On the Structure of Skein Modules of 3-Manifolds

Rhea Palak Bakshi
University of California, Santa Barbara

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Goals:

- Understand the Kauffman bracket skein module of 3-manifolds.

- Look at some structural results about skein modules.

- Provide the structure of the KBSM of $H_1 \# H_1$ and $S^1 \times S^2 \# S^1 \times S^2$.

OUTLINE

- What are skein modules?
- The (n, ∞) - skein module
- Kauffman bracket skein module
- Some computed examples of the KBSM
- Properties of the KBSM
- Main result about the KBSM of $S^1 \times S^2 \# S^1 \times S^2$
- Future directions

What are skein modules?

Alexander - Conway polynomial (1969)

Jones polynomial (1984)

HOMFLYPT polynomial (1984)

Kauffman polynomial (1985)

Kauffman bracket polynomial (1985)

Dubrovnik polynomial (1985)

Vassiliev - Gusarov invariant (1989)

Invariants of links in S^3 and Skein Relations

WHAT ARE SKEIN MODULES?

- Introduced by Józef H. Przytycki in 1987. Independently by Vladimir Turaev in 1988.
- Generalisations of polynomial link invariants in S^3 to M^3.
- They are 3-manifold invariants and link invariants.
- There are many different skein modules!

EXAMPLES OF SKEIN MODULES

- The framing skein module
- The q-homology skein module
- The q-homotopy skein module
- The HOMFLYPT skein module
- The Kauffman skein module
- The Vassiliev-Gusarov skein module
- The Dubrovnik skein module
- The Bar-Natan skein module
- The Kauffman bracket skein module
- Skein lasagna modules

Algebraic Geometry

Hyperbolic Geometry

Mirror Symmetry

Quantum Cluster algebras

Temperley - Lieb algebras

Topological Quantum Field Theories

Skein Modules

Hecke algebras

Hopf algebras

Witten - Reshetikhin - Turaev Invariants

Representation Theory

Homological invariants

AJ Conjecture

The (n, ∞)-skein module

Oriented   Commutative with unity

$\mathcal{S}_{n,\infty}(M^3; R)$

R\{ambient isotopy classes of unoriented framed links in M^3 including Ø\}

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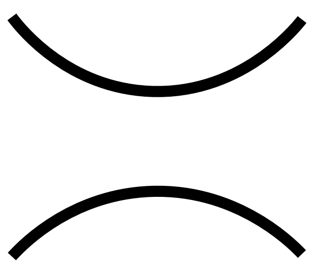
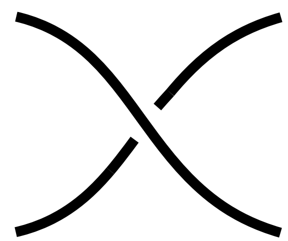
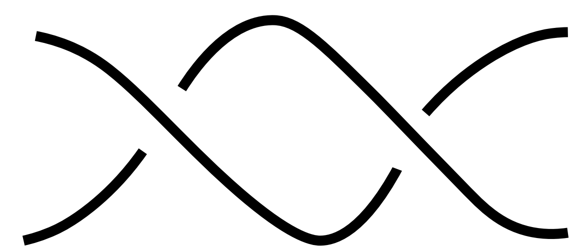
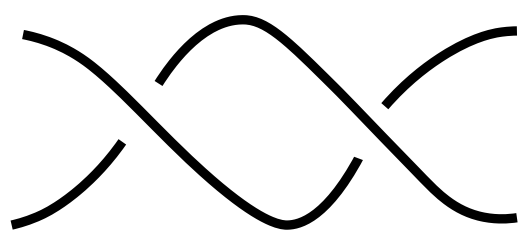
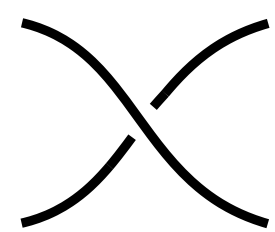
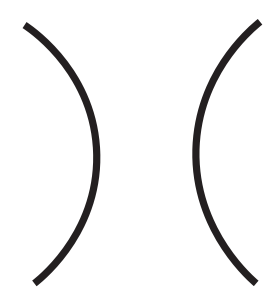
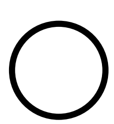
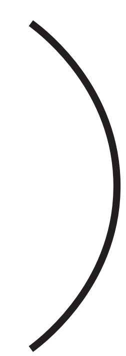
1. b_0  $+b_1$  $+b_2$  $+ \dots$

$+b_{n-1}$  $\bullet\ \bullet\ \bullet$  $+b_\infty$ 

2. $L \sqcup$  $+ tL$

Trivial Framed Link

3.  $+a$ 

$n = 2$: **Linear skein modules**

Eg. Kauffman bracket skein module

$n = 3$: **Quadratic skein modules**

Eg. HOMFLYPT skein module, Kauffman skein module

$n = 4$: **Cubic skein modules**

Kauffman bracket skein module

$b_0 = -A$, $b_1 = 1$, $b_\infty = -A^{-1}, a = A^3$, and $t = A^2 + A^{-2}$

The Kauffman bracket skein module

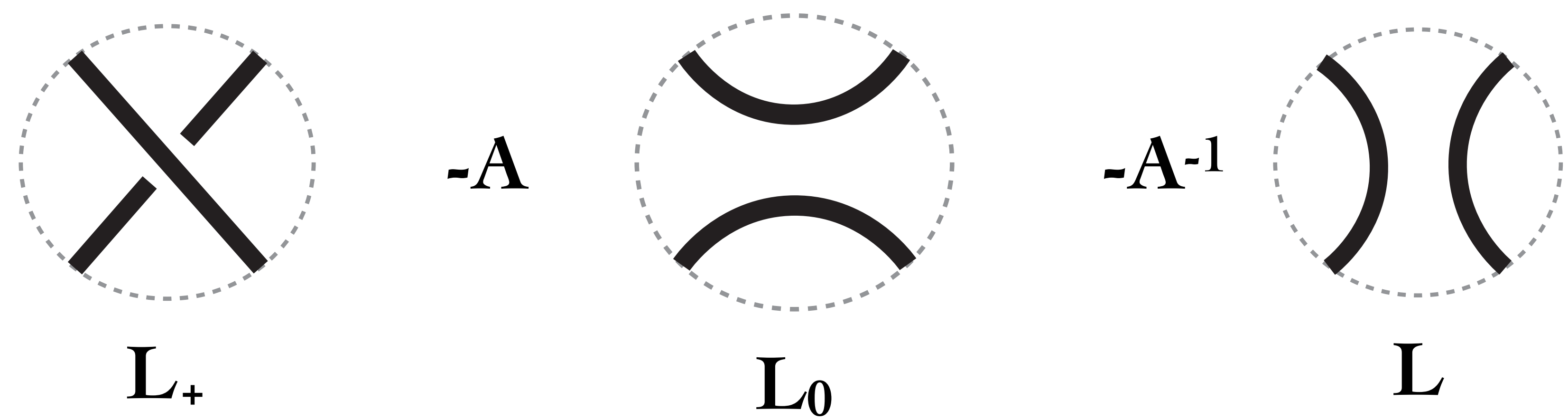
Oriented

Commutative with unity

$\mathbb{Z}[A^{\pm 1}]$, $\mathbb{Q}(A)$

$$\mathcal{S}_{2,\infty}(M^3; R, A) = \frac{R\{\text{ambient isotopy classes of unoriented framed links in } M^3 \text{ including } \emptyset\}}{}$$

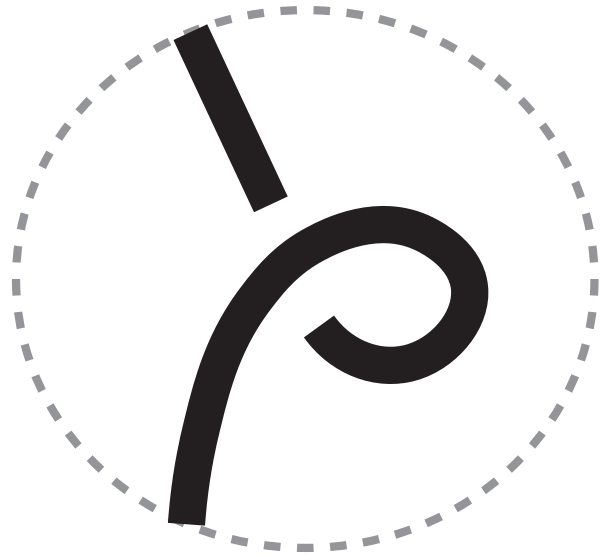
Invertible in R

1. 

2. $L \sqcup \bigcirc + (A^2 + A^{-2})L$

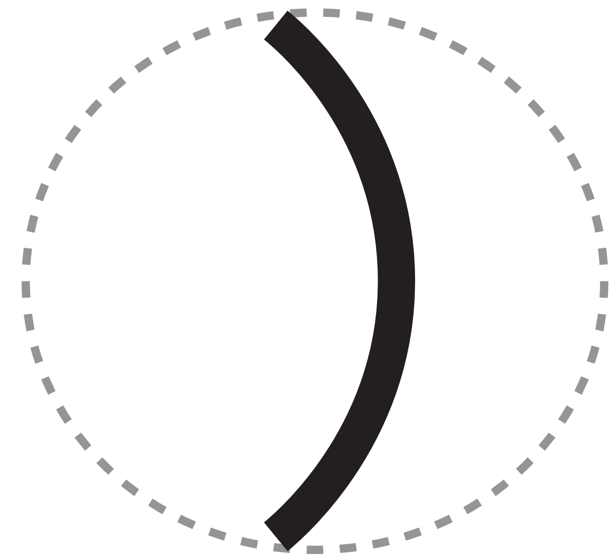
Trivial Framed Link

- In $\mathcal{S}_{2,\infty}(M^3; R, A)$



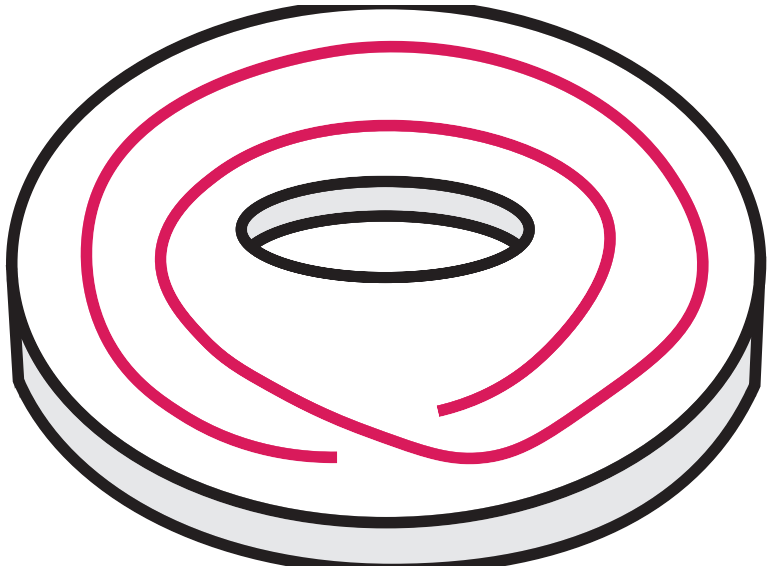
$\mathbf{L}^{(1)}$

$+\mathbf{A}^3$

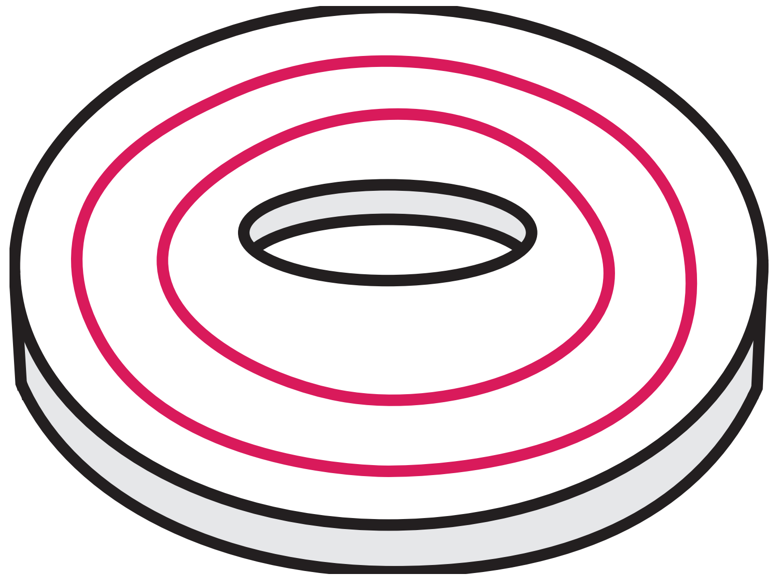


$\mathbf{L}$

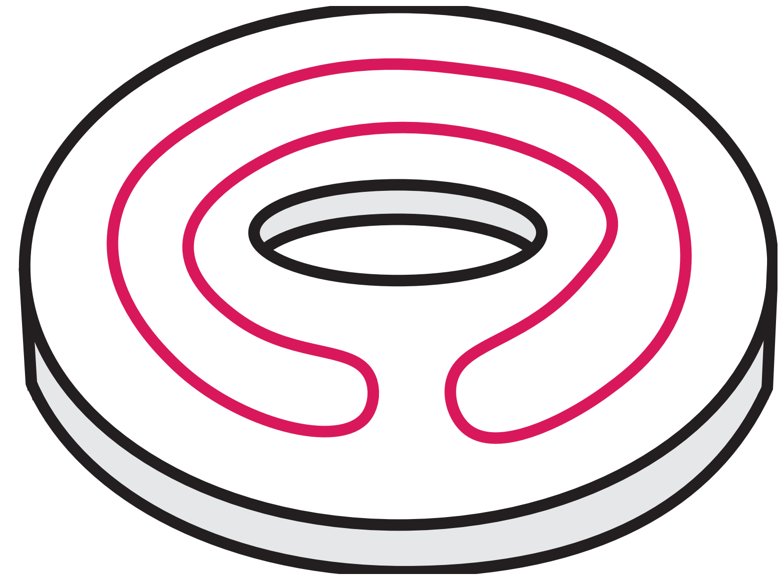
- For simplicity we use the notation when $\mathcal{S}_{2,\infty}(M^3)$ when $R = \mathbb{Z}[A^{\pm 1}]$.



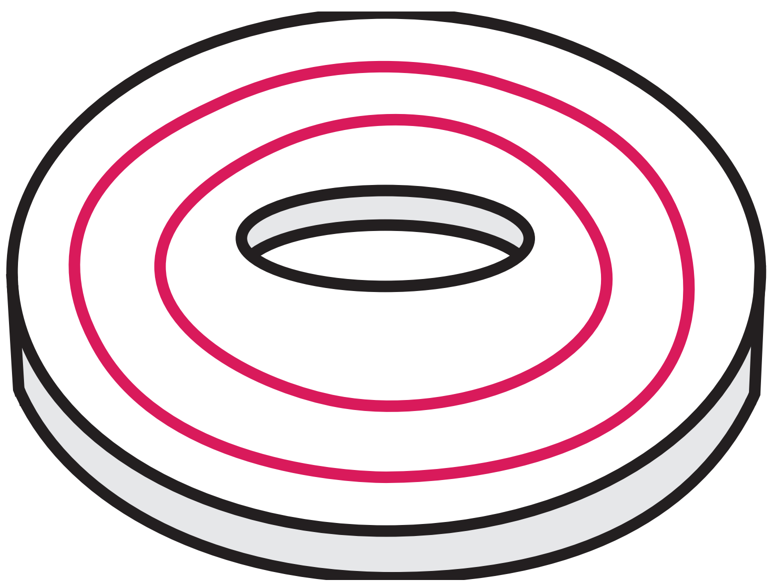
$= A$



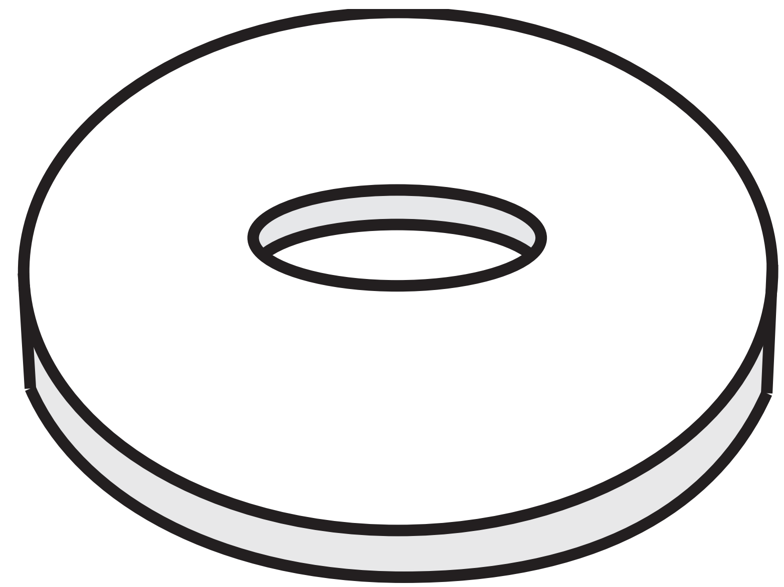
$+ A^{-1}$



$= A$



$+ A^{-1}(-A^{2}-A^{-2})$



$$= Ax^2 + (-A - A^{-3}) \cdot \emptyset$$

Examples of Kauffman Bracket Skein Modules

SOME COMPUTED EXAMPLES

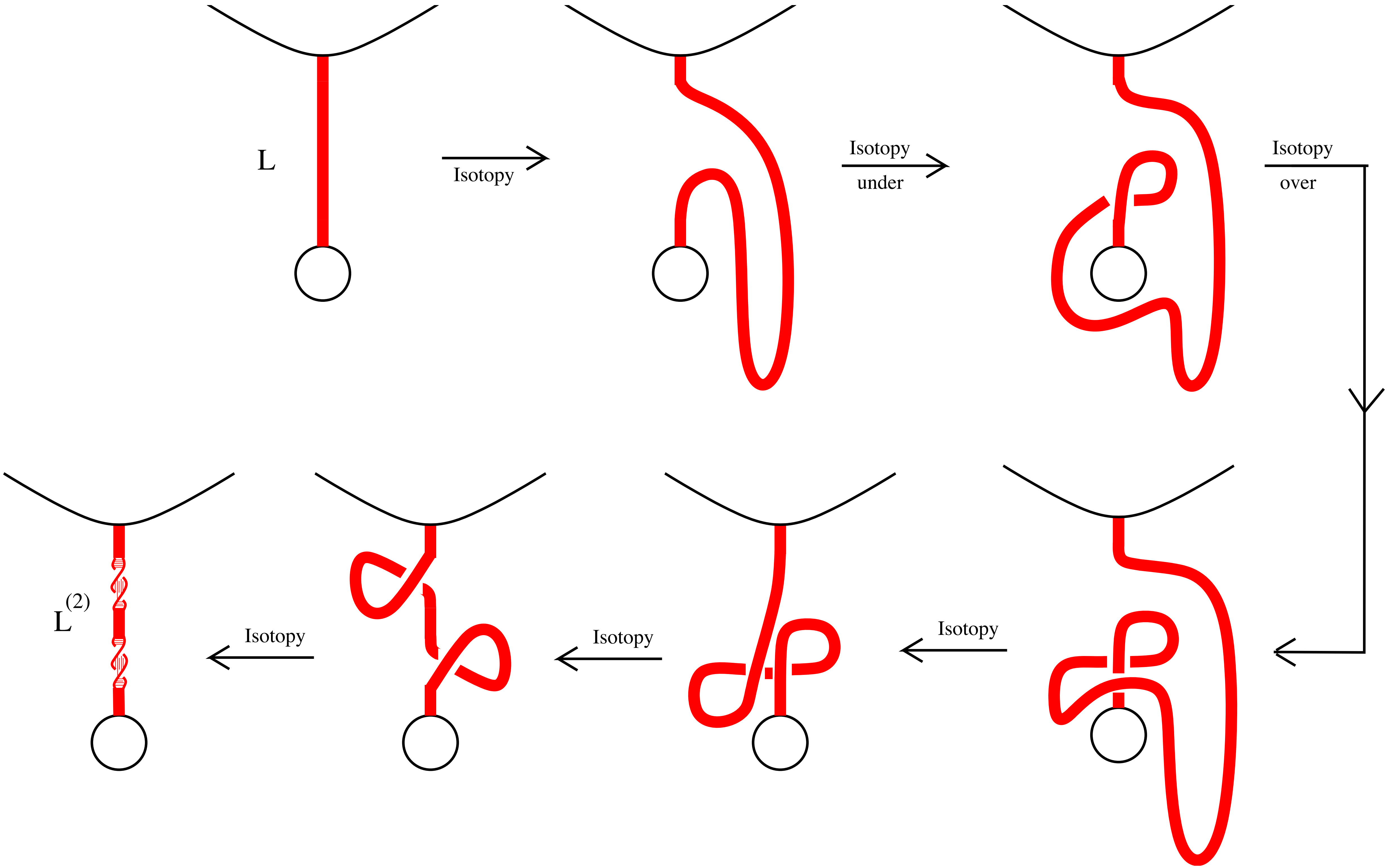
- $\mathcal{S}_{2,\infty}(S^3) = \mathbb{Z}[A^{\pm 1}]\emptyset$.

- (Hoste - Przytycki, 1989): For $p \geq 1$, $\mathcal{S}_{2,\infty}(L(p,q))$ is a free $\mathbb{Z}[A^{\pm 1}]$ -module and it has $\lfloor p/2 \rfloor + 1$ free generators.

- (Hoste - Przytycki, 1990): $\mathcal{S}_{2,\infty}(S^1 \times S^2)$ is an infinitely generated module.

$$\mathcal{S}_{2,\infty}(S^1 \times S^2)) = \mathbb{Z}[A^{\pm 1}] \oplus \bigoplus_{i=1}^{\infty} \frac{\mathbb{Z}[A^{\pm 1}]}{1 - A^{2i+4}}.$$

- (Hoste - Przytycki, 1992): $\mathcal{S}_{2,\infty}(W)$ is infinitely generated, torsion free, but not free.



Dirac trick for a knot illustrated using a light bulb

- (Bullock, 1997): KBSM of M^3 obtained from an integral surgery on the trefoil knot is a finitely generated $\mathbb{Z}[A^{\pm 1}]$-module if and only if the manifolds contains no essential surface.

- (Lê, 2006): Computed the KBSMs of the exteriors of 2-bridge knots. Special cases of the exteriors of 2-bridge knots had been computed earlier by Bullock and Lofaro.

- (Gilmer - Harris, 2007): The KBSM of the quaternionic manifold is a finitely generated R -module, where R is the ring $\mathbb{Z}[A^{\pm 1}]$ localized by inverting all the cyclotomic polynomials. The basis consists of five elements.

- (Harris, 2010): Computed the KBSMs of some Dehn fillings of (2,2n) torus knots over $\mathbb{Z}[A^{\pm 1}]$ and showed that they are finitely generated.

- (Dąbkowski-Mroczkowski, 2009): $\mathcal{S}_{2,\infty}(\Sigma_{0,3} \times S^1)$ is an infinitely generated free $\mathbb{Z}[A^{\pm 1}]$ -module.

- (Carrega, 2016): $\mathcal{S}_{2,\infty}(T^3; \mathbb{Q}(A), A)$ is a finitely generated $\mathbb{Q}(A)$ -module with 9 generators. In 2018, Gilmer showed that these generators are linearly independent.

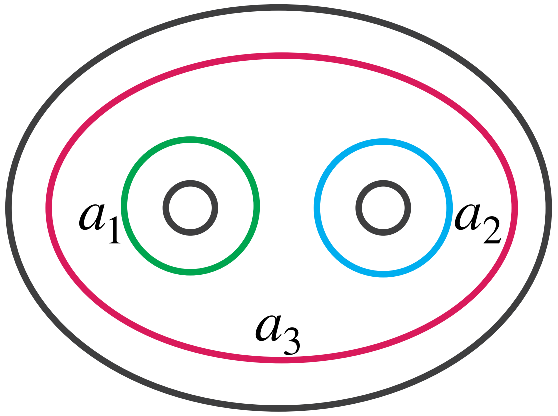
- (Detcherry, 2019): Gave examples of infinite families of hyperbolic 3-manifolds whose KBSMs are finitely generated.

- (Detcherry-Wolff, 2020): For any compact closed oriented surface Σ of genus $g \geq 2$, $\mathcal{S}_{2,\infty}(\Sigma \times S^1; \mathbb{Q}(A), A)$ is a finite dimensional $\mathbb{Q}(A)$-module with dimension $2^{2g+1} + 2g - 1$. Earlier, in 2018, Gilmer and Masbaum had shown that the dimension of this skein module is at least $2^{2g+1} + 2g - 1$.

Theorem (Przytycki, 1987): $\mathcal{S}_{2,\infty}(\Sigma \times I)$ is a free module generated by the empty link $\emptyset$ and links in Σ which have no trivial components. This applies in particular to handlebodies, since $H_n = \Sigma_{0,n+1} \times I$.

- $\mathcal{S}_{2,\infty}(\Sigma_{0,2} \times I; R, A)$ is free & infinitely generated by the curves $\{x^i\}_{i=0}^{\infty}$, $x^0 = \emptyset$.

- $\mathcal{S}_{2,\infty}(\Sigma_{0,3} \times I; R, A)$ is a free & infinitely generated by the monomials $\{a_1^i a_2^j a_3^k\}_{i,j,k \geq 0}$. $a_1^0 a_2^0 a_3^0 = \emptyset$



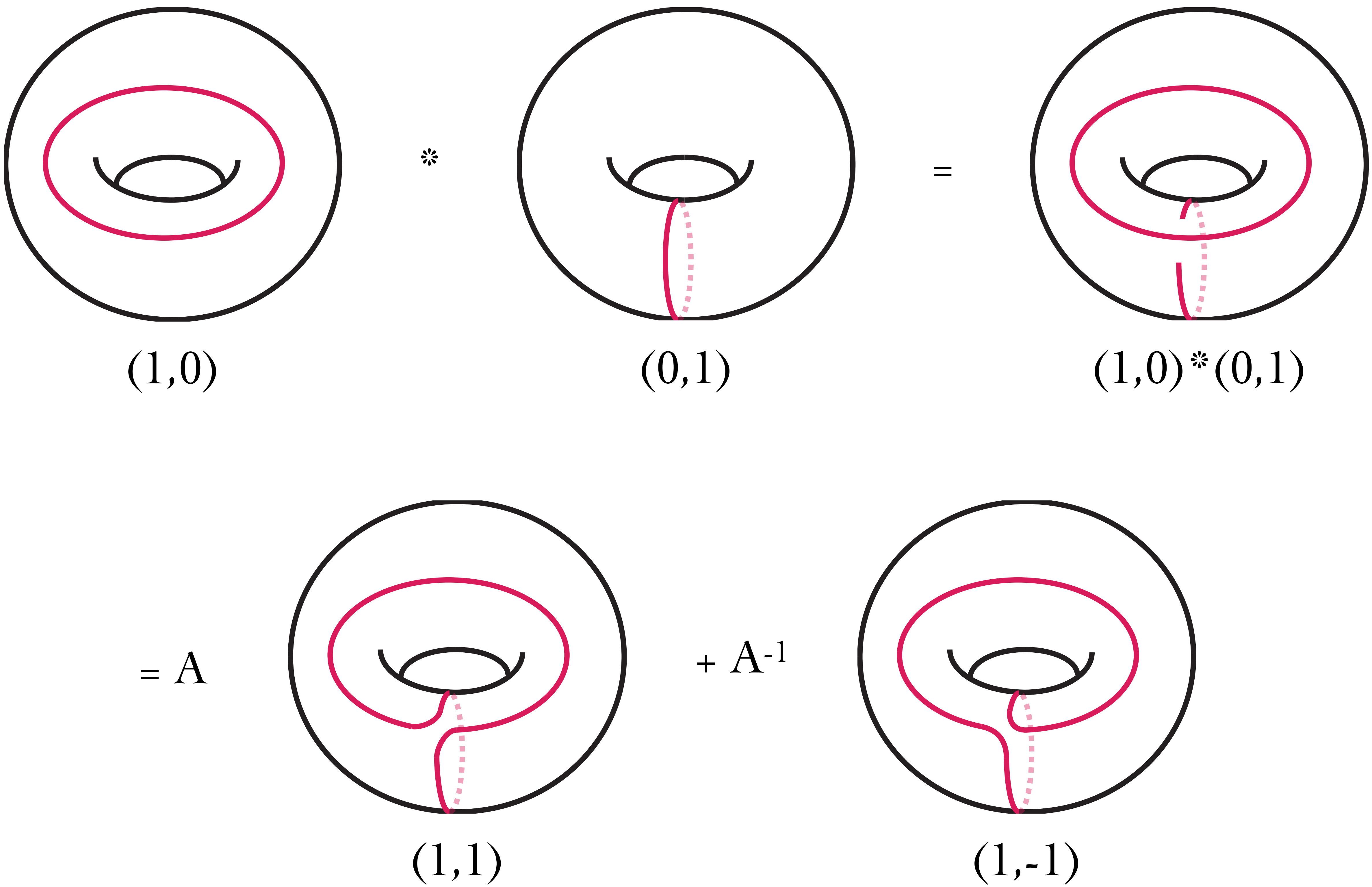
- $\mathcal{S}_{2,\infty}(\Sigma_{1,1} \times I; R, A) \cong \mathcal{S}_{2,\infty}(\Sigma_{0,3} \times I; R, A)$.

- $\mathcal{S}_{2,\infty}(T^2 \times I; R, A)$ is a free R-module generated by $\emptyset$, all (p, q)-curves, and their parallel copies on the torus. $gcd(p, q) = 1$.

- $\mathcal{S}_{2,\infty}(\mathbb{R}P^2 \hat{\times} I; R, A) \cong \mathcal{S}_{2,\infty}(\mathbb{R}P^3; R, A) \cong R \oplus R$.

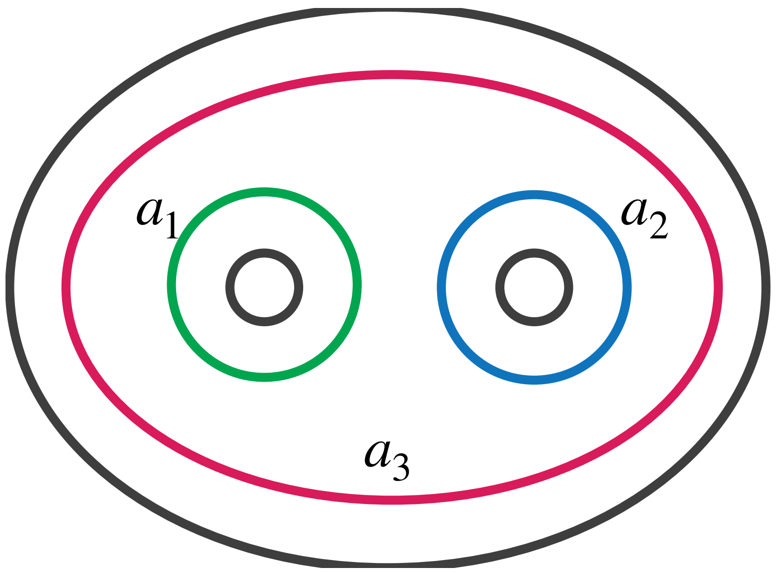
Digression to Skein Algebras

- The KBSM of $\Sigma \times I$ can be enriched with an algebra structure.
- The empty link serves as the multiplicative identity.
- $L_1 \cdot L_2 :=$ place L_1 over L_2 that is, $L_1 \subset \Sigma \times (\frac{1}{2}, 1)$ and $L_2 \subset \Sigma \times (0, \frac{1}{2})$


(1,0)
*
(0,1)
=
(1,0)*(0,1)
= A
(1,1)
+ A-1
(1,-1)

Theorem (Bullock - Przytycki, 2000)

- $\mathcal{S}^{alg}(S^2) \cong \mathcal{S}^{alg}(\Sigma_{0,1}) \cong \mathbb{Z}[A^{\pm 1}]$.
- $\mathcal{S}^{alg}(\Sigma_{0,2}) \cong \mathbb{Z}[A^{\pm 1}][x]$.
- $\mathcal{S}^{alg}(\Sigma_{0,3}) \cong \mathbb{Z}[A^{\pm 1}][a_1, a_2, a_3]$.



- $\mathcal{S}^{alg}(\Sigma_{g,b})$ is commutative when $g = 0$ and $b \leq 3$.
- $\mathcal{S}^{alg}(\Sigma_{g,b})$ is commutative when $A = \pm 1$.

$SL(2,\mathbb{C})$-character variety of M: $\chi(M) := Hom(\pi_1(M), SL(2,\mathbb{C}))//SL(2,\mathbb{C})$

Coordinate ring of $\chi(M)$: $\mathbb{C}[\chi(M)]$

Theorem (Bullock, Przytycki - Sikora 1997)

$\mathcal{S}^{alg}(\Sigma; \mathbb{C}, -1)$ is isomorphic to $\mathbb{C}(\chi(F))$.

Properties of Skein Modules

PROPERTIES OF KBSMs

(Przytycki 1987)

- $i : M^3 \hookrightarrow N^3$ orientation preserving $\Rightarrow i_* : \mathcal{S}_{2,\infty}(M^3; R, A) \longrightarrow \mathcal{S}_{2,\infty}(N^3; R, A)$ homomorphism.

- $N^3 = M^3 \cup 3 - handle \Rightarrow \mathcal{S}_{2,\infty}(M^3; R, A) \xrightarrow{\text{iso}} \mathcal{S}_{2,\infty}(N^3; R, A)$.

- $N^3 = M^3 \cup_\gamma 2 - handle \Rightarrow \mathcal{S}_{2,\infty}(M^3; R, A) \xrightarrow{\text{epi}} \mathcal{S}_{2,\infty}(N^3; R, A)$. In particular,

$$\mathcal{S}_{2,\infty}(N^3; R, A) = \frac{\mathcal{S}_{2,\infty}(M^3; R, A)}{\langle L - sl_\gamma(L) \rangle}.$$

- $N^3 = M^3 \cup 0 - handle \Rightarrow \mathcal{S}_{2,\infty}(N^3; R, A) = \mathcal{S}_{2,\infty}(M^3; R, A) \otimes R$.

- Any compact oriented M^3 is obtained from H_m by adding 2- and 3-handles. Thus, the generators of $\mathcal{S}_{2,\infty}(M;R,A)$ are the generators of $\mathcal{S}_{2,\infty}(H_m;R,A)$ and the relators of $\mathcal{S}_{2,\infty}(M;R,A)$ are obtained by handle sliding relations.

- $M = H_1 \cup_F H_2$, $\mathcal{S}_{2,\infty}(M;R,A) = \mathcal{S}_{2,\infty}(H_1;R,A) \otimes_{\mathcal{S}^{alg}(\Sigma;R,A)} \mathcal{S}_{2,\infty}(H_2;R,A)$. (McLendon 2004)

Theorem (Przytycki, 2000): If M^3 and N^3 are compact, then

$$\mathcal{S}_{2,\infty}(M^3 \# N^3;\mathbb{Q}(A)) = \mathcal{S}_{2,\infty}(M^3;\mathbb{Q}(A)) \otimes \mathcal{S}_{2,\infty}(N^3;\mathbb{Q}(A)).$$

Theorem [Witten's conjecture, Gunningham - Jordan - Safronov, 2023]

The Kauffman bracket skein module for any closed oriented 3-manifold over $\mathbb{Q}(A)$ is finite dimensional.

- $\mathcal{S}_{2,\infty}(S^1 \times S^2, \mathbb{Q}(A)) \cong \mathbb{Q}(A)$
- This theorem does not hold when $R = \mathbb{Z}[A^{\pm 1}]$.
- The KBSM of $S^1 \times S^2$ over $\mathbb{Z}[A^{\pm 1}]$ is infinitely generated.

Marché's Generalisation of Witten's Conjecture

Consider closed, compact M^3. Then there exists an integer $d \geq 0$ and finitely generated $\mathbb{Z}[A^{\pm 1}]$ -modules N_k so that

$$\mathcal{S}_{2,\infty}(M^3) = (\mathbb{Z}[A^{\pm 1}])^d \oplus \bigoplus_{k \geq 1} N_k,$$

where N_k is a $(A^k - A^{-k})$ -torsion module for each k .

- A byproduct of the proof by Detcherry and Wolff is that torsion elements of $\mathcal{S}_{2,\infty}(\Sigma \times S^1; \mathbb{Z}[A^{\pm 1}])$ are always of $(A^k - A^{-k})$ -torsion for some $k \geq 1$.

$$\mathcal{S}_{2,\infty}(S^1 \times S^2)) = \mathbb{Z}[A^{\pm 1}] \oplus \bigoplus_{i=1}^{\infty} \frac{\mathbb{Z}[A^{\pm 1}]}{1 - A^{2i+4}}.$$

- (B., 2022): This conjecture is not true.

- $\mathcal{S}_{2,\infty}(\mathbb{R}P^3 \# \mathbb{R}P^3)$ is a counterexample.

$\mathcal{S}_{2,\infty}(\mathbb{R}P^3 \# \mathbb{R}P^3) = \mathbb{Z}[A^{\pm 1}] \oplus \mathbb{Z}[A^{\pm 1}] \oplus \mathbb{Z}[A^{\pm 1}][t]/S$, where S is a submodule of $\mathbb{Z}[A^{\pm 1}][t]$ generated by the following two relations:

1. $(A^{n+1} + A^{-(n+1)})(S_n(t) - 1) - 2(A + A^{-1})) \sum_{k=1}^{n/2} A^{n+2-4k}$, for $n \geq 2$ even,

2. $(A^{n+1} + A^{-(n+1)})(S_n(t) - t) - 2t \sum_{k=1}^{(n-1)/2} A^{n+1-4k}$, for $n \geq 3$ odd.

Here $t = -A^{-3}x$, that is, t represents x with a negative full-twist, where x denotes the knot that runs parallel to the core of the solid torus $S^1 \times D^2$, and $S_n(t)$ denotes the Chebyshev polynomial of the second kind.

- Mroczkowski and his work on arrow diagrams.

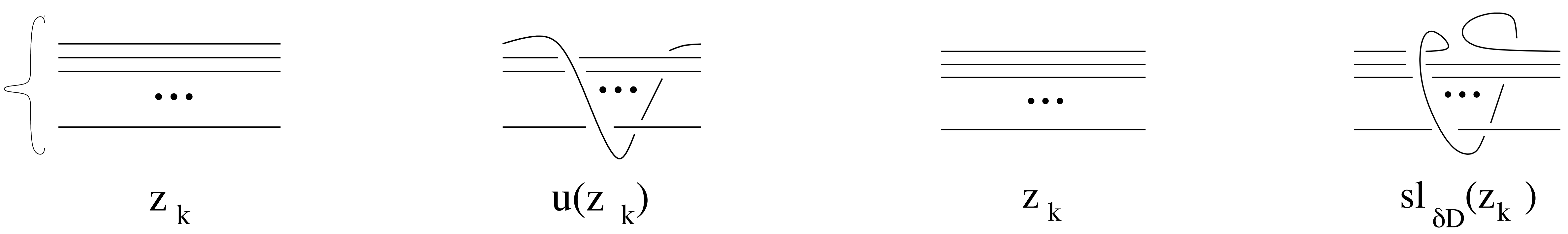
Q. What is the structure of the KBSM over $\mathbb{Z}[A^{\pm 1}]$?

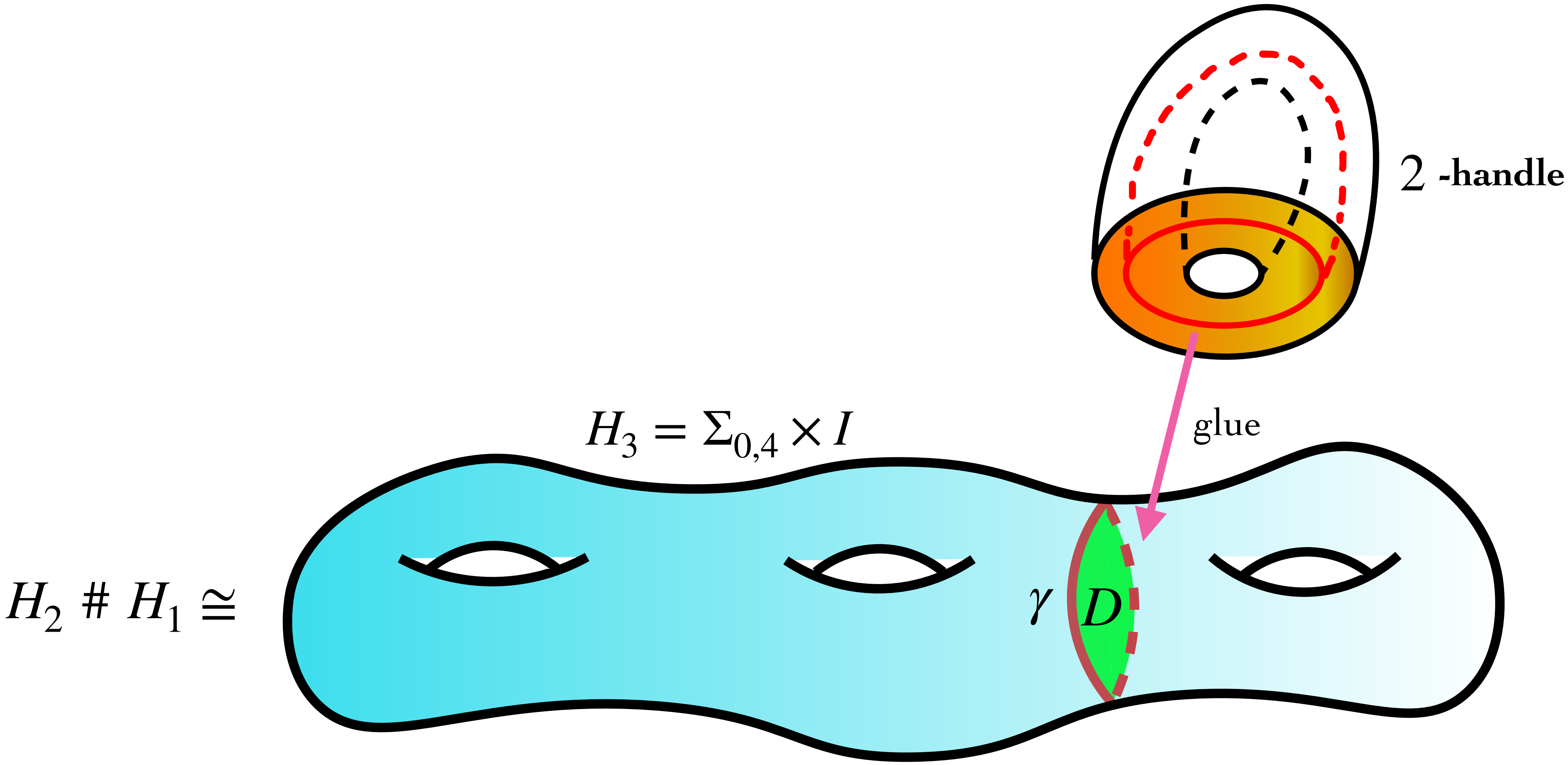
First step: Connected sums of handlebodies.

Theorem (Przytyki, 2000)

$$\mathcal{S}_{2,\infty}(H_n \# H_m) \cong \mathcal{S}_{2,\infty}(H_{n+m})/\mathcal{J},$$

- $\mathcal{J} = z_k - A^6 u(z_k)$ for any even $k \geq 2$,
- z_k are basis elements of of $\mathcal{S}_{2,\infty}(\Sigma_{0,3} \times I)$ with geometric intersection number k with the disc D separating H_n and H_m.





2 -handle
glue
$H_3 = \Sigma_{0,4} \times I$
$H_2 \# H_1 \cong$
γ
D

(B., 2022): This theorem is incorrect. The counterexample is given by $H_n \# H_m$ $n \geq 2, m \geq 1$ and we the submodule $\mathcal{I}$ should be replaced by a strictly bigger submodule to obtain the equality stated in the theorem.

Theorem (B., 2022)

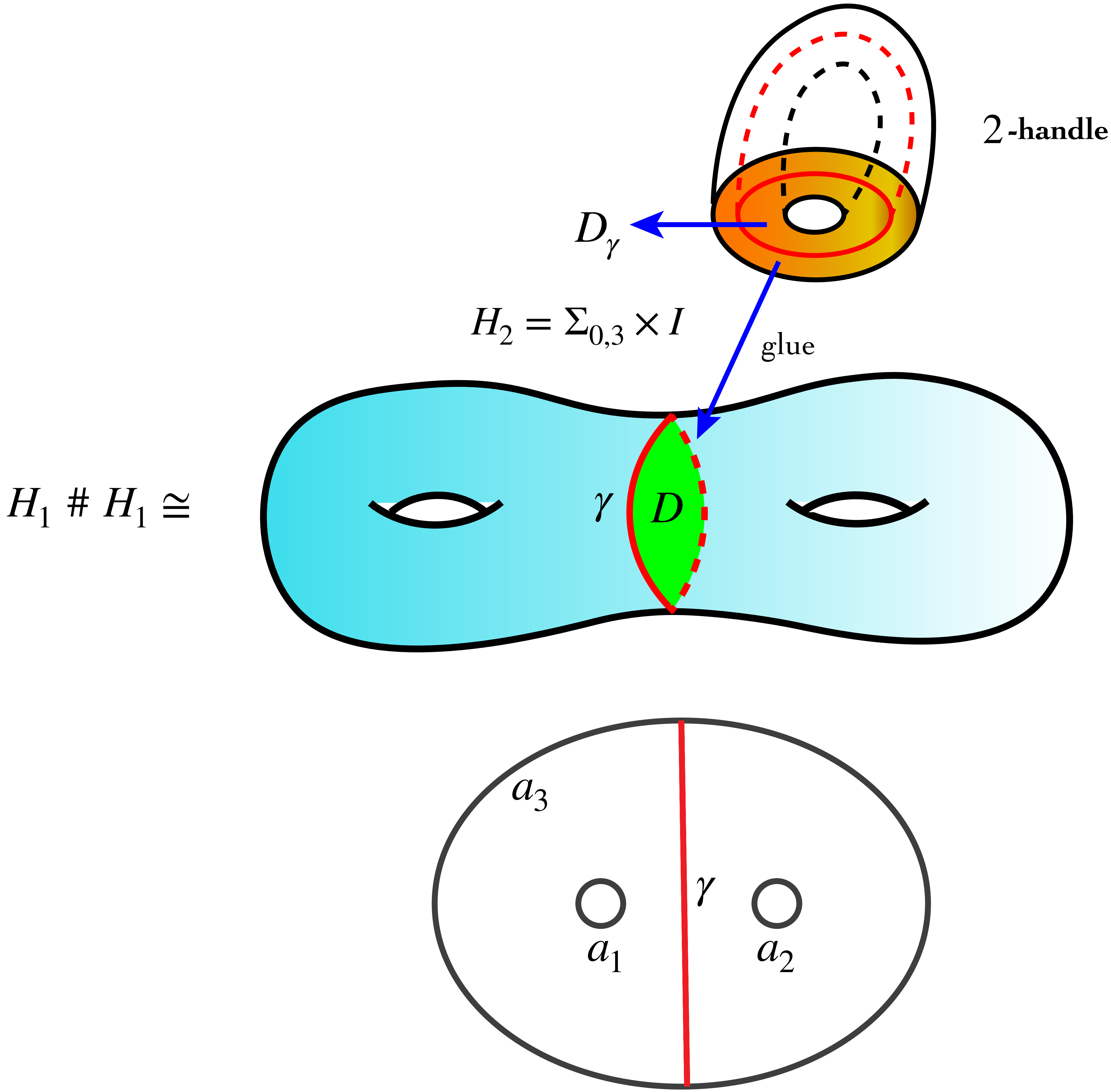
1. The natural epimorphism $i_* : \mathcal{S}_{2,\infty}(H_3)/\mathcal{I} \longrightarrow \mathcal{S}_{2,\infty}(H_2 \# H_1)$ is not an isomorphism.
2. In general, the natural epimorphism $i_* : \mathcal{S}_{2,\infty}(H_{n+m})/\mathcal{I} \longrightarrow \mathcal{S}_{2,\infty}(H_n \# H_m)$, where $n + m \geq 3$ is not an isomorphism.

Theorem (B. - Lê - Przytyki, to appear)

$$\mathcal{S}_{2,\infty}(H_1 \# H_1) \cong \mathcal{S}_{2,\infty}(H_2)/\mathcal{J},$$

- $\mathcal{J} = z_k - A^6 u(z_k)$ for any even $k \geq 2$,
- z_k are basis elements of of $\mathcal{S}_{2,\infty}(\Sigma_{0,3} \times I)$ with geometric intersection number k with the disc D separating H_1 and H_1.

z_k $u(z_k)$ z_k $sl_{\delta D}(z_k)$



Projection of $H_1 \# H_1$ onto $\Sigma_{0,3}$

Theorem (B. - Lê - Przytycki, to appear)

The natural epimorphism $i_* : \mathcal{S}_{2,\infty}(H_2)/\mathcal{I} \longrightarrow \mathcal{S}_{2,\infty}(H_1 \# H_1)$ is an isomorphism. In particular,

$$\mathcal{S}_{2,\infty}(H_1 \# H_1) = \frac{\mathcal{S}_{2,\infty}(H_2)}{\langle z_k - A^6 u(z_k) \rangle}.$$

RELATIVE KAUFFMAN BRACKET SKEIN MODULES

$\mathcal{S}_{2,\infty}((M^3, \partial M^3), \{x_i\}_1^{2n}) =$ $R\{$ambient isotopy classes of relative unoriented framed links in $M^3\}$

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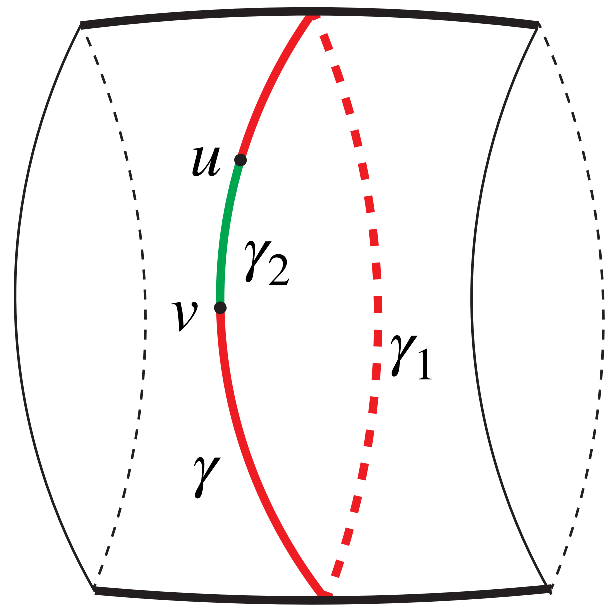
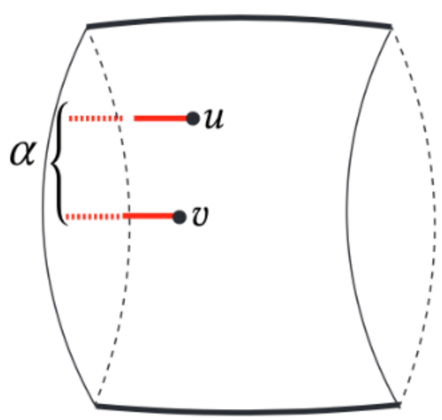
1. Kauffman bracket skein relation
2. Trivial component relation

Theorem (Przytycki 1987) Let $\partial F \neq \emptyset$, and let $\{x_i\}_1^{2n}$ be $2n$ oriented framed points that all lie on $\partial F \times \{\frac{1}{2}\}$. Then $\mathcal{S}_{2,\infty}(M, \{x_i\}_1^{2n}; R, A)$ is a free R-module whose basis is composed of relative links in $F \times \{\frac{1}{2}\}$ without trivial components.

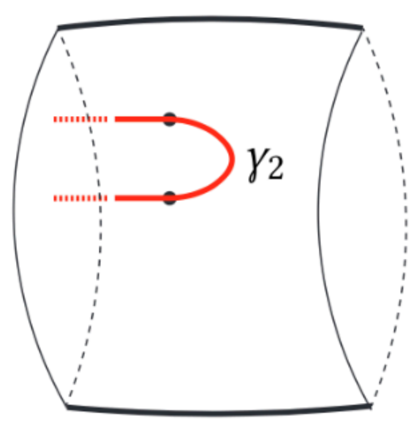
IDEA OF THE PROOF



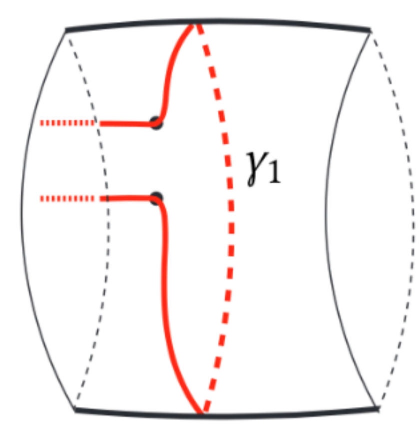
Annular neighbourhood of γ in ∂M



(A) Relative curve in $\mathcal{S}_{2,\infty}(M; u, v)$



(B) $\alpha \cup \gamma_2$

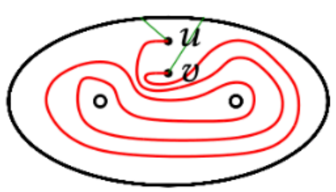


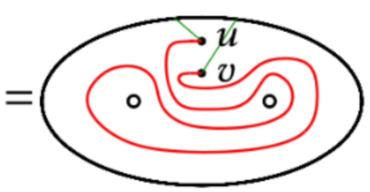
(C) $\alpha \cup \gamma_1$

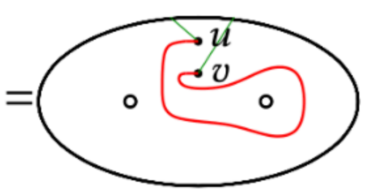
- $\omega : \mathcal{S}_{2,\infty}(H_2; u, v) \longrightarrow \mathcal{S}_{2,\infty}(H_2)$ given by $\omega(\alpha) = \alpha \cup \gamma_1 - \alpha \cup \gamma_2$.

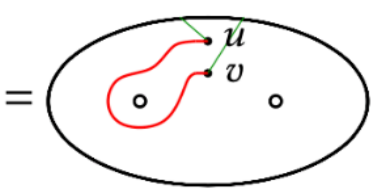
- The image under ω of the generators of $\mathcal{S}_{2,\infty}(H_2; u, v)$ form the generating set of all the handle sliding relations in $\mathcal{S}_{2,\infty}(H_2)$.

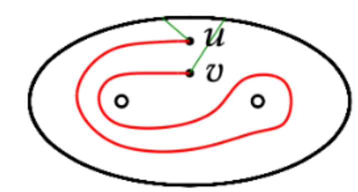
- Proposition: The set $\{c_L a_3^{i_3}, c_R a_3^{i_3}, c_0 a_3^{i_3}, c_1 a_3^{i_3}\}$ forms a basis of $\mathcal{S}_{2,\infty}(F_{0,3} \times I; u, v)$.

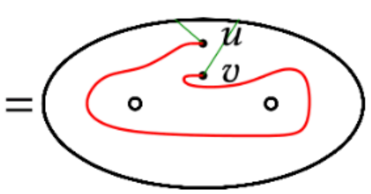
$c_{-2} =$


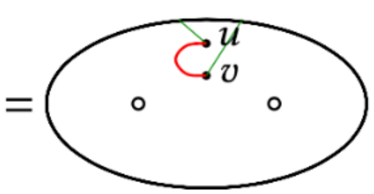
$c_{-1} =$


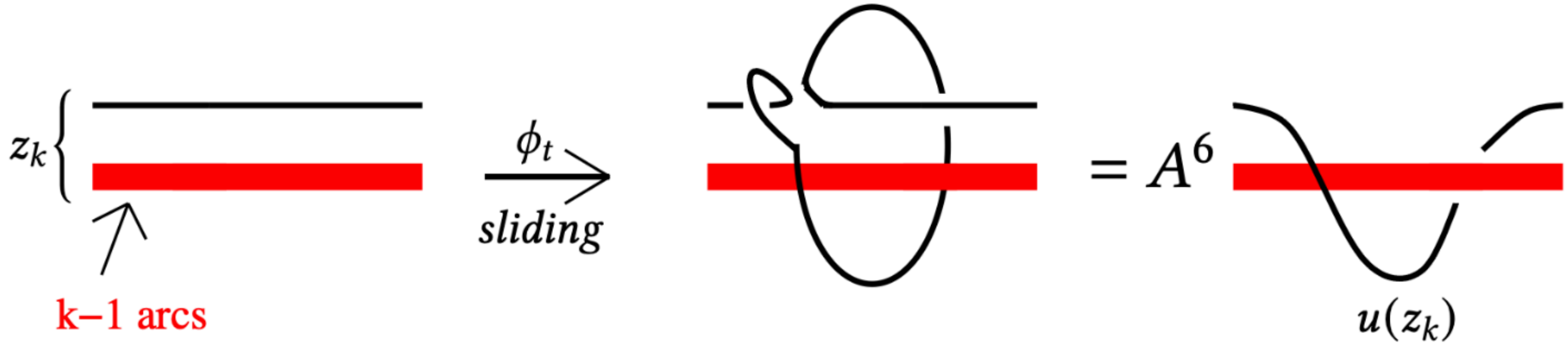
$c_0 =$


$c_1 =$


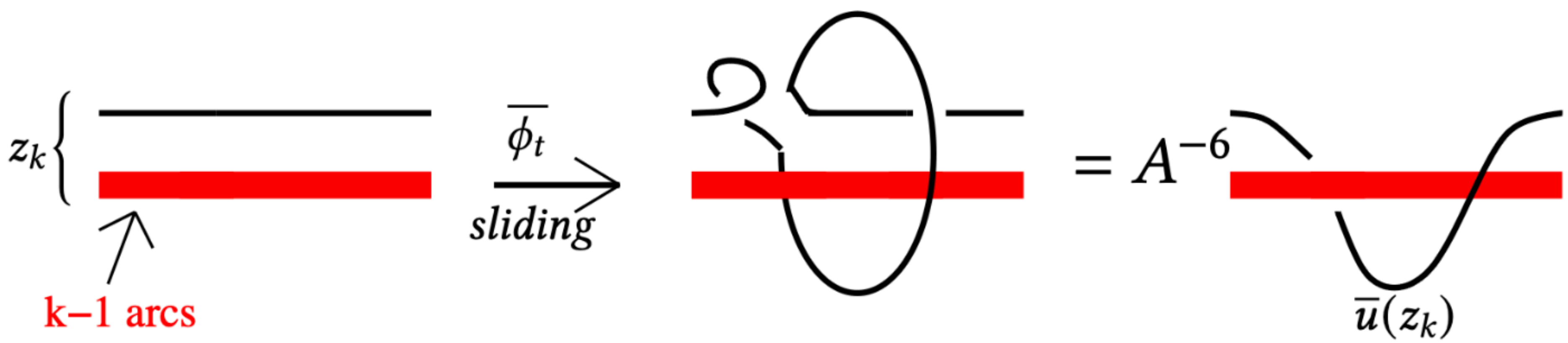
$c_2 =$


$c_L =$


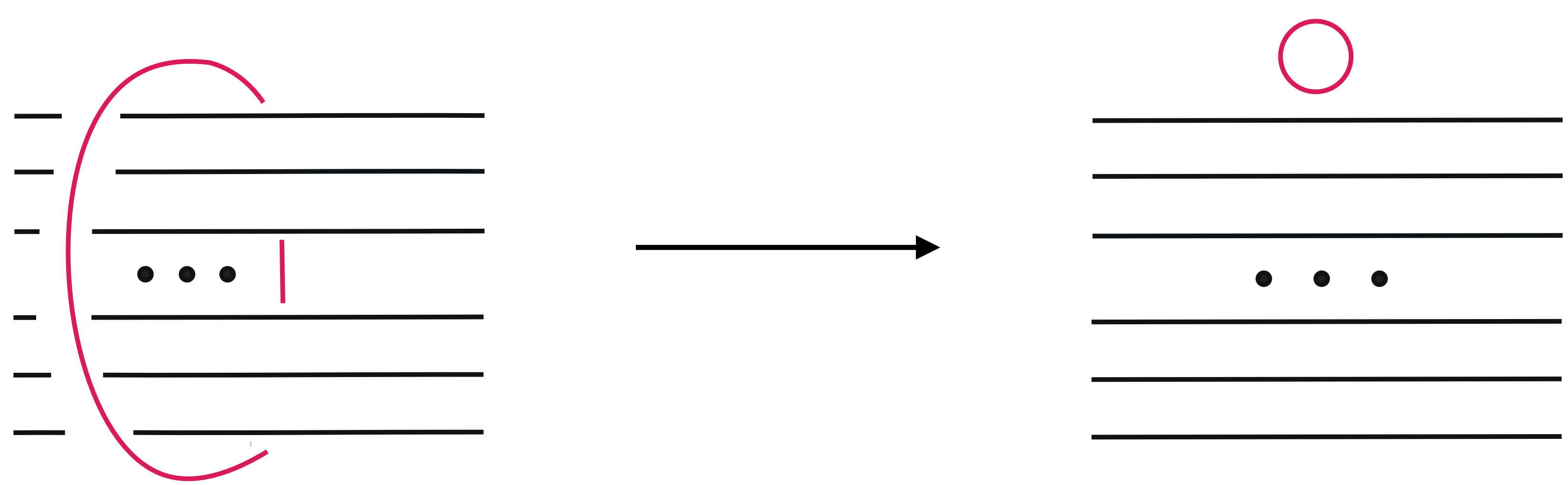
$c_R =$




Positive 2-handle sliding on the upper arc, denoted by ϕ_t .



Negative 2-handle sliding on the upper arc, denoted by $\bar{\phi}_t$.



Ring sliding.

- Negative 2-handle sliding on the upper arc is spanned by $\{\omega(c_L a_3^{i_3})\}_{i_3 \geq 0}$.
- $\omega(c_R a_3^{i_3})$ corresponds to ring sliding in which a ring encircling $a_3^{i_3}$ is taken out and becomes the unknot.
- $\omega(c_1 a_3^{i_3}) = 0$.
- $\omega(c_0 a_3^{i_3})$ follows from the ring sliding relation described in part (1).
- Ring sliding follows from negative 2-handle sliding on the top arc.
- Thus, $\mathcal{S}_{2,\infty}(H_1 \# H_1) = \dfrac{\mathcal{S}_{2,\infty}(H_2)}{\langle z_k - A^6 \bar{u}(z_k) \rangle}$.

HANDLE SLIDING RELATIONS

$$n = 1 : (A^4 - A^{-4})S_1(a_3) + (A^2 - A^{-2})S_1(a_1)S_1(a_2) = 0.$$

$$n = 2 : (-1)(A^6 - A^{-6})S_2(a_3) + (A^2 - A^{-2})S_2(a_1)S_2(a_2) = 0.$$

$$n = 3 : (A^8 - A^{-8})S_3(a_3) + (A^2 - A^{-2})S_3(a_1)S_3(a_2) = 0.$$

$$n = 4 : (-1)(A^{10} - A^{-10})S_4(a_3) + (A^2 - A^{-2})S_4(a_1)S_4(a_2) = 0.$$

Theorem (B. - Lê - Przytycki, to appear)

The handle sliding relations are represented by Chebyshev polynomials. In particular,

$$\mathcal{S}_{2,\infty}(H_1 \# H_1) = \frac{\mathbb{Z}[A^{\pm 1}][a_1, a_2, a_3]}{\mathcal{J}},$$

where $\mathcal{J}$ is a submodule of $\mathbb{Z}[A^{\pm 1}][a_1, a_2, a_3]$ generated by

$$(-1)^{n+1}\{n+1\}S_n(a_3) + \{1\}S_n(a_1)S_n(a_2), n \geq 1.$$

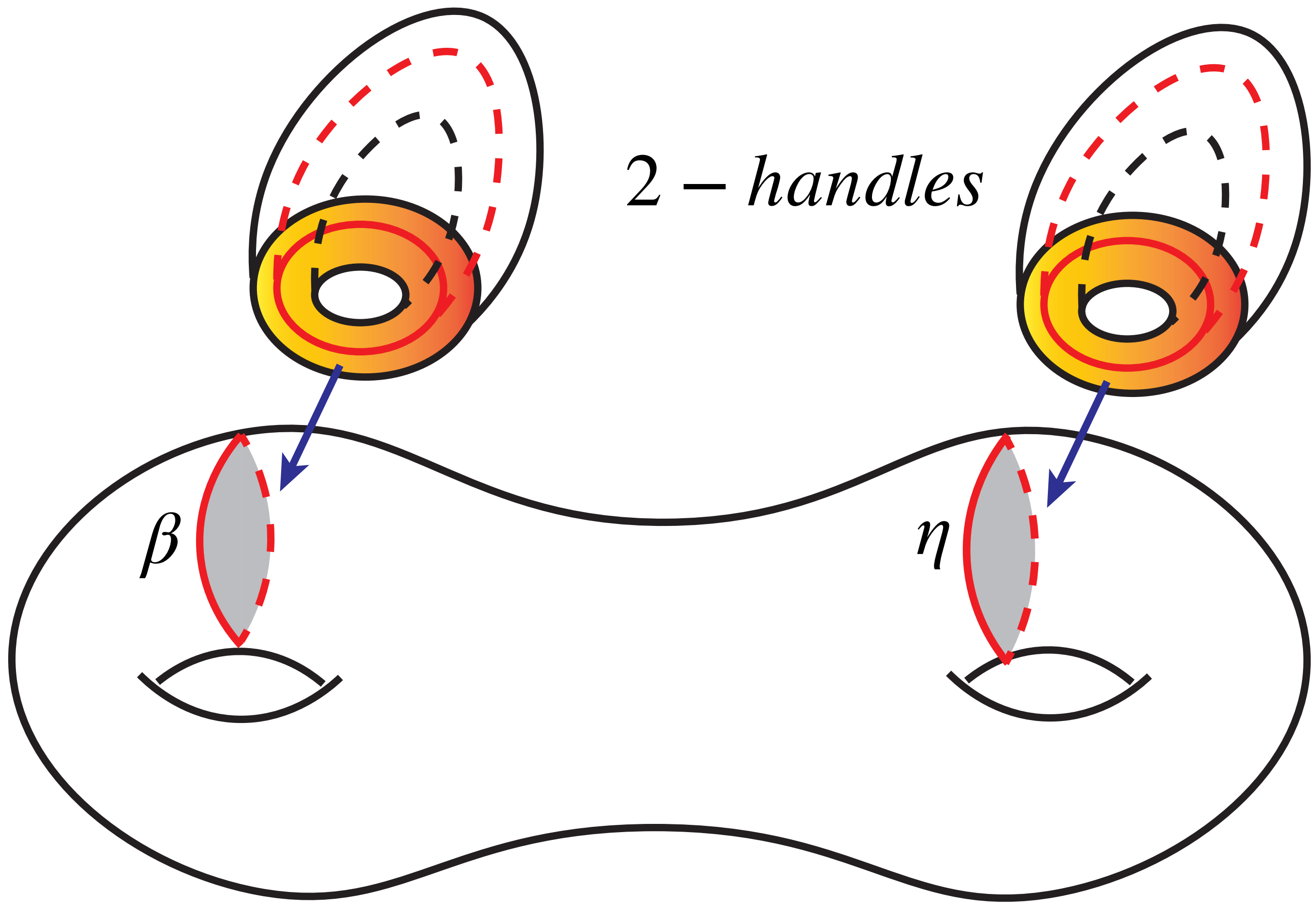
Here, S_n is the n^{th} Chebyshev polynomial of the second kind and $\{k\} = A^{2k} - A^{-2k}$.

The Skein Module of

$$\mathcal{S}_{2,\infty}(S^1 \times S^2 \# S^1 \times S^2)$$

THE KBSM OF $S^1 \times S^2 \# S^1 \times S^2$

$S^1 \times S^2 \# S^1 \times S^2 \cong$



$H_2 = F_{0,3} \times I$

Then, $\mathcal{S}_{2,\infty}(S^1 \times S^2 \# S^1 \times S^2) \cong \dfrac{\mathcal{S}_{2,\infty}(H_2)}{\mathcal{J}}$.

Theorem (B. - Kim - Shi - Wang, 2025)

$$C_{m,n}S_q(a_2) = (-A^{m+n+2} + A^{-m-n-2})S_m(a_1)S_n(a_3)S_q(a_2)$$
$$+(-A^{m+n} + A^{-m-n})S_{m-1}(a_1)S_{n-1}(a_3)S_{q-1}(a_2)$$
$$+(-A^{m+n} + A^{-m-n})S_{m-1}(a_1)S_{n-1}(a_3)S_{q+1}(a_2)$$
$$+(-A^{m+n-2} + A^{-m-n+2})S_{m-2}(a_1)S_{n-2}(a_3)S_q(a_2)$$

for all $m, n, q \geq 0$, *where* $S_{-2}(x) = -1$, $S_{-1}(x) = 0$.

Theorem (B. - Kim - Shi - Wang, 2025)

$\mathcal{S}_{2,\infty}(S^1 \times S^2 \# S^1 \times S^2) \cong \mathbb{Z}[A^{\pm 1}]\langle S_l(a_1)S_m(a_2)S_n(a_3)\rangle / \mathcal{J}$, where

$$\mathcal{J} = \langle C_{m,n}S_q(a_2), \overline{C}_{m,n}S_q(a_1)\rangle.$$

Non-splitness into free and torsion modules

- (Hoste - Przytycki, 1990): $\mathcal{S}_{2,\infty}(S^1 \times S^2)$ is an infinitely generated module.

$$\mathcal{S}_{2,\infty}(S^1 \times S^2)) = \mathbb{Z}[A^{\pm 1}] \oplus \bigoplus_{i=1}^{\infty} \frac{\mathbb{Z}[A^{\pm 1}]}{1 - A^{2i+4}},$$

where the free part is generated by the empty link ∅.

Theorem (B. - Kim - Shi - Wang, 2025)

$\mathcal{S}_{2,\infty}(S^1 \times S^2 \# S^1 \times S^2)$ does not split into the sum of free and torsion submodules.

Idea of Proof

- If $\mathcal{S}_{2,\infty}((S^1 \times S^2) \# (S^1 \times S^2))$ can be decomposed to free and torsion parts. Then the empty link is the one and only generator for the free part.

- We have a module homomorphism $\iota : \mathcal{S}_{2,\infty}((S^1 \times S^2) \# (S^1 \times S^2)) \longrightarrow \mathbb{Z}[A^{\pm 1}]$. Denote $\iota(S_{n_1}(a_1)S_{n_3}(a_3)S_{n_2}(a_2))$ by α_{n_1,n_3,n_2}.

- Show that $\alpha_{n,n,n} \neq 0$ and breadth$(\alpha_{0,0,0}) >$ breadth$(\alpha_{1,1,1}) > \cdots >$ breadth$(\alpha_{n,n,n}) > \cdots$

- However, since breadth$(\alpha_{n,n,n}) \in \mathbb{N} \cup \{0\}$, we get a contradiction.

Torsion Elements in $\mathcal{S}_{2,\infty}(S^1 \times S^2 \# S^1 \times S^2)$

Theorem (B. - Kim - Shi - Wang, 2025)

The set of elements of the form

$$\Sigma_{i=0}^{m+n+1}(A^{-m-n-1+2i})S_m(a_1)S_n(a_3)S_q(a_2)$$
$$+\Sigma_{i=0}^{m+n-1}(A^{-m-n+1+2i})S_{m-1}(a_1)S_{n-1}(a_3)S_{q+1}(a_2)$$
$$+\Sigma_{i=0}^{m+n-1}(A^{-m-n+1+2i})S_{m-1}(a_1)S_{n-1}(a_3)S_{q-1}(a_2)$$
$$+\Sigma_{i=0}^{m+n-3}(A^{-m-n+3+2i})S_{m-2}(a_1)S_{n-2}(a_3)S_q(a_2)$$

is a family of $(1 - A^2)$-torsion elements, for all $m, q \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{Z}$

- $\mathcal{S}_{2,\infty}(S^1 \times S^2)$ is an infinitely generated module.

$$\mathcal{S}_{2,\infty}(S^1 \times S^2)) = \mathbb{Z}[A^{\pm 1}] \oplus \bigoplus_{i=1}^{\infty} \frac{\mathbb{Z}[A^{\pm 1}]}{1 - A^{2i+4}},$$

Theorem (B. - Kim - Shi - Wang, 2025)

Consider the embedding $j : (S^1 \times S^2) - D^3 \to (S^1 \times S^2) \# (S^1 \times S^2)$ that sends the class of the longitude curve to a_1 and the meridian to β. Then, $(1 - A^{2i+4}) j_*(e_i') = 0$ and $j_*(e_i') \neq 0$ in $\mathcal{S}_{2,\infty}((S^1 \times S^2) \# (S^1 \times S^2))$ for all i.

FUTURE DIRECTIONS

- What is the structure of the KBSM of $\#_k(S^1 \times S^2)$?
- Generalise this to $H_1 \# L(p,q)$ and $L(p_1,q_1) \# L(p_2,q_2)$.
- What is the structure of the KBSM of $H_n \# H_m$?
- What is the structure of the KBSM of Seifert - fibered manifolds?
- When is the Kauffman bracket skein module of 3−manifold the sum of free and torsion modules?

Conjecture

- The KBSM of any closed, prime, oriented 3-manifold can be decomposed into the direct sum of free modules and torsion modules.

- The KBSM of two nontrivial closed, oriented 3 -manifolds does not split into the sum of free and torsion modules.

We Did Not Touch Upon…

- The volume conjecture
- The AJ conjecture
- Kauffman bracket skein algebras and their generalisations
- SL_n skein modules and algebras
- Categorification of skein modules
- Skein modules in higher dimensions