

Belavin-Knizhnik theorem

Superstring partition function

$$Z_g = \int_{\mathcal{M}_g} \frac{\mu \wedge \bar{\mu}}{(\operatorname{Ber} \bar{Q})^{5}}$$

3. Supermoduli Space $\mathcal{M}_{g,n,0}$ with n NS punctures

$$A_{g,n} = \int_{\mathcal{M}_{g,n,0}} \left\langle \prod_{j=1}^{n} V_j \right\rangle \nu \wedge \bar{\nu} \quad , \quad V_j = \frac{\text{vertex-operator insertion}}{\text{(VOI)}}$$

Still have $\mu \in \lambda_{3/2} \otimes \lambda_{1/2}^{-5}$.

s, Mumford form

Issue:

$$\operatorname{Ber} R'\pi_*(\omega^{\otimes 3}_{\mathcal{C}/\mathcal{M}}) \neq \omega_{\mathcal{M}_{g,n,0}} \text{ for } n>0$$

In fact, $\operatorname{Ber} \omega_{\mathcal{M}_{g,n,0}} \cong \operatorname{Ber} R\pi_*(\omega^{\otimes 3}(NS))$, NS is the divisor on $\mathcal{C}_{g,n,0}$ corresponding to n NS punctures, $\deg NS = n$.

So, $\mu \in \omega_{\mathcal{M}_{g,n,0}} \otimes \left(\mathrm{Ber}\, R\pi_*\left(\omega^3_{\mathcal{C}/\mathcal{M}}(NS)\big|_{NS}\right)\right)^{-1} \otimes \lambda_{1/2}^{-5}$

$$\left(\begin{array}{l} 0 \to \omega^3_{\mathcal{C}/\mathcal{M}} \to \omega^3_{\mathcal{C}/\mathcal{M}}(NS) \to \omega^3_{\mathcal{C}/\mathcal{M}}(NS)\big|_{NS} \to 0 \\ \Rightarrow \underbrace{\mathrm{Ber}\, R\pi_* \omega^3_{\mathcal{C}/\mathcal{M}}}_{\lambda_{3/2}} \otimes \mathrm{Ber}\, R\pi_*\left(\omega^3_{\mathcal{C}/\mathcal{M}}(NS)\big|_{NS}\right) \cong \underbrace{\mathrm{Ber}\, R\pi_*\left(\omega^3_{\mathcal{C}/\mathcal{M}}(NS)\right)}_{\omega_{\mathcal{M}_{g,n,0}}} \end{array}\right)$$

<u>Fact</u>: Vertex-operator insertions V_j come from $\mathrm{Ber}\, R\pi_*\left(\omega_{\mathcal{C}/\mathcal{M}}\big|_{NS}\right)$

<u>Prop</u>: $\mathrm{Ber}\, R\pi_*\left(\omega_{\mathcal{C}/\mathcal{M}}\big|_{NS}\right) = \mathrm{Ber}\, R\pi_*\left(\omega^3_{\mathcal{C}/\mathcal{M}}(NS)\big|_{NS}\right)$

Thus, $$\int_{\mathcal{M}_{g,n,0}} \frac{\left(\prod_{j=1}^{n} V_j\right) \mu \wedge \bar{\mu}}{(\det \operatorname{Im} Q)^5} =: A_{g,n,0}$$ makes sense, and this is a <u>scattering amplitude</u>

$A_{g,n,0} / Z_g = \frac{\text{probability of scattering } n \text{ superstrings through}}{\text{a genus } g \text{ SRS}}$

4. Genus 0, $n \geq 3$ NS punctures

Simplifications: an SRS of genus 0 is (up to isom) $X = \mathbb{C}P^{1|1}$ with odd distribution $\mathcal{D}$ generated by $\frac{\partial}{\partial \zeta} + \zeta \frac{\partial}{\partial z}$, where $(z|\zeta)$ coordinates on the main chart $\mathbb{C}^{1|1} \subset \mathbb{C}P^{1|1}$

$$\mathcal{M}_{0,n,0} = \mathrm{Conf}_n(\mathbb{C}P^{1|1}) / POSp(1|2)$$

$\dim_{\mathbb{C}} \mathcal{M}_{0,n,0} = (n-3|n-2)$, affine, has global coordinates $(z_4, \dots, z_n | \zeta_3, \dots, \zeta_n)$

For $X = \mathbb{C}P^{1|1}$:

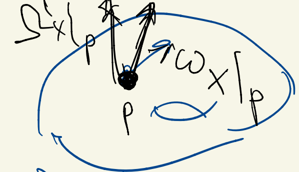
$$\lambda_{1/2} = \mathrm{Ber}\, H^\bullet(X, \omega_X) = \mathrm{Ber}\, H^0(X, \omega_X) \otimes \left(\mathrm{Ber}\, H^1(X, \omega_X)\right)^{-1}$$

$$= \mathrm{Ber}(0) \otimes \mathrm{Ber}(H^0(X, \mathcal{O}_X)) \cong \mathbb{C} \otimes \mathbb{C} = \mathbb{C}$$

thus, no need to worry about Q, $\overline{Q}$, $(\mathrm{Ber}\, \overline{Q})^5$, etc.

Theorem: Given sections V_j of $\mathrm{Ber}\, H^0(X, \omega_X \| NS)$ (interpreted as VOIs) for $j = 1, \dots, n$, the super Mumford form μ gives a holomorphic volume form on $\mathcal{M}_{0,n,0}$:

$$\left(\prod_{j=1}^{n} V_j(z_j \mid \xi_j)\right) \nu \quad , \text{ where}$$



$$\nu = -\frac{1}{2^{n-2}} \left(z_3 - z_1 - \tfrac{1}{2}\xi_3\xi_1\right)\left(z_3 - z_2 - \tfrac{1}{2}\xi_3\xi_2\right)[dz_4 \dots dz_n \mid d\xi_3 \dots d\xi_n]$$

$z_i = z_i^0$ for $i = 1, 2, 3.$ and $\xi_i = \xi_i^0$ for $i = 1, 2$.

<u>Cor.</u> $$A_{0,n,0} = \int_{\mathcal{M}_{0,n,0}} \left\langle \prod_{j=1}^{n} V_j(z_j, \bar z_j \mid \xi_j, \bar\xi_j) \right\rangle \nu \wedge \bar\nu \, .$$

5. Genus 0, $n \geq 4$ Ramond punctures

An SRS with n Ramond punctures is neither an SRS, nor with punctures!

$(X^{1|1}, \mathcal{D})$, $\mathcal{D}^{0|1} \subset T_X^{1|1}$, but $[-,-]: \mathcal{D} \otimes \mathcal{D} \to T/\mathcal{D}$ is not an isom-m, but gives an isom-m $\mathcal{D} \otimes \mathcal{D} \xrightarrow{\sim} T/\mathcal{D}(-R) \hookrightarrow T/\mathcal{D}$