Modular tensor categories via local modules

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Talk based on `arXiv:2408.06314` with Kenichi Shimizu

Plan of the talk

1. Motivation
2. General categorical results
3. Applications to finite tensor categories
4. Examples

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Motivation

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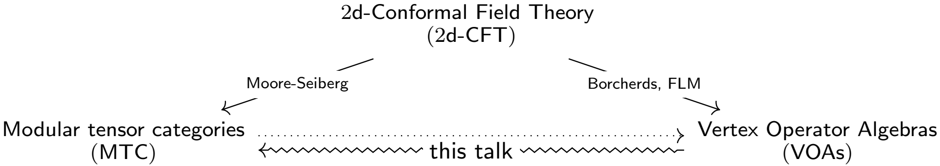

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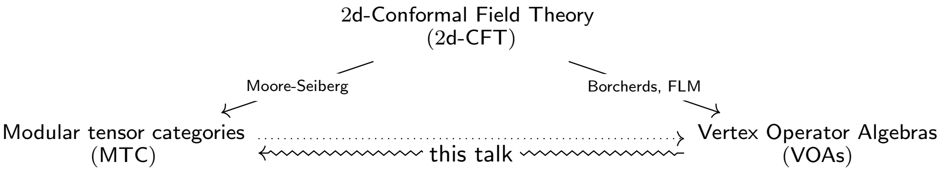

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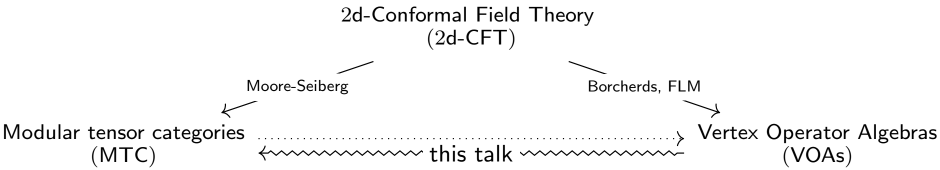

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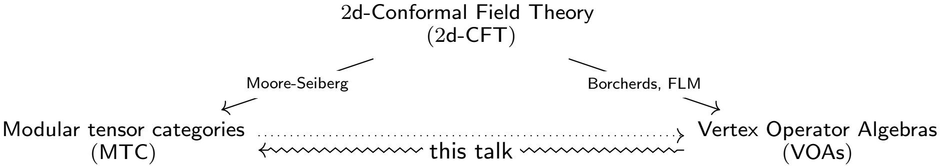

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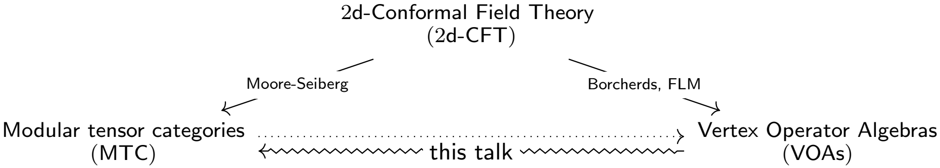

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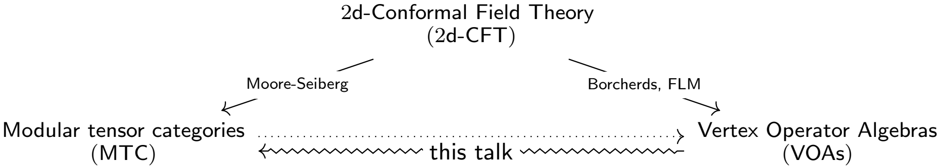

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General categorical results

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- The category of A-bimodules, denoted as ${}_A\mathcal{C}_A$, is monoidal. For $M, N \in {}_A\mathcal{C}_A$, the tensor product $M \otimes_A N$ is defined as

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An object $X \in \mathcal{C}$ is called **left rigid** if $\exists\, X^* \in \mathcal{C}$ along with maps $\mathrm{ev} : X \otimes X^* \to \mathbb{1}$ and $\mathrm{coev} : \mathbb{1} \to X^* \otimes X$ satisfying snake relations.

A monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ is called **left closed** if the endofunctor $- \otimes X$ of $\mathcal{C}$ admits a right adjoint $\forall\, X \in \mathcal{C}$. We denote the right adjoint as $\underline{\mathrm{Hom}}^l(X, -)$.

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$$\mathrm{Hom}_{\mathcal{C}}(W \otimes X, Y) \cong \mathrm{Hom}_{\mathcal{C}}(W, \underline{\mathrm{Hom}}^l(X, Y)) \qquad (W, Y \in \mathcal{C}).$$

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Let $\mathcal{C}$ be a ribbon monoidal category. If A is a commutative algebra with $\theta_A = \mathrm{Id}_A$ and $A^* \in (\mathcal{C}_A^{\mathrm{loc}})'$, then $\mathcal{C}_A^{\mathrm{loc}}$ is ribbon with the same twist as $\mathcal{C}$.

Proof. As $\theta_A = \mathrm{Id}$, θ is a twist on $\mathcal{C}_A^{\mathrm{loc}}$.
Let $i_{X,A} : X^\dagger = \underline{\mathrm{Hom}}_A(X, A) \to A \otimes X^*$ denote the canonical injection. Then,

$$i_{X,A} \circ \theta_{X^\dagger} = \theta_{A\otimes X^*} \circ i_{X,A} = c^A_{X^*,A} \circ c^A_{A,X^*} \circ (\theta_A \otimes \theta^*_X) \circ i_{X,A}$$
$$= c^A_{X^*,A} \circ c^A_{A,X^*} \circ i_{X,A} \circ \underline{\mathrm{Hom}}_A(\theta_X, A) = c^A_{X^*,A} \circ c^A_{A,X^*} \circ i_{X,A} \circ (\theta_X)^\dagger.$$

As $A^* \in (\mathcal{C}_A^{\mathrm{loc}})'$, we obtain $i_{X,A} \circ \theta_{X^\dagger} = i_{X,A} \circ (\theta_X)^\dagger$. Since $i_{X,A}$ is a monomorphism, it follows that $\theta_{X^\dagger} = (\theta_X)^\dagger$. □

Summary

Let $\mathcal{C}$ be a rigid monoidal category with equalizers and coequalizers. Let (A, σ) be a commutative algebra in $\mathcal{Z}(\mathcal{C})$. Then

$$ {}_A\mathcal{C}_A \text{ rigid} \quad \Longrightarrow \quad \mathcal{C}_A^{\sigma} \text{ is rigid} $$

Let $\mathcal{C}$ be a rigid braided monoidal category with equalizers and coequalizers. Let A be a commutative algebra in $\mathcal{C}$. Then,

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Applications to finite tensor categories

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Proof. As A is simple, ${}_A\mathcal{C}_A$ is rigid. By our general result, $\mathcal{C}_A^\sigma$ is rigid. In fact, $\mathcal{C}_A^\sigma$ is a full subcategory of ${}_A\mathcal{C}_A$ closed under subquotients, $\oplus$, $\otimes$ and duals. Thus, the first claim follows. The second claim is proved in a similar manner.

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$$i_+ : \mathcal{C} \to \mathcal{Z}(\mathcal{C}), \;\; X \mapsto (X, c_{X,-}) \qquad \text{and} \qquad i_- : \overline{\mathcal{C}} \to \mathcal{Z}(\mathcal{C}), \;\; X \mapsto (X, c^{-1}_{-,X})$$

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There are braided equivalences

$$\mathcal{C}_A^{\mathrm{loc}} \boxtimes \overline{\mathcal{C}} \longrightarrow (\mathcal{C} \boxtimes \overline{\mathcal{C}})_{A \boxtimes \mathbb{1}}^{\mathrm{loc}} \longrightarrow \mathcal{Z}(\mathcal{C})_{i_+(A)}^{\mathrm{loc}} \longrightarrow \mathcal{Z}(\mathcal{C}_A)$$

The middle equivalence is induced by the equivalence $\mathcal{C} \boxtimes \overline{\mathcal{C}} \to \mathcal{Z}(\mathcal{C})$ which we have since $\mathcal{C}$ is non-degenerate. The last equivalence is due to Schauenburg.

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From the above equivalence, we can deduce that an object of $X \in (\mathcal{C}_A^{\mathrm{loc}})'$ yields an object of $X \boxtimes \mathbb{1}$ in the Müger center of $\mathcal{Z}(\mathcal{C}_A)'$. However, $\mathcal{Z}(\mathcal{C}_A)$ has trivial Müger center. Thus, $\mathcal{C}_A^{\mathrm{loc}}$ is non-degenerate. □

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Examples

Simple current algebras

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- Then the semisimple subcategory of $\mathcal{C}$ spanned by G is isomorphic to the pointed braided fusion category $\mathcal{C}(G, q)$ where the quadratic form $q : G \to \Bbbk^\times$ given by $c_{X_g,X_g} = q(g)\,\mathrm{Id}$.

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