USC Dornsife
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College of Letters, Arts and Sciences

Nonsemisimple Topological Quantum Computation

Topological Quantum Field Theory Club

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August 13th, 2025

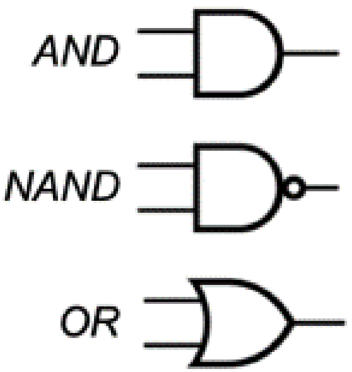
Classical versus Quantum Computation

Classical Computation

- Classical bits 0, 1
- Computes functions

$$f : \{0,1\}^n \rightarrow \{0,1\}$$

- Basic Logic Gates



X	Y	OUT
0	0	0
0	1	0
1	0	0
1	1	1

X	Y	OUT
0	0	0
0	1	0
1	0	0
1	1	1

X	Y	OUT
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1	0	0
1	1	1

- Universal computation with AND, OR, NOT

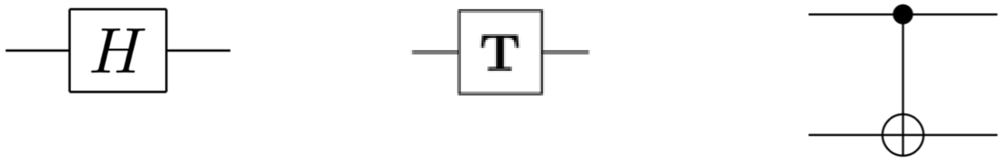
Quantum Computation

- Basis vectors $|0\rangle$ and $|1\rangle$ of a 2-dimensional Hilbert space $\mathcal{H}$
- Computation performed by unitary transformation $U^\dagger = U^{-1}$

$$U : \mathcal{H} \rightarrow \mathcal{H}$$

- Basic logic: a choice of certain matrices that can be physically implemented

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \quad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix} \quad \mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$



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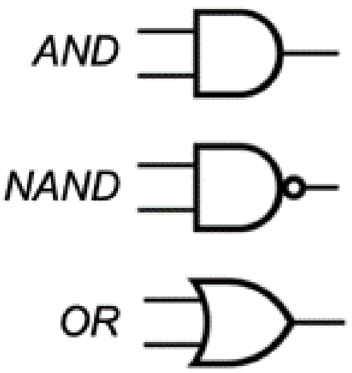

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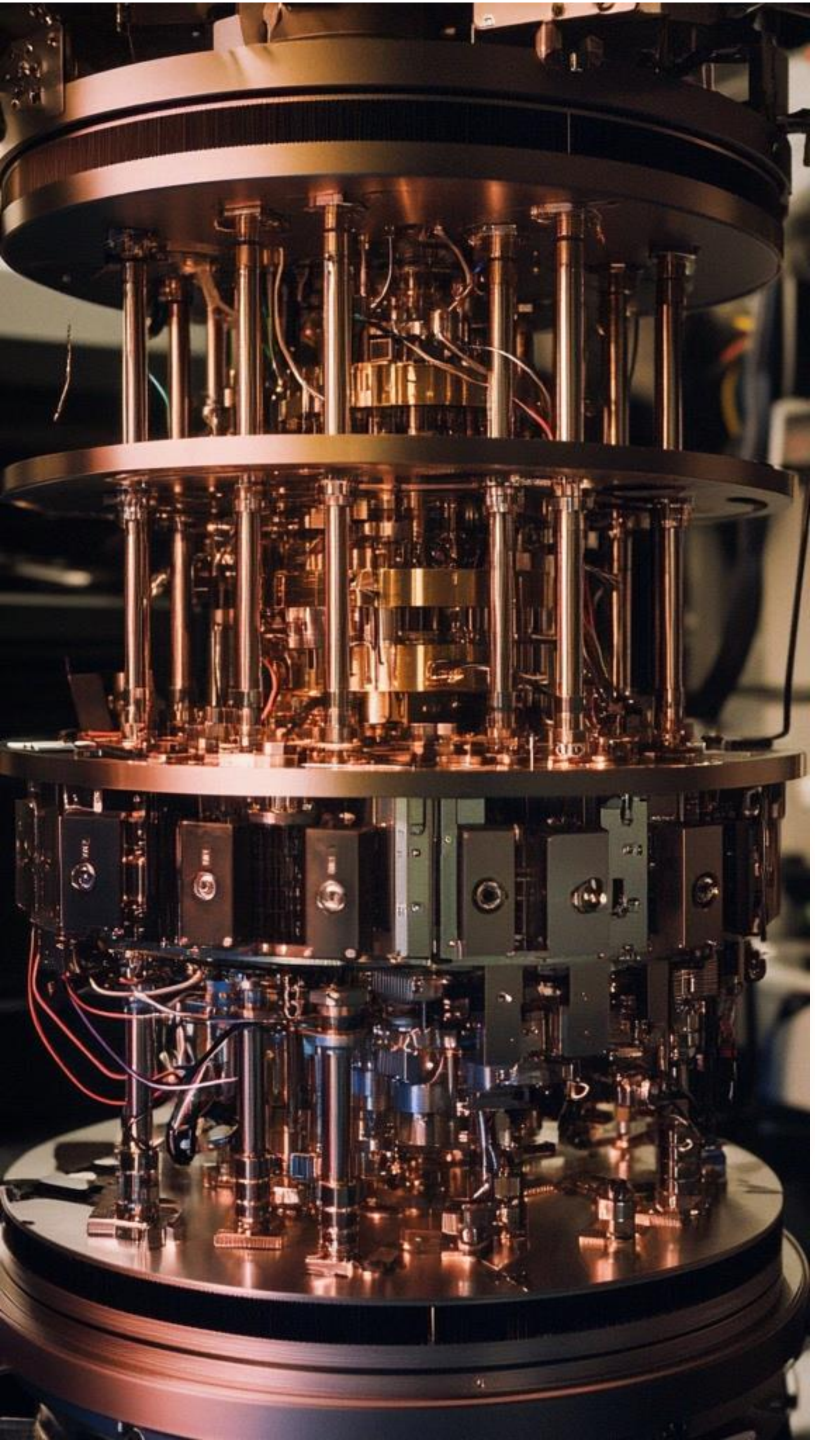
- Basic logic: a choice of certain matrices that can be physically implemented
- Universal quantum computer if we can approximate any U to arbitrary accuracy using basic gates



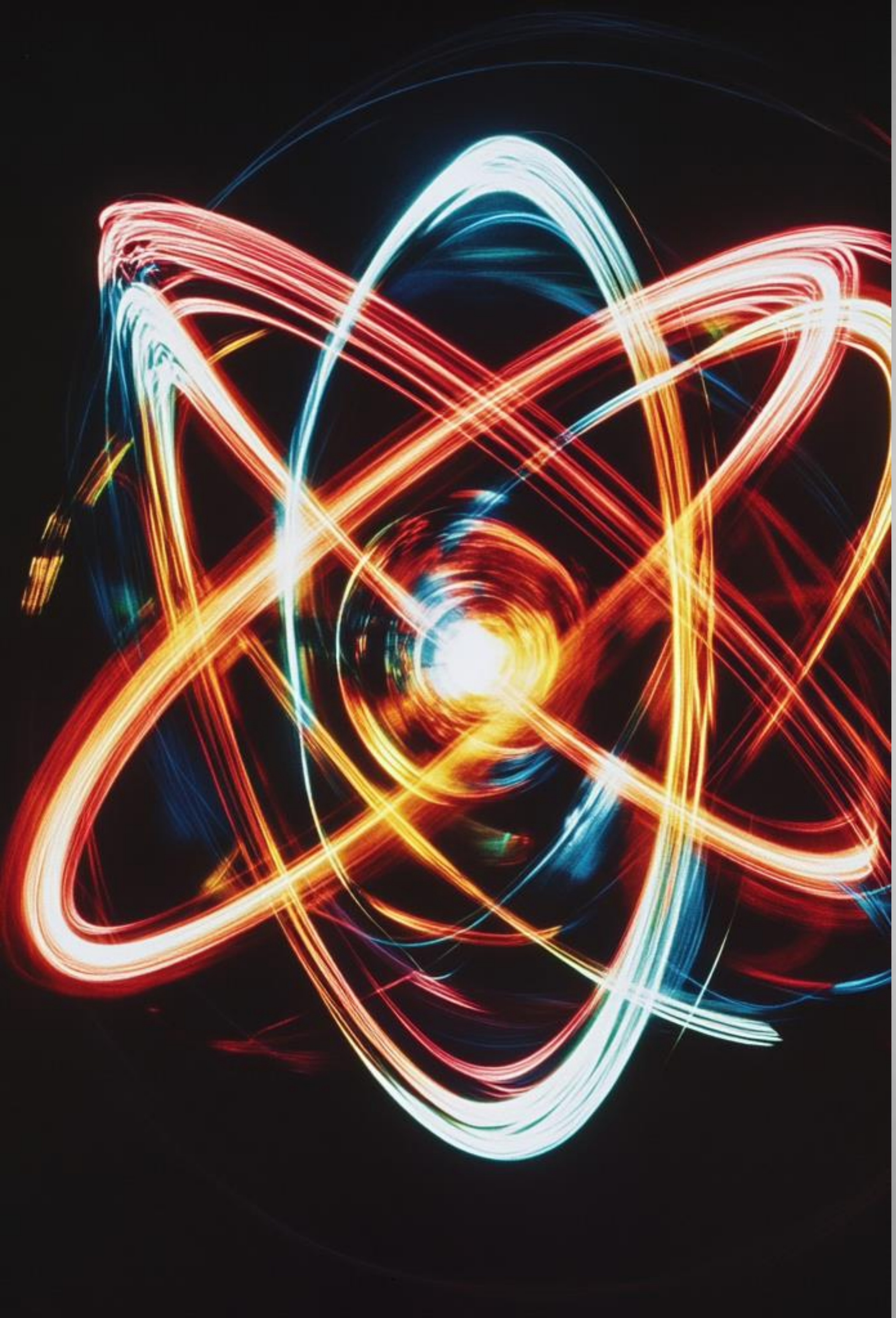
Advantages of quantum computation

- Unlike classical bits that are either 0 or 1, **qubits** exist in a superposition of both 0 and 1 simultaneously

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad \text{where } |\alpha|^2 + |\beta|^2 = 1$$

- **Superposition** allows a quantum computer to process multiple possible outcomes of a computation at once. This exponentially increases the power of quantum computers for certain tasks compared to classical systems.

- Shor's algorithm for factoring large numbers and Grover's search algorithm demonstrate the potential power of quantum computers



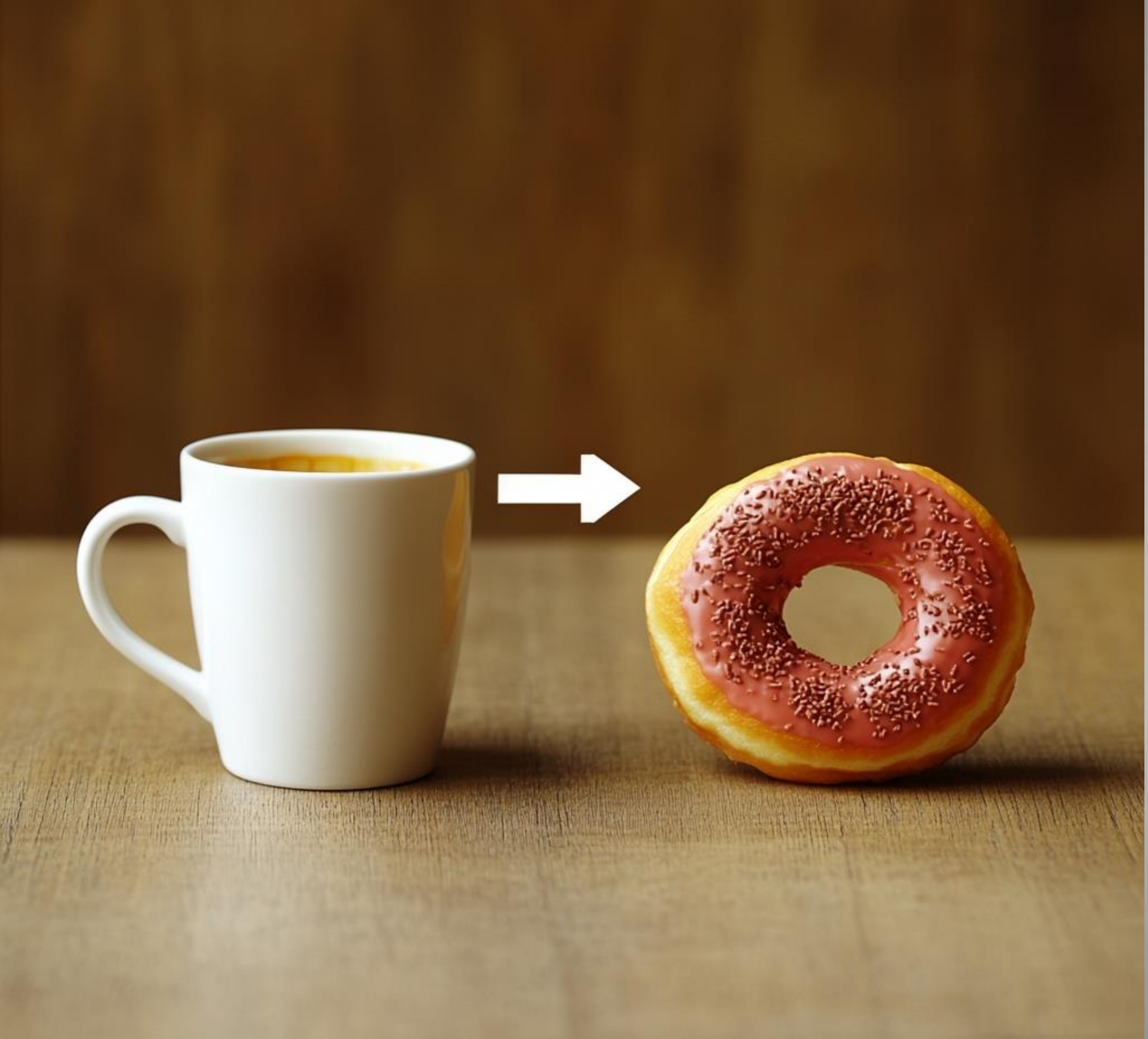
Challenges in Quantum Computing

Quantum Decoherence

One of the main challenges in quantum computing is **decoherence**, where the quantum system interacts with its environment, causing the delicate quantum states (superposition and entanglement) to collapse into classical states. This leads to errors in computation.

Use topology to get fault-tolerant quantum computation

Alexei Kitaev

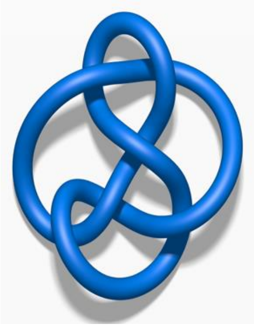


Topology in Quantum Computation

LEVERAGE TOPOLOGY TO OVERCOME DECOHERENCE

Quantum Algorithms for the Jones Polynomial

Computing Jones polynomials



$$-q^3 + 2q^2 - 2q + 3 - 2q^{-1} + 2q^{-2} - q^{-3}$$

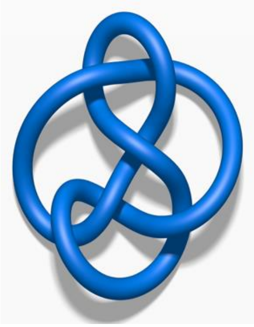
Approximating the Jones polynomial at certain roots of unity is as hard as the hardest problem that can be efficiently solved by a quantum computer

Mike Freedman Michael Larsen Zhenghan Wang

Quantum Algorithms for the Jones Polynomial

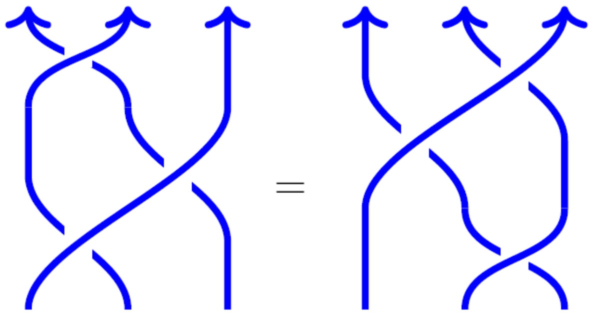
Computing Jones polynomials

Prime Factorization



$$-q^3 + 2q^2 - 2q + 3 - 2q^{-1} + 2q^{-2} - q^{-3}$$

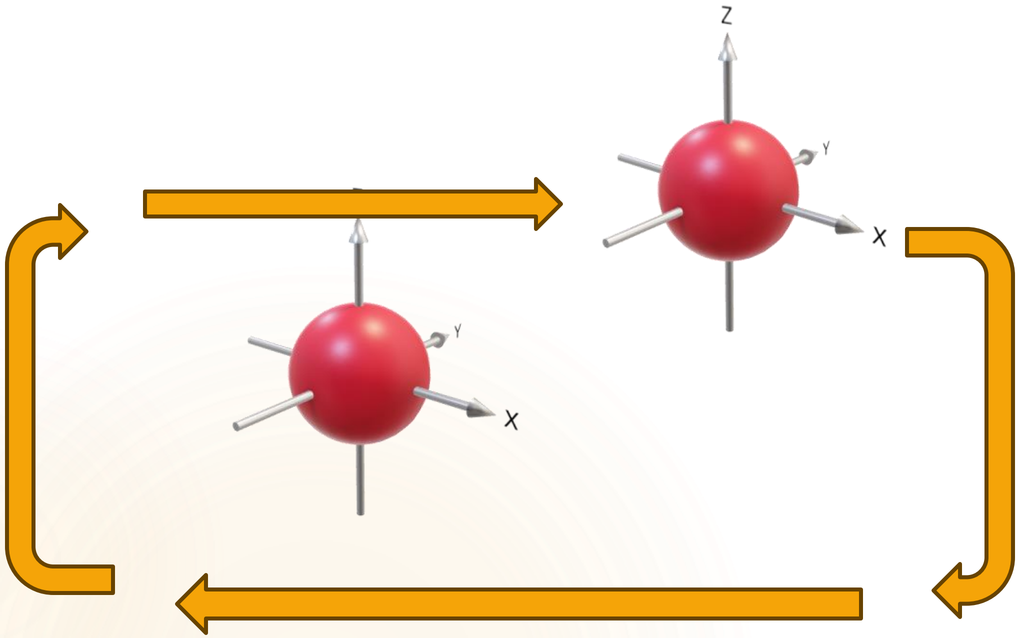
1389864140741
= 1178803 x 1179047



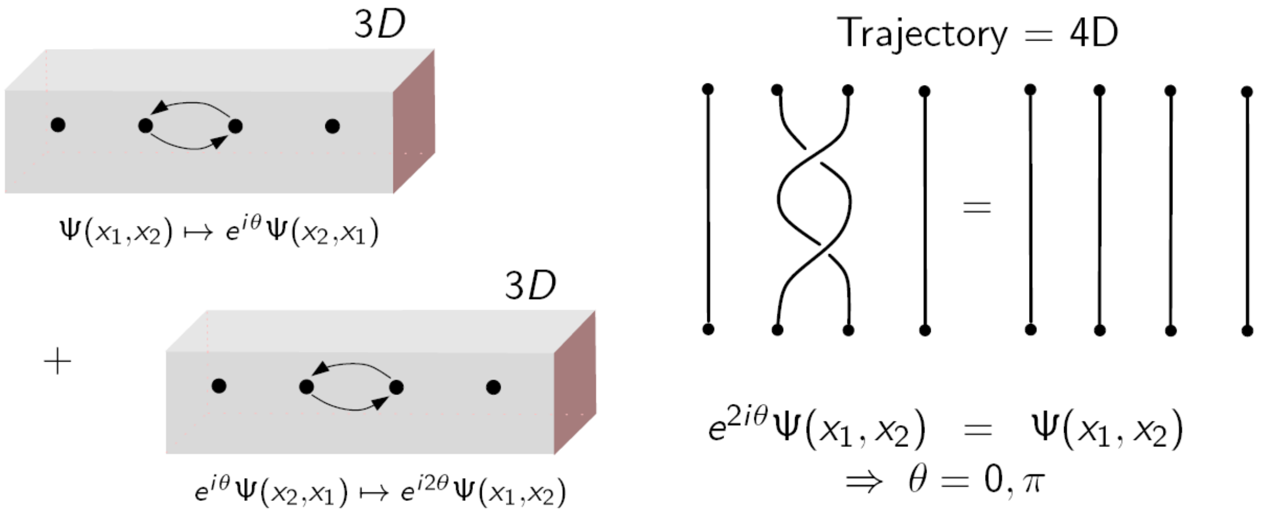
Braiding in physical systems

ANYONS IN TWO-DIMENSIONAL SYSTEMS

Particles in 3D



Path deforms continuously to constant path



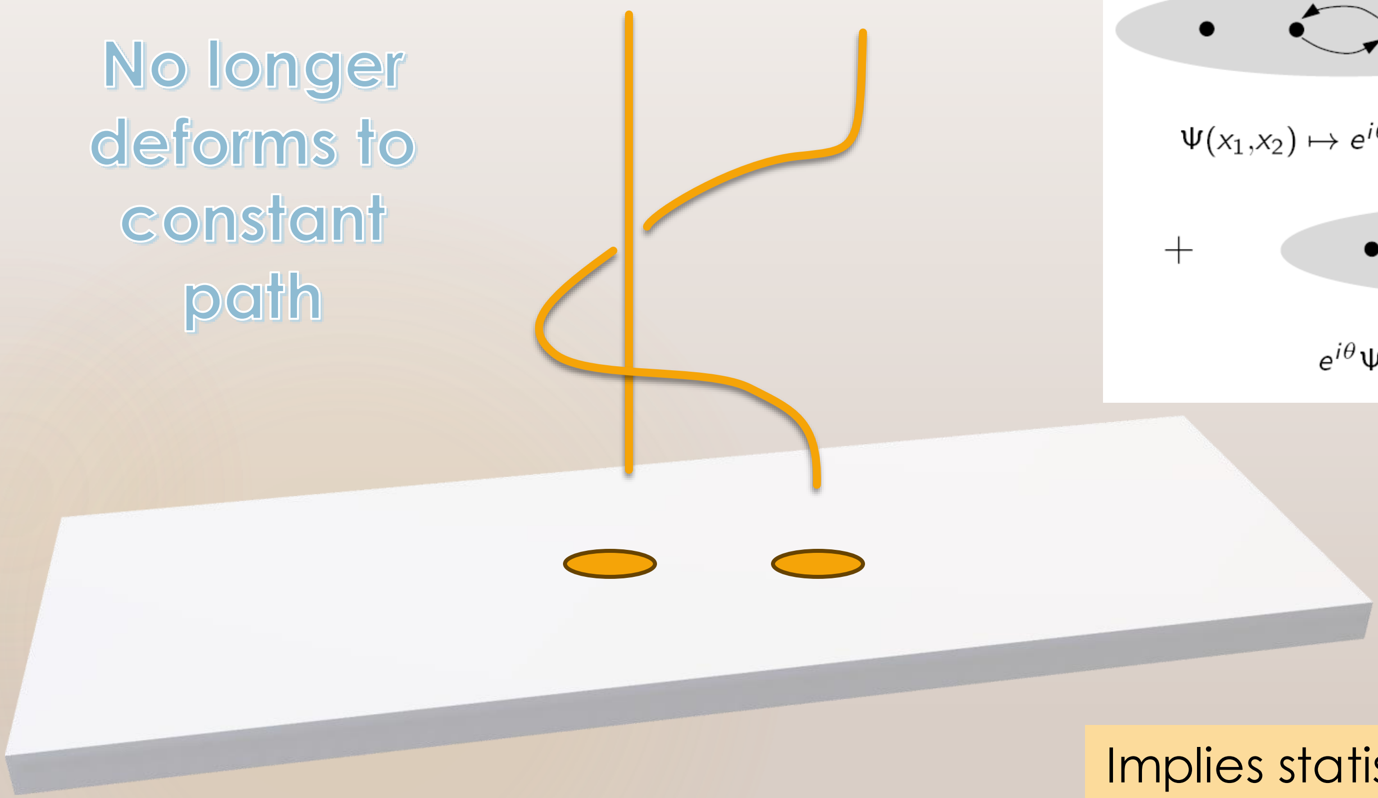
Implies two types of point particle statistics in 3D

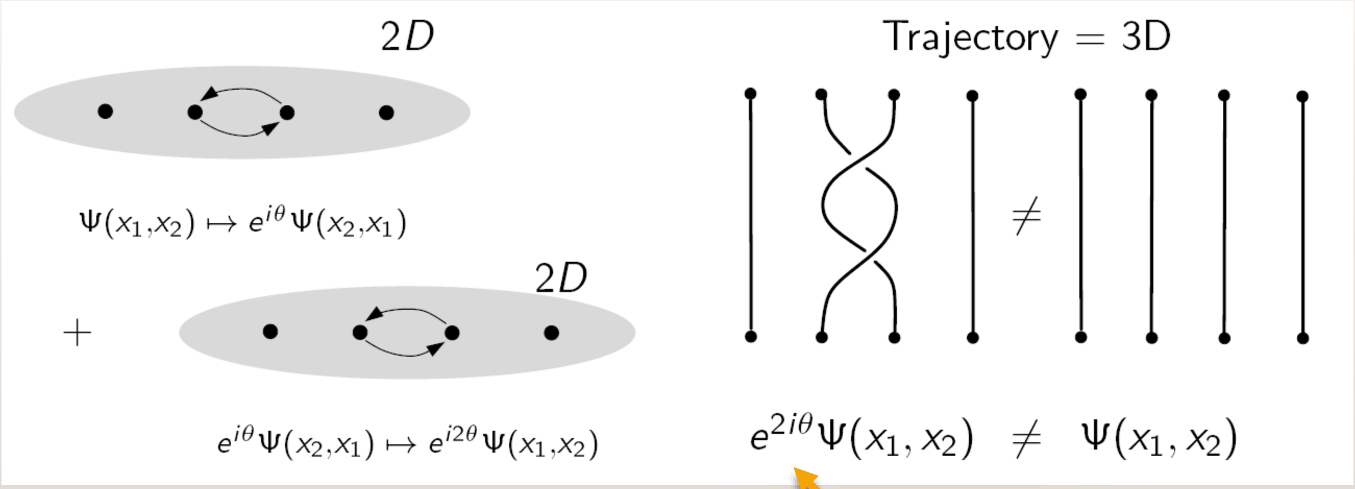
$\theta = 0, \Psi(x_1,x_2) = \Psi(x_2,x_1)$ ➔ Bosons

$\theta = \pi, \Psi(x_1,x_2) = -\Psi(x_2,x_1)$ ➔ Fermions

Particles in 2D

No longer deforms to constant path





Can be a nontrivial phase!

Implies statistics can change by an arbitrary phase

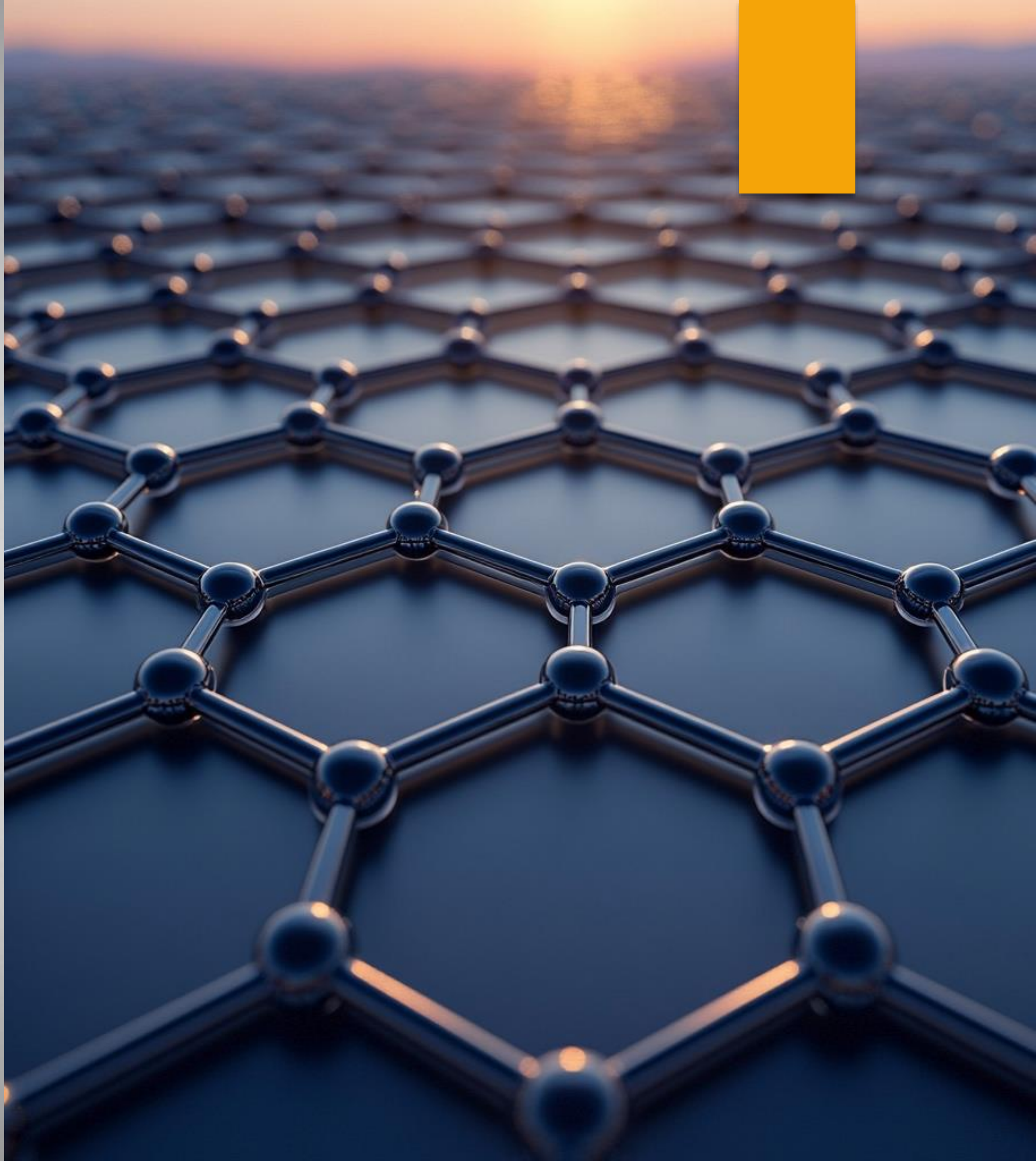
$\Psi(x_1,x_2) = e^{i\theta}\Psi(x_2,x_1)$ → Anyons (abelian)

Or more generally

$\Psi(x_1,x_2) = U\,\Psi(x_2,x_1)$ → non abelian Anyons

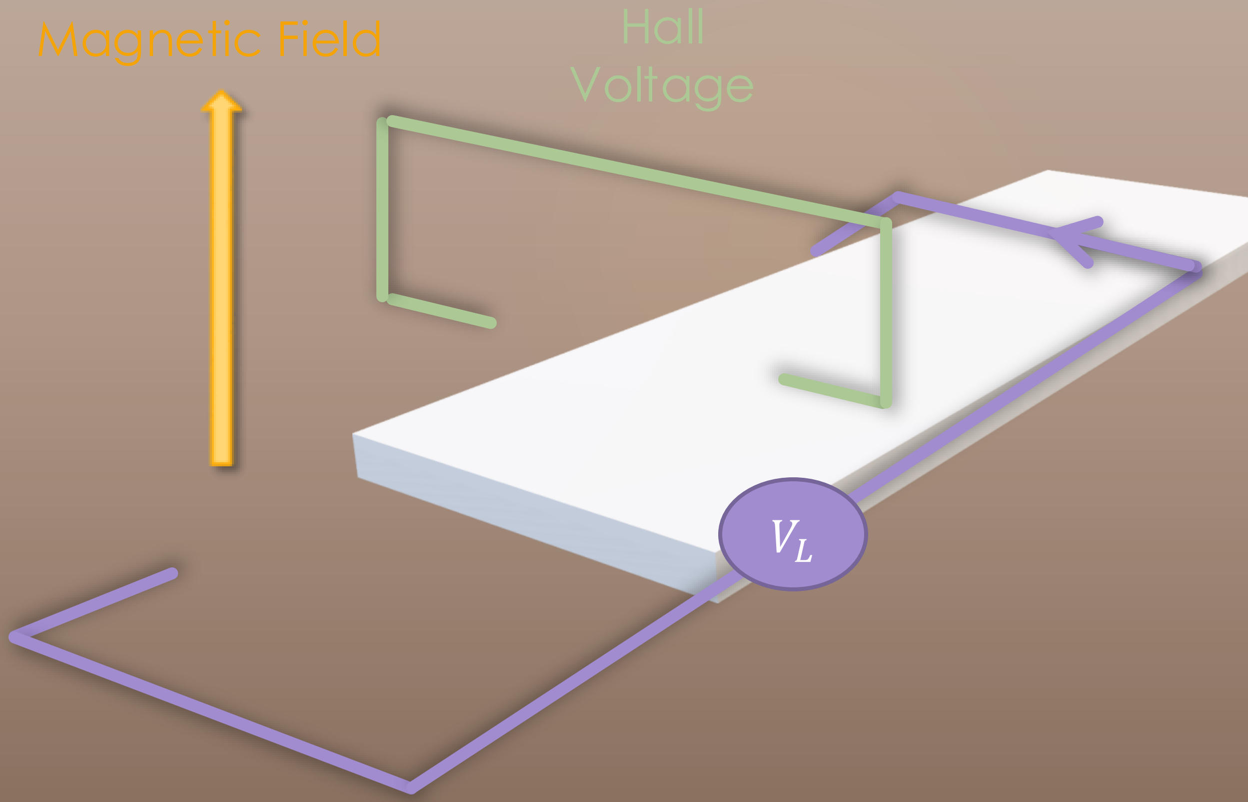
Need physical system where braiding relates to the Jones polynomials

CHERN SIMONS THEORY IN MANY BODY PHYSICS

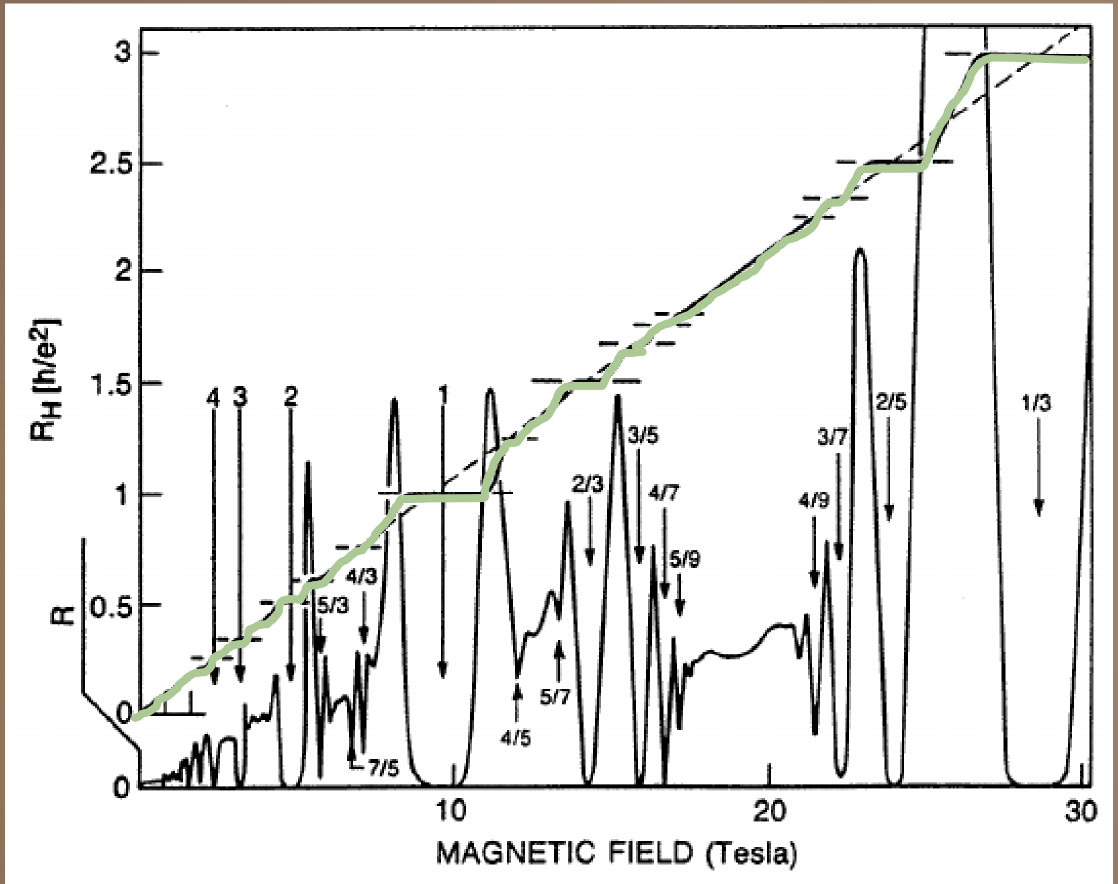


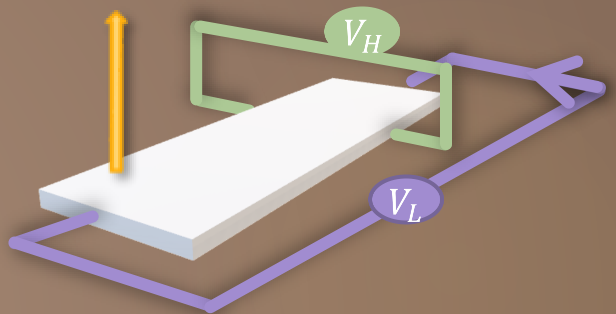
Quantum Hall Effect

- 2-dimensional electron system
- Low temperature
- High Magnetic Field
- High purity/low disorder
- Depends on a parameter called the filling fraction ν

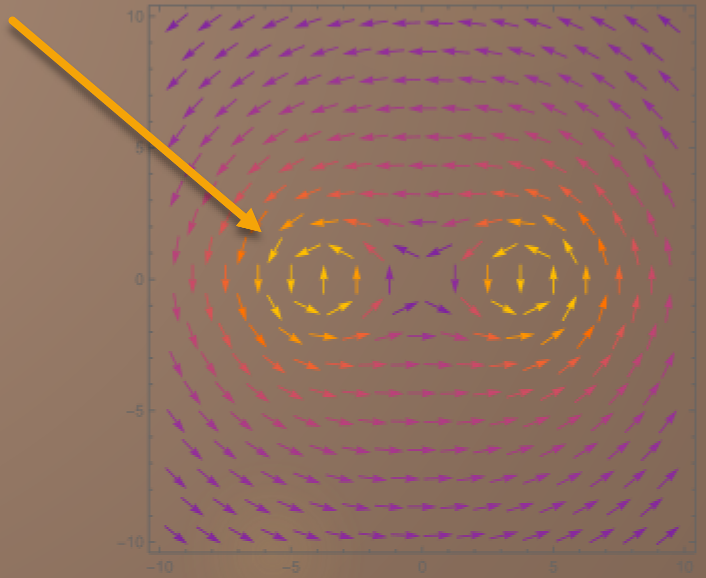


Quantum Hall Effect

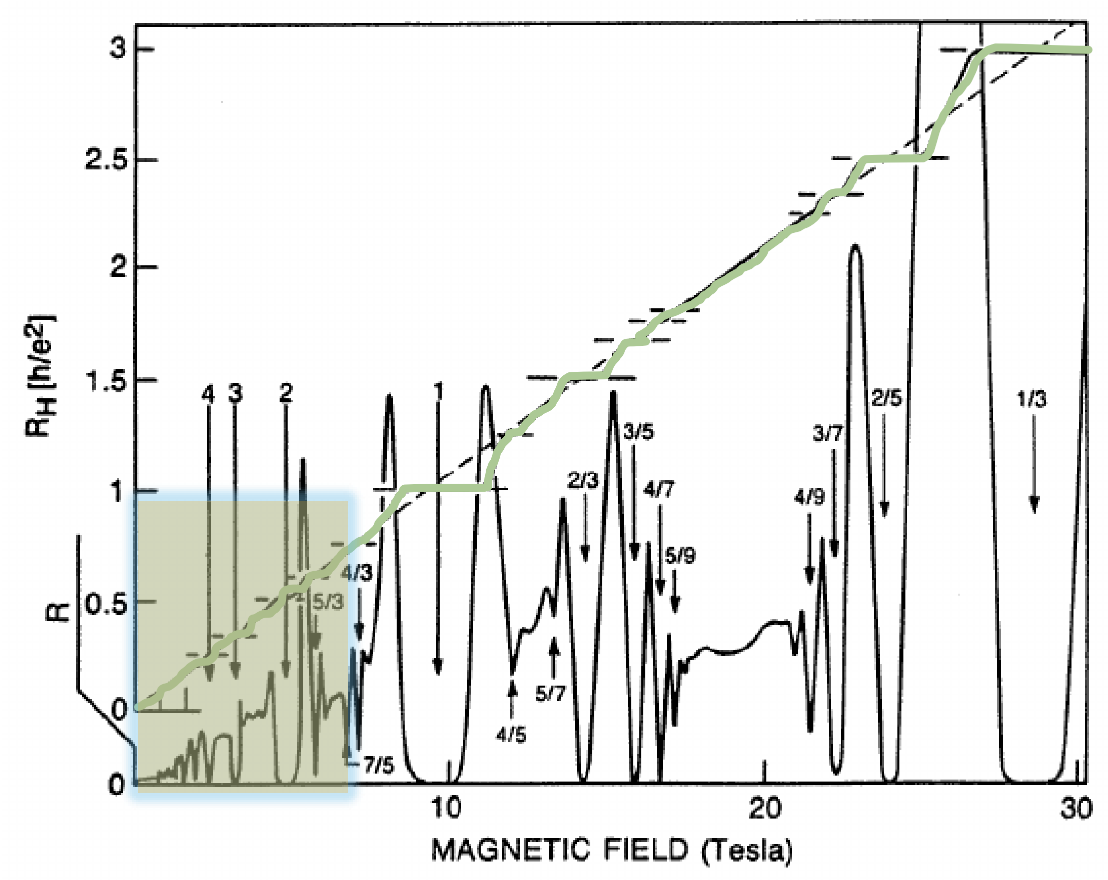


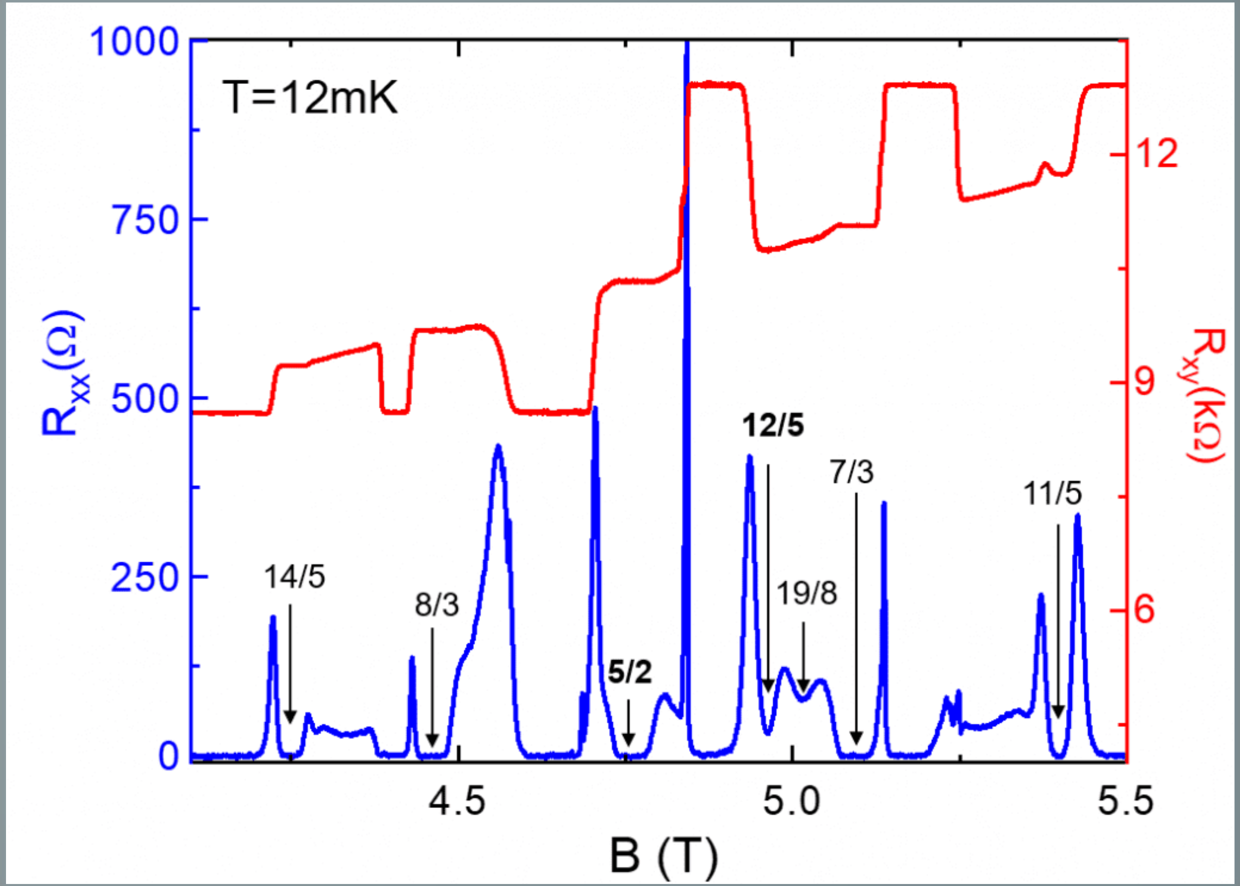


Vortices = anyons



Quantum Hall Effect

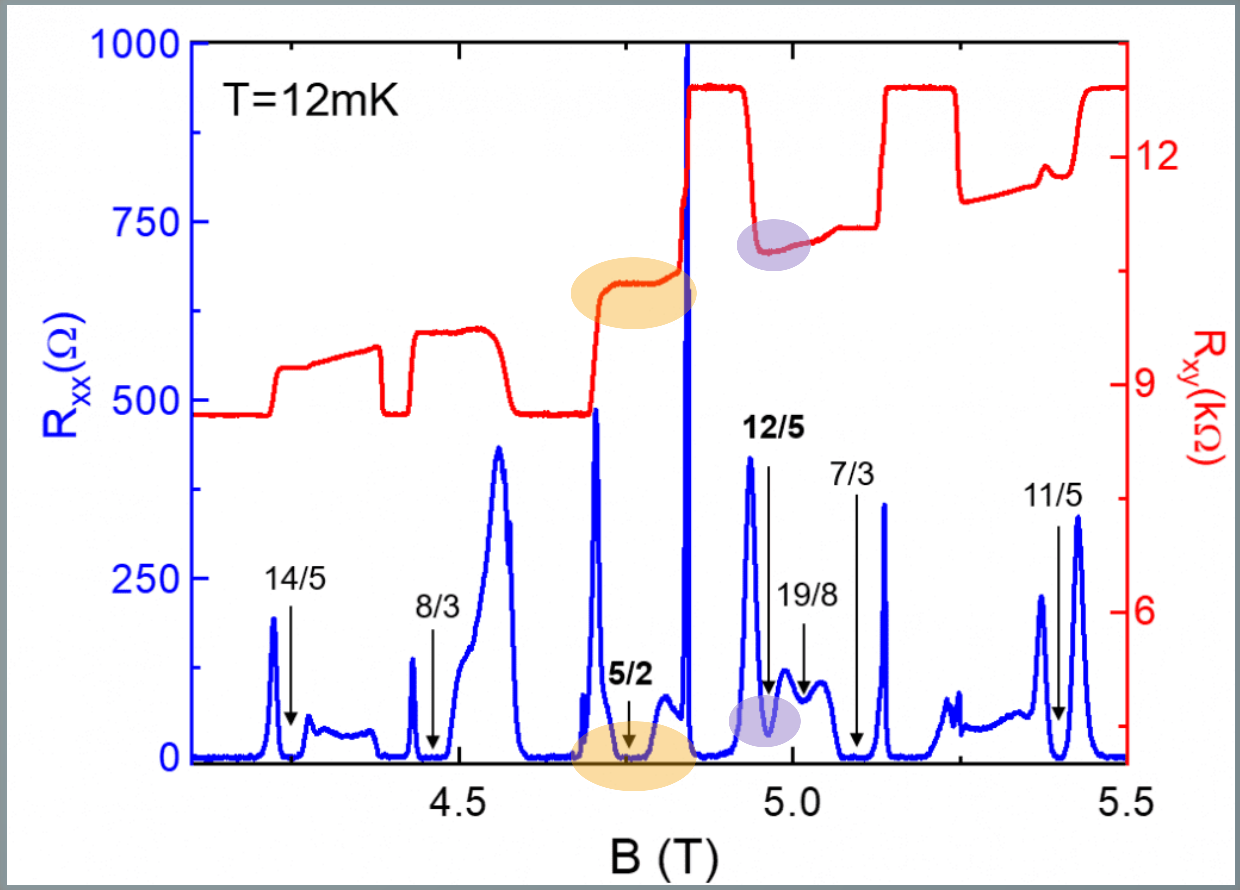




Quantum Hall Effect

Topological phases at different filling fractions described by different values of quantum parameter in the Jones polynomial

- $\nu = \frac{5}{2}$ corresponds to $q = e^{\frac{2\pi i}{8}}$
  - Anyons called ***Ising anyons***
- $\nu = \frac{12}{5}$ corresponds to $q = e^{\frac{2\pi i}{10}}$
  - Anyons called ***Fibonacci anyons***

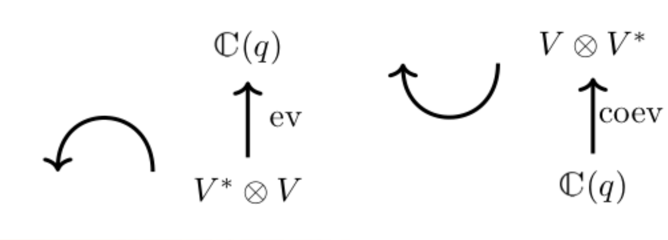


Mathematical description of TQFTs

MODULAR TENSOR CATEGORIES

Witten-Reshetikhin-Turaev TQFTs

3D TQFTs ⟷ **Modular Tensor Category**

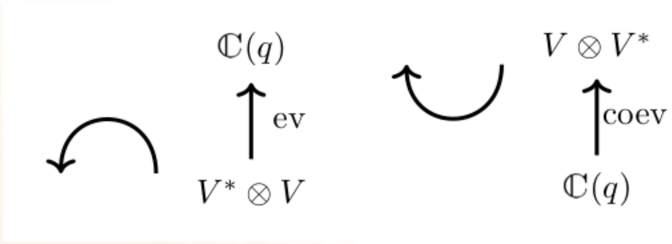


- Finite # of simple objects $X_0, X_1, \dots, X_n$
- Tensor products $X_i \otimes X_j = \oplus_k X_k$
- Duals X_i^*

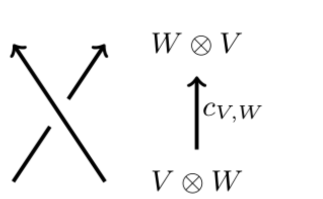
Semisimple Fusion Rules

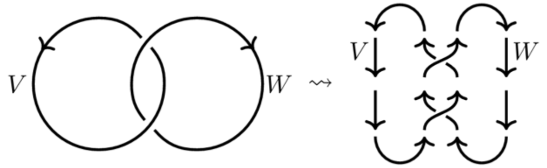
Witten-Reshetikhin-Turaev TQFTs

3D TQFTs ⟷ **Modular Tensor Category**



- Finite # of simple objects $X_0, X_1, \ldots, X_n$
- Tensor products $X_i \otimes X_j = \oplus_k X_k$
- Duals X_i^*
- Braiding
- Nondegeneracy





Gives invariants of knots colored by simple objects

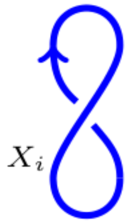
Witten-Reshetikhin-Turaev TQFTs

3D TQFTs ⟷ **Modular Tensor Category**

M a 3-manifold

- Obtained from surgery along a link $L \subset S^3$

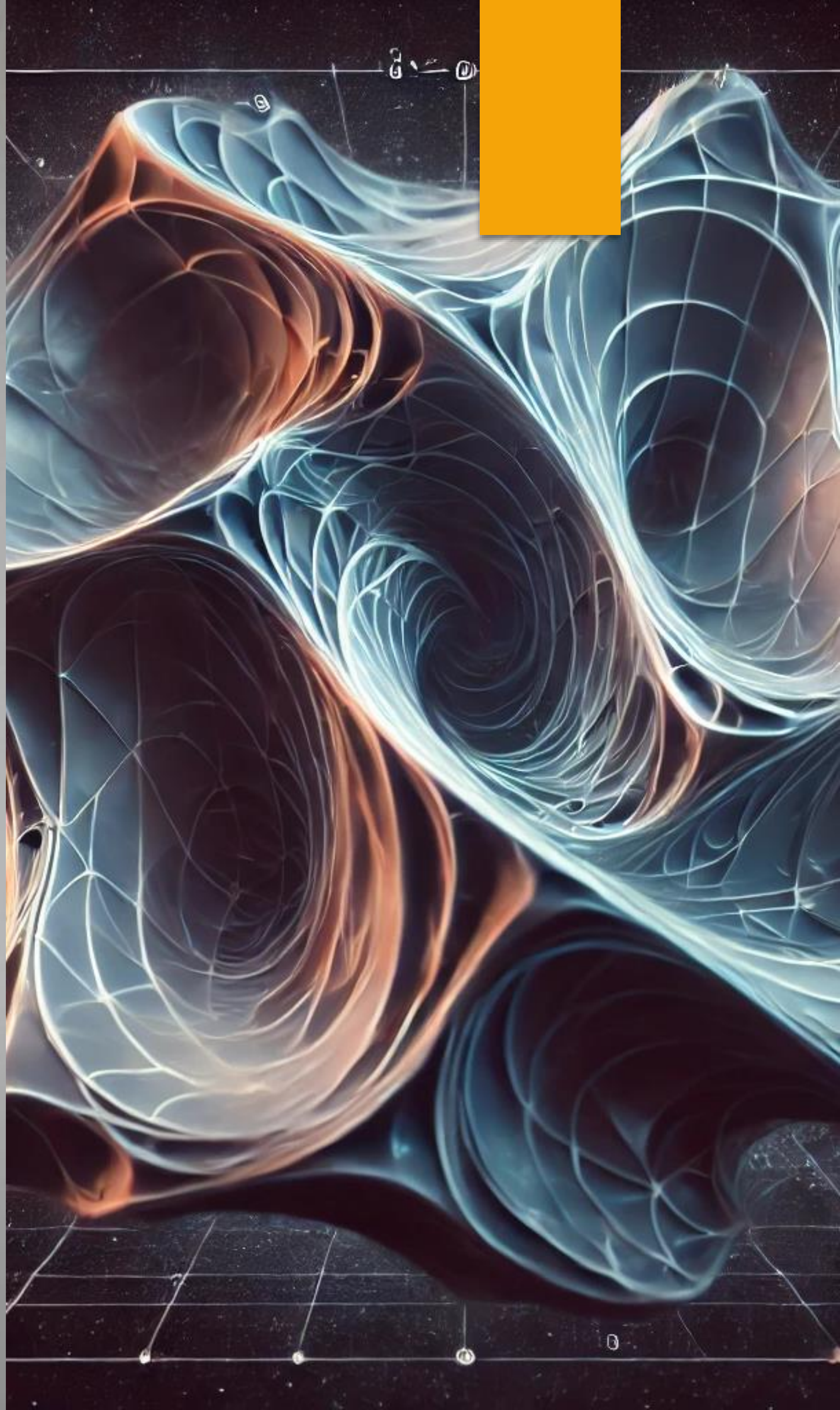
Kirby Color: $\Omega = \sum_i \mathrm{qdim}(X_i) X_i$

$\mathrm{qdim}(X_i) :=$ 

$Z_{WRT}(M) = \Omega$-colored link invariant x (framing correction)

Where do modular tensor categories come from?

QUANTUM GROUPS – BUT NOT COMPLETELY STRAIGHTFORWARD



Quantum spin representations

Generic parameter

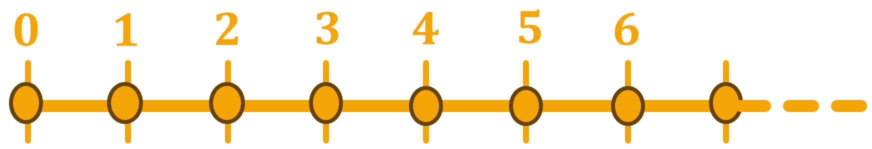
The quantum group $U_q(sl_2)$ is the $\mathbb{C}(q)$-algebra

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2 E,$$
$$KFK^{-1} = q^{-2}F, \quad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}$$

- Simple objects indexed by $\mathbb{N}_{\geq 0}$

$$S_n = \langle v_n, FV_n, \ldots, F^n v_n \rangle$$
$$Ev_n = 0, \ KV_n = q^n v_n$$

Spin Representations



$$\begin{tikzcd} v_{-n} \arrow[r, bend left, "E"] & \cdots \arrow[l, bend left, "F"] & v_{-\lambda+2} \arrow[r, bend left, "E"] & v_\lambda \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] & v_{\lambda+2} \arrow[l, bend left, "F"] & \cdots \arrow[r, bend left, "E"] & v_n \arrow[l, bend left, "F"] \end{tikzcd}$$

- Semisimple, braided tensor category, ∞ **number of simples**

- $\mathrm{qdim}(\boldsymbol{S_n}) = [\boldsymbol{n}+\mathbf{1}] = \frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}}$

Bad for 3-manifolds
Fine for knot invariants

Chop down some representations

- $\overline{U}_q(sl_2)$ is the specialization to a primitive $2r$th root of unity $q = e^{\frac{2\pi i}{2r}}$ and $E^r = F^r = 0$
- This actually makes the problem worse!

Simple-Projective Modules

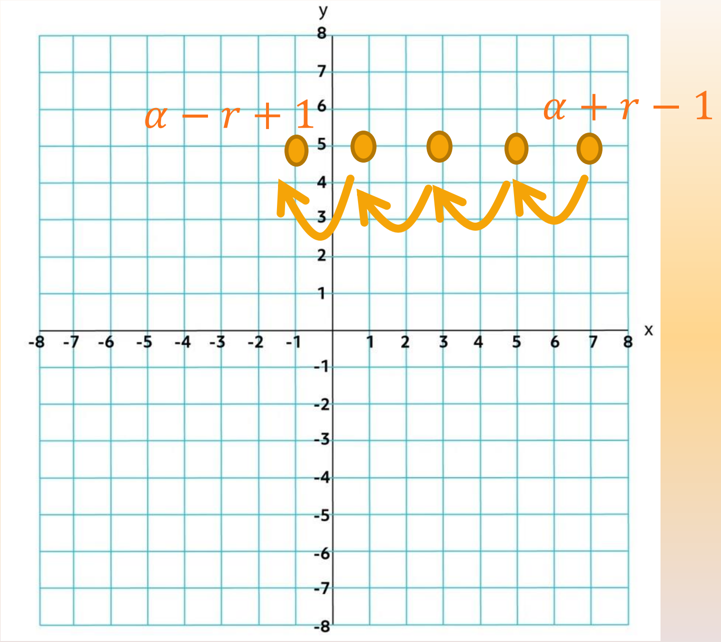
$V_\alpha = < v_0, v_1, \dots, v_{r-1} >$ for $\alpha \in \mathbb{C}$

Generated by a highest weight vector $\alpha + r - 1$

$$K.v_i = q^{(\alpha+r-1-2i)} v_i, \quad E.v_i = \frac{\{i\}\{i-\alpha\}}{\{1\}^2} v_{i-1}, \quad F.v_i = v_{i+1}$$

They are all simple for $\alpha \in (\mathbb{C} - \mathbb{Z})/2r\mathbb{Z}$

$$\mathrm{qdim}(V_\alpha) = q^{\alpha+r-1}(1 + q^2 + q^4 + \dots + q^{2(r-1)}) = 0$$

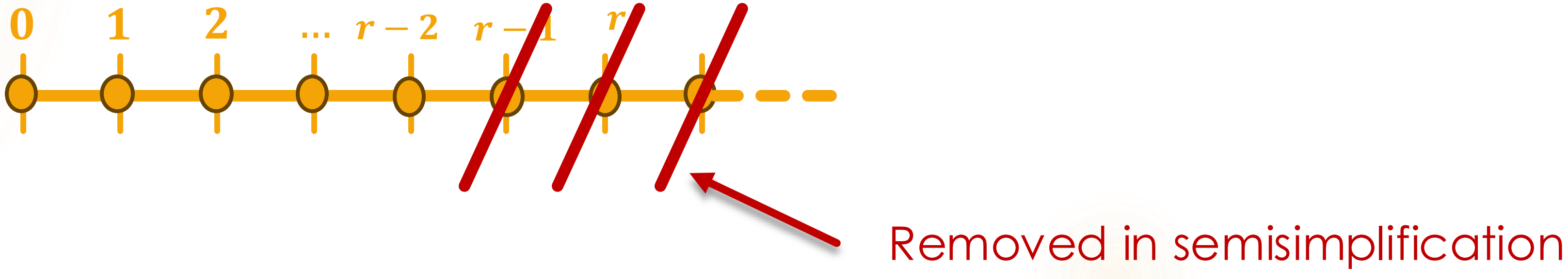


Semisimplification

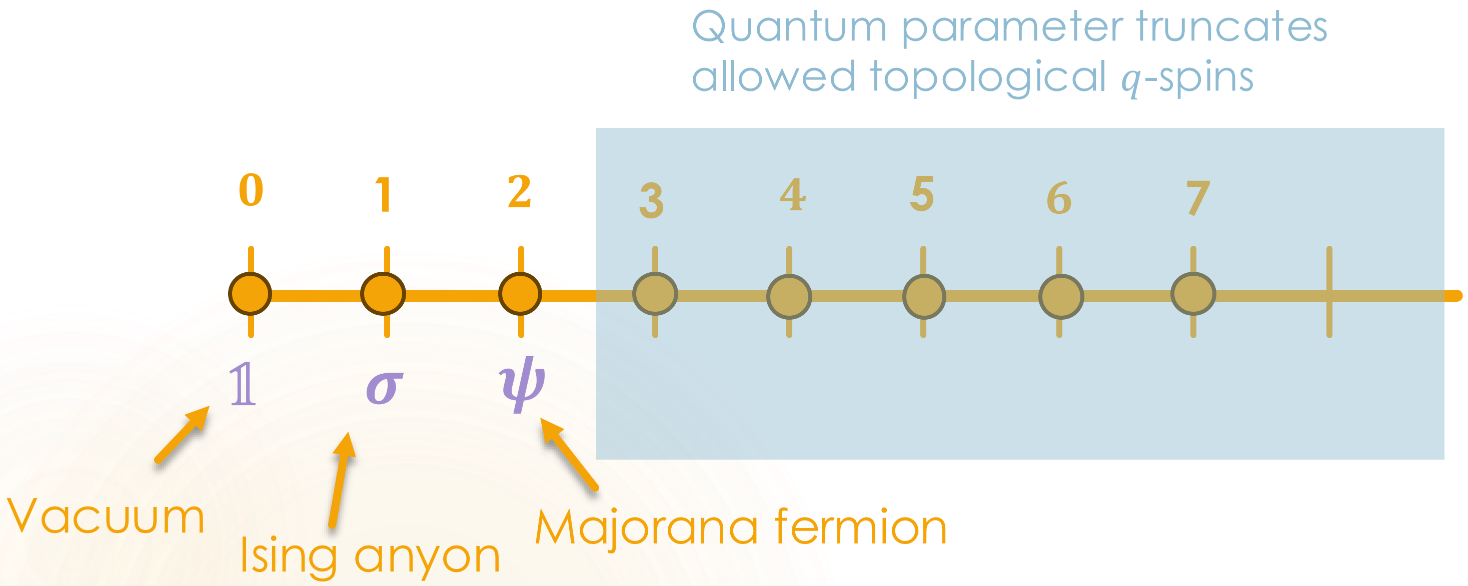
- Specialize to a primitive $2r$th root of unity $q = e^{\frac{2\pi i}{2r}}$ and $E^r = F^r = 0$
- To get a <u>semisimple</u> category with <u>finite</u> number of simple objects, throw away all quantum dimension zero representations

Previously simple modules now have quantum dimension zero

$$\mathrm{qdim}(S_{r-1}) = [r] = \frac{q^r - q^{-r}}{q - q^{-1}} = q^{-r}\,\frac{(q^{2r}-1)}{q - q^{-1}} = 0$$

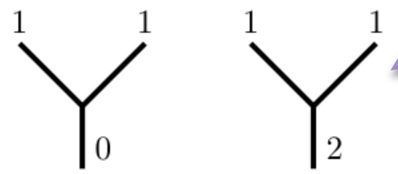


Example: Ising Category ($q = e^{\frac{2\pi i}{8}}$)

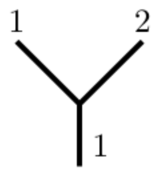


- $S_1 \otimes S_1 = S_0 \oplus S_2$

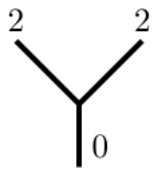
Unique map
$S_2 \to S_1 \otimes S_1$



- $S_1 \otimes S_2 = S_1 \oplus \cancel{S_3}$



- $S_2 \otimes S_2 = S_0 \oplus \boxed{S_2 \oplus S_4}$



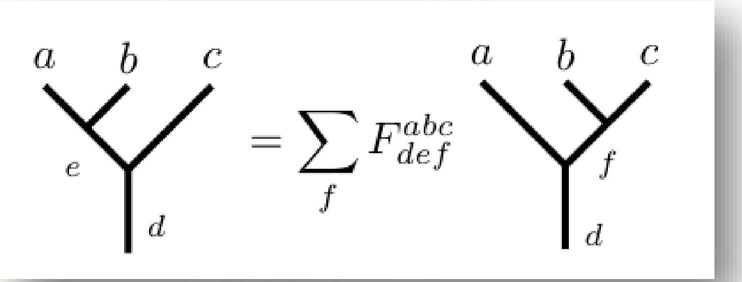
Nonsemisimple
rep qdim = 0

Where are the qubits?

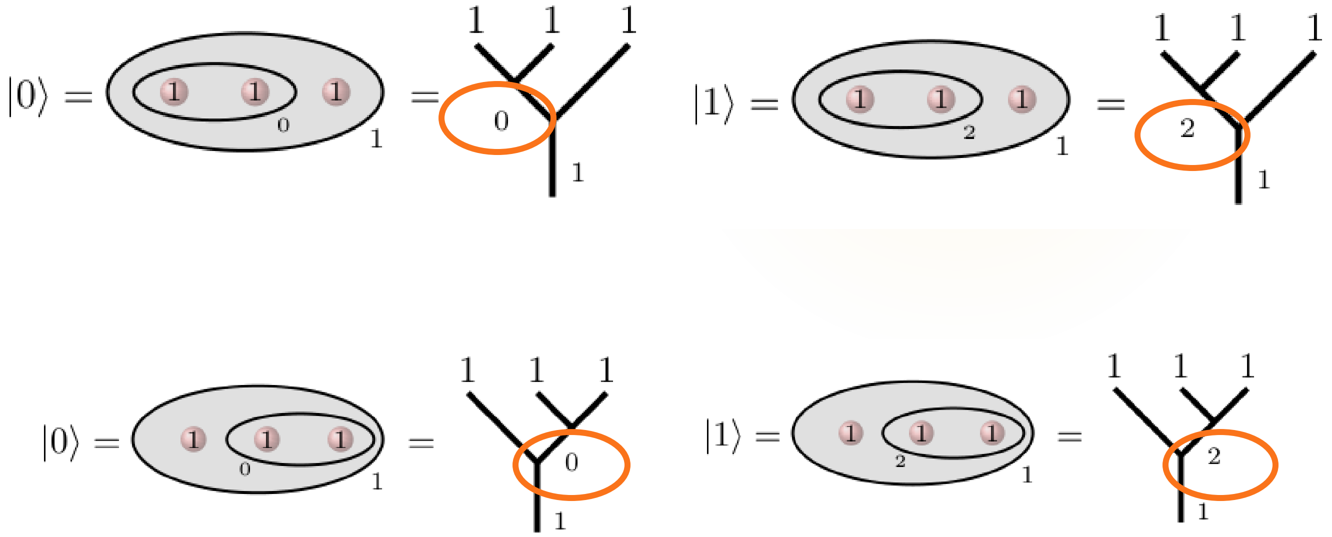
Idea: qubits are encoded into collective quantum states of a group of anyons

$S_1 \otimes S_1 = S_0 \oplus S_2$

- Hilbert space $\mathrm{Hom}(S_1, S_1^{\otimes 3})$
- Basis coming from fixed choices of trivalent vertices
- Just one possible choice of basis
- Change of basis using `F-symbols'

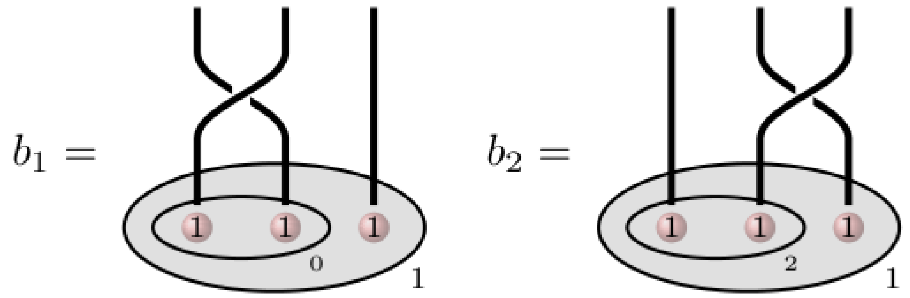


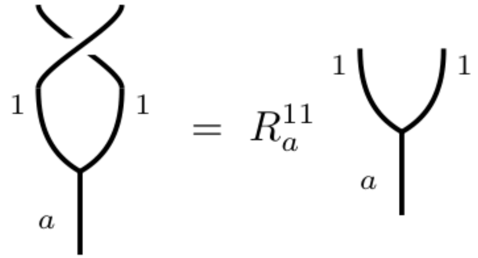
Three anyons = one qubit



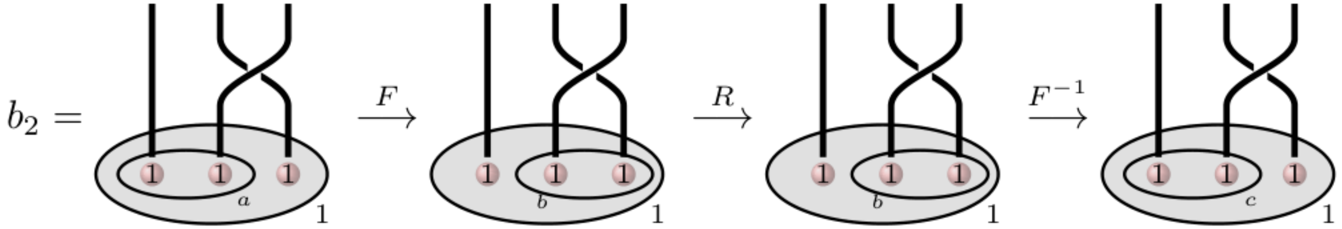
Quantum gates via braiding

- Basic gates on our qubit become elementary swaps of anyons
- Because our anyons are nonabelian these braids give nontrivial unitary transformations
- Stacking elementary braids and their inverses give more complex unitary transformations





$$b_1|0\rangle = R^{11}_0|0\rangle$$
$$b_1|1\rangle = R^{11}_1|1\rangle$$



Quantum gates via braiding

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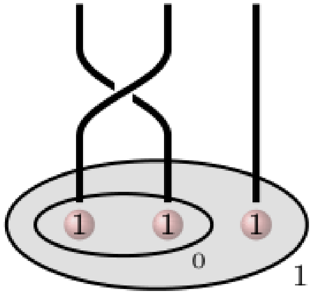
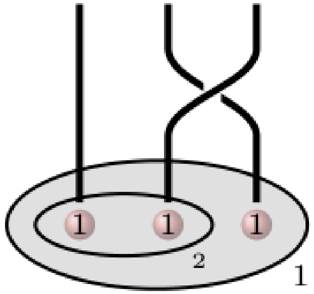
$b_1 =$ 

$b_2 =$ 

$$b_1 = \begin{pmatrix} -1 & 0 \\ 0 & i \end{pmatrix} \qquad b_2 = \begin{pmatrix} 1 & i \\ i & i \end{pmatrix}$$

<u>ISING ANYONS NOT UNIVERSAL</u>
Only generate Pauli matrices
No irrational Phases

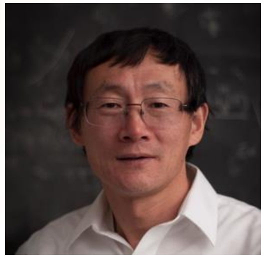
Not dense in SU(2)

Universal Quantum Computation

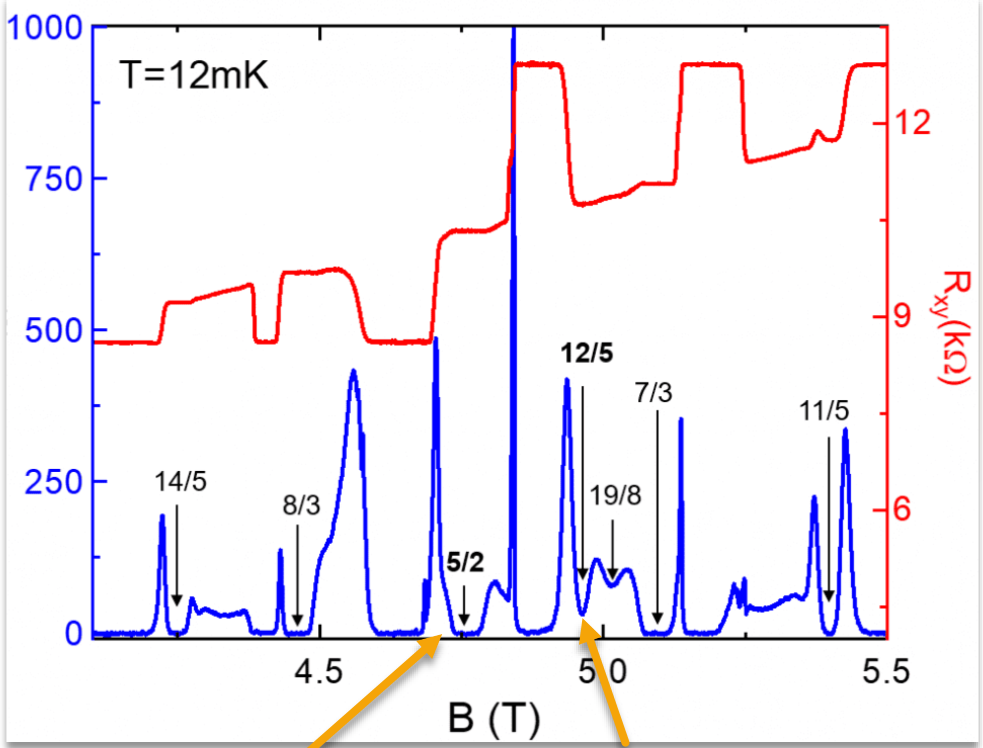
- Unfortunately, **Ising anyons are not universal for quantum computation** (despite being the best candidate for study in physical systems)
- However, Freedman-Larson-Wang showed that Fibonacci anyons at $\nu = \frac{12}{5}$ are universal for quantum computation

Mike Freedman

Michael Larsen



Zhenghan Wang



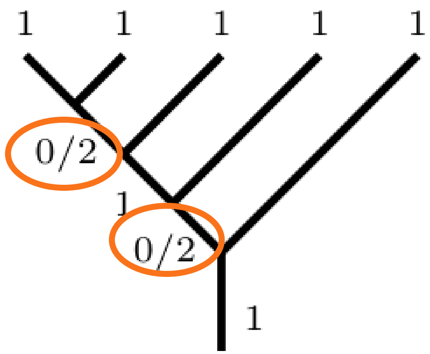
Not universal

Universal

Corresponds to SU(2) at a 10th root of unity

Quantum gates via braiding

- Ising anyons do give entangling gates between pairs of qubits



Two qubits

$\mathrm{Hom}(S_1, s_1^{\otimes 5})$

Can the standard TQFT approach be enhanced to make Ising anyons universal?

New structures in low-dimensional topology

PUSHING THE BOUNDARIES OF TOPOLOGY AND QUANTUM COMPUTATION

New ‘nonsemisimple’ TQFTs

- Very recently, new kinds of topological quantum field theories have been discovered that are closely connected to Chern-Simons theories

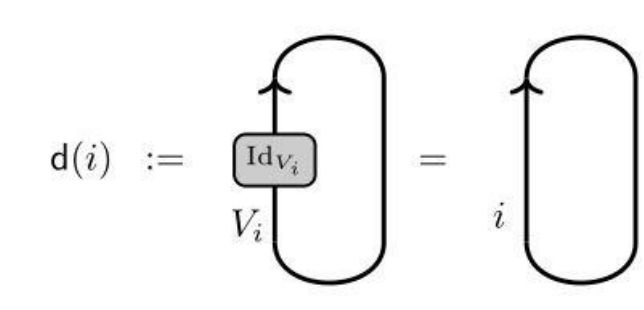

Blanchet


Costantino


Geer


Patureau-Mirand

- Blanchet, Constantino, Geer, Patureau-Mirand found a way to `renormalize’ quantum dimensions



Renormalized dimension

Given a closed ribbon graph labelled by objects of $\mathcal{C}$

Given some set of simples $\mathrm{A} \subset \mathcal{C}$ define modified trace (A,d)

$$d : A \to \mathbb{C}^* \qquad \mathrm{d(V) = d(V^*)} \qquad \mathrm{d(V) = d(V')} \qquad \mathrm{V} \cong V'$$

if

$$d(V)\langle T_V \rangle = d(V')\,\langle T_{V'} \rangle$$

for all ribbon graphs with at least one color from A

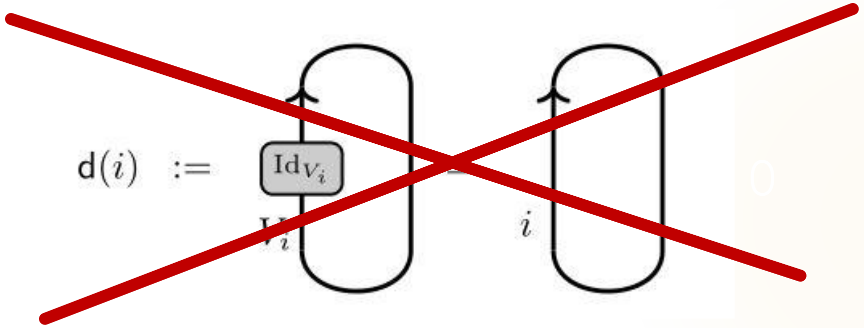
EX: for $\overline{U_q}(sl2)$ we have

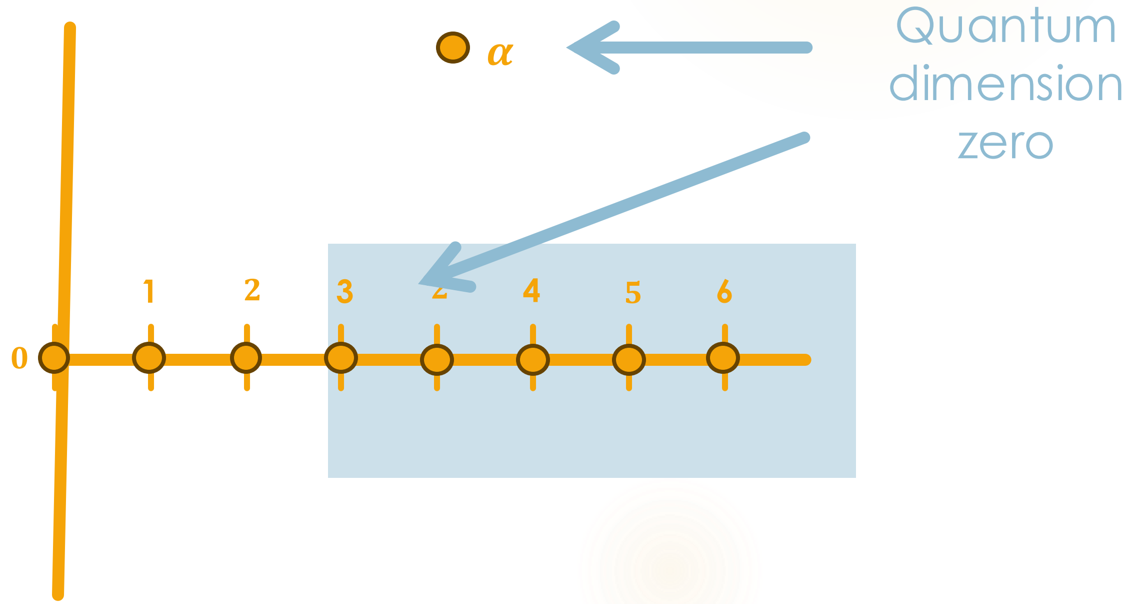
$$d(V_\alpha) = \frac{q^\alpha - q^{-\alpha}}{q^{r\alpha} - q^{-r\alpha}} \neq 0$$

Nonsemisimple TQFT

- Prior approaches to (2+1)-dimensional TQFT throws out all topological spins with quantum dimension zero

- Before semisimplification we have quantum dimension zero particles for generic values of $\alpha \in \mathbb{C} - \mathbb{Z}$





Nonsemisimple TQFT

- Leverages quantum dimension zero objects
- Renormalizes the trace
- Gives renormalized WRT invariants defined using modified dimensions

Semisimple WRT

- can't distinguish lens spaces
- Dehn twist on torus has finite order

Nonsemisimple WRT

- Can distinguish lens spaces
- Dehn twist on a torus has infinite order

Does improved topological power translate into more powerful models of topological quantum computation?

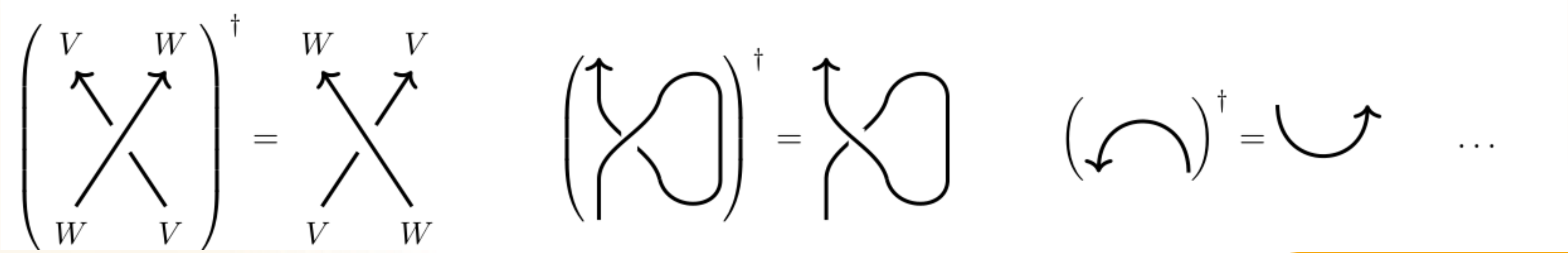
Unitarity

For physics applications, it is important that TQFTs are Hermitian and Unitary

- A <u>dagger</u> on a tensor category sends $f: V \to W$ to $f^{\dagger}: W \to V$ satisfying

$$f^{\dagger\dagger} = f, \qquad (f \otimes g)^{\dagger} = f^{\dagger} \otimes g^{\dagger}, \qquad (g \circ f)^{\dagger} = f^{\dagger} \circ g^{\dagger}$$

- A Hermitian ribbon category C is a ribbon category with † satisfying



- Gives an inner product $\langle f, g \rangle$: $\mathrm{Hom}(V,W) \otimes \mathrm{Hom}(V,W) \to \mathbb{C}$

$$f \otimes g \to tr(f^{\dagger} g)$$

Unitary TQFTs → Homs are Hilbert spaces

Hermitian → $\langle f, g \rangle = \overline{\langle g, f \rangle}$

→ Nondegenerate

Unitary → $\langle , \rangle$ is positive definite

Are nonsemisimple TQFTs Physical?

Theorem(Geer, L-, Patureau-Mirand, Sussan)
Nonsemisimple TQFTs are Hermitian but (indefinite) unitary

Geer

Patureau-Mirand

Sussan

- Leads to Hilbert spaces with indefinite forms
- Leads to Hamiltonian theories that are `pseudo' unitary
- Related to logarithmic conformal field theory
- May be boundary of a unitary theory

Nonsemisimple Ising Model

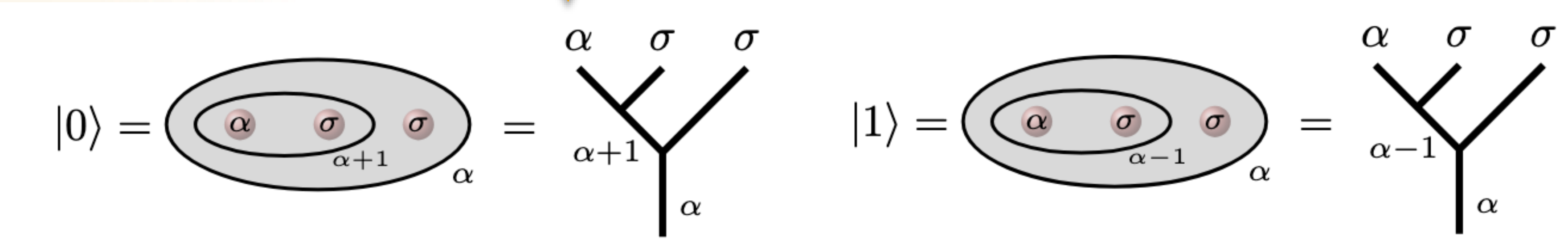
Consider representations of $\overline{U}_q(sl_2)$ at $q = e^{\frac{2\pi i}{8}}$

- Need to encode our qubits into Hom spaces
- Want density for SU(2)

Add a single negligible representation

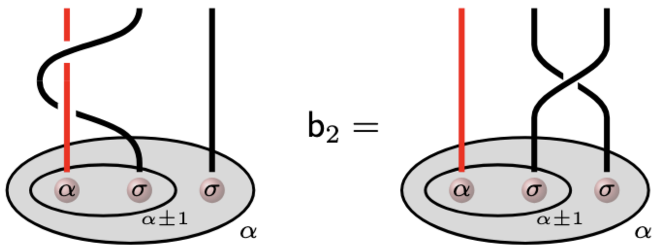
Neglecton???

New q-dim 0 particle



Universal nonsemisimple Ising anyons

- Inner product is positive definite for α in appropriate range
- Unlike the semisimple model, single qubit operations are dense in SU(2)



$$\mathsf{b}_1^2 = -q\,(\mathsf{b}_1^{\alpha\sigma\sigma})^2 = \begin{pmatrix} q^{\alpha} & 0 \\ 0 & q^{-\alpha} \end{pmatrix},$$

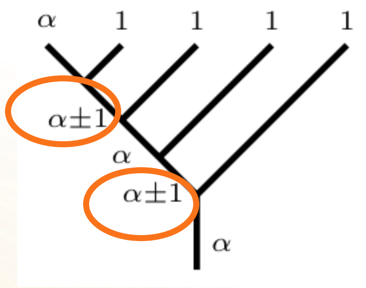
$$\mathsf{b}_2 = q^{-\frac{3}{2}}\,\mathsf{b}_2^{\alpha\sigma\sigma} = q^{-1}\begin{pmatrix} \frac{1+q^2}{1-q^{2\alpha}} & q^{-1}\frac{\sqrt{\mathbf{B}_{\alpha+1}}}{\sqrt{\mathbf{B}_{\alpha-1}}} \\ q^{-1}\frac{\sqrt{\mathbf{B}_{\alpha+1}}}{\sqrt{\mathbf{B}_{\alpha-1}}} & \frac{1+q^2}{1-q^{-2\alpha}} \end{pmatrix}$$

$$\mathbf{B}_{\alpha+1} := \frac{\sqrt{2}}{-1+\cot\frac{\pi(\alpha+1)}{4}}$$

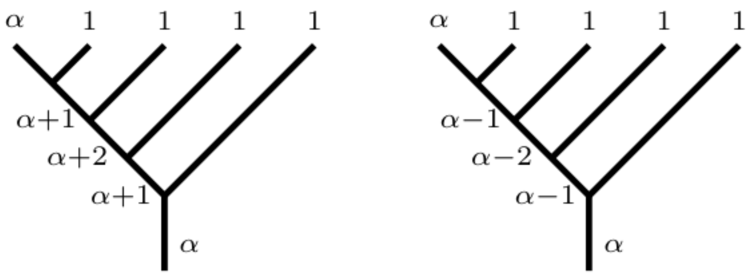
$$\mathbf{B}_{\alpha-1} := \frac{\sqrt{2}}{-1+\cot\frac{\pi\alpha}{4}},$$

Universal nonsemisimple Ising anyons

- Multiqubit Hilbert space $\mathrm{Hom}(S_\alpha, V_\alpha \otimes S_1^{2n})$



4-dimensional Computation space



2-dimensional Non-Computation space

- There exists α where computational space is positive definite
- Can construct low leakage entangling gates

More powerful computation

Theorem (Iulianelli, Kim, L, Sussan)
For nonsemisimple **Ising TQFT**

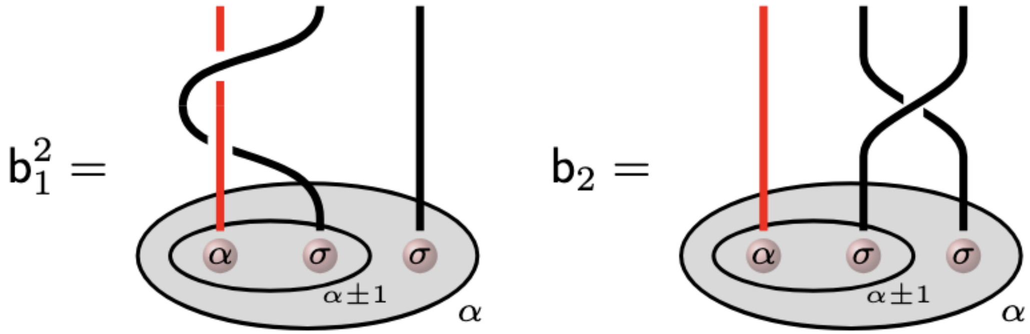
- Qubit spaces can be encoded into (positive-definite) computational Hilbert spaces

- Nonsemisimple Ising model is **universal** for quantum computation


Filippo Iulianelli


Sung Kim


Josh Sussan



Open questions

- Condensed matter systems supporting new anyons?
- What can new developments in topology tell us about topological phases and opportunities for new quantum technologies