Symmetries of link homology.

Joint with:
You Qi
Louis-Hadrien Robert
Emmanuel Wagner

July 23, 2025

Outline

- Review of foams.
- $\mathfrak{sl}_2$-actions on foams.
- $\mathfrak{sl}_2$-actions on link homology.
- Characteristic p.

Motivation

There is a history of actions of algebraic structures on link homology:

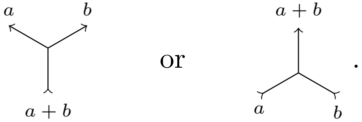
- Lipschitz-Sarkar defined Steenrod operations on Khovanov homology.
- Khovanov-Rozansky defined a Witt-type action on HOMFLYPT homology.
- p-DG structures give rise to categorification of knot homologies at roots of unity.
- Shumakovitch and Wang defined an operator e on Khovanov-Rozansky $\mathfrak{sl}_p$ homology in characteristic p. This proved reduced homology is independent of basepoint. They also showed how to recover unreduced homology.

Motivation

- Elias-Qi defined an $\mathfrak{sl}_2$-action on the Soergel categories.
- Elias-Qi and Khovanov defined an $\mathfrak{sl}_2$-action on categorified quantum groups. Connected to this talk by work of Grlj–Lauda.
- Grigsby-Licata-Wehrli defined an $\mathfrak{sl}_2$-action on annular Khovanov homology. Queffelec-Rose extended this to annular Khovanov-Rozansky homology.
- Gorsky-Hogancamp-Mellit defined an $\mathfrak{sl}_2$-action on y-ified homology. This gives equality of dimensions categorifying symmetries of the HOMFLYPT polynomial.

Review of foams and webs

- Let Γ be an oriented, trivalent graph in $\mathbb{R}^2$.
- Edges are labeled by $\mathbb{N}$ and Γ has the following flow conditions at each vertex:



- $MOY_N(\Gamma)$ is a Laurent polynomial assigned to each closed web which leads to the $\mathfrak{gl}_N$ Reshetikhin-Turaev invariant of a link L.

Review of foams and webs

- Let $F \subset \mathbb{R}^3$ be a foam. It's a collection of facets (compact oriented surfaces labeled by $\mathbb{N}_{\geq 0}$) glued along boundary points such that each $p \in F$ has a closed neighborhood homeomorphic to one of the following:
  1. A disk (when p belongs to exactly one facet).
  2. $Y \times [0,1]$ where Y is the neighborhood of a merge or split vertex in a web. This singular set is called a binding. (This happens when p belongs to exactly 3 facets).
  3. "Ping-pong table". This singular set is called a singular vertex. (This happens when p belongs to exactly 6 facets).

Review of foams and webs

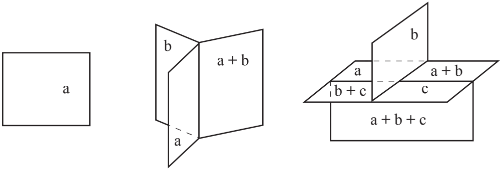


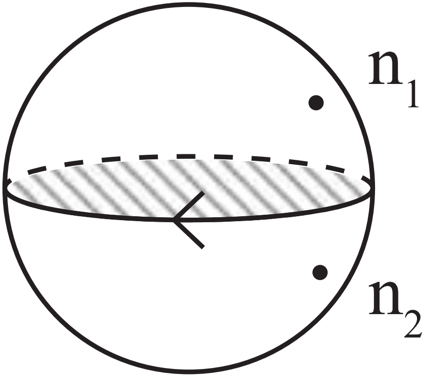
FIGURE 1.2.1. Three types of points on a foam

Review of foams and webs

- The label of each facet $\ell(f)$ is called its thickness and it satisfies a flow condition.
- Each facet f has a decoration $P_f \in Sym_{\ell(f)}$.
- Let $F_2 =$ collection of facets, $F_1 =$ collection of bindings, $F_0 =$ collection of singular points.
- A foam F has a degree $deg_N(F)$.

Review of foams and webs

Θ – foam

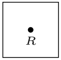


Review of foams and webs

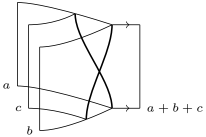
We define a basic foam to be one of the following:

- A trace of isotopy. This is $h_t(\Gamma)$ where $h\colon \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ is a smooth isotopy and Γ is a web.
- A foam equal to $\Gamma \times [0,1]$ outside of a cylinder and inside one of the following:

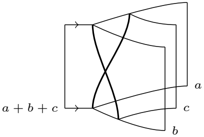
Review of foams and webs



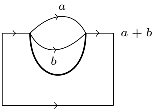
polynomial
$\deg(R)$



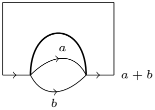
associativity
0



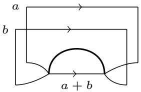
associativity
0



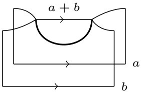
digon-cup
$-ab$



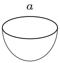
digon-cap
$-ab$



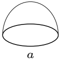
unzip
ab



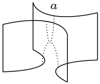
zip
ab



cup
$-a(N-a)$



cap
$-a(N-a)$



saddle
$a(N-a)$

Review of foams and webs

- Later we will think of a foam in terms of a decomposition into basic foams.
- Robert and Wagner defined an evaluation of foams with values in the ring of symmetric functions $\mathbb{Z}[E_1, \dots, E_N]$.

Review of foams and webs

- Let $X_1, \dots, X_N$ be indeterminates.
- Let $\mathbb{P} = \{1, \dots, N\}$ be a set of pigments.
- A $\mathfrak{gl}_N$-coloring c of a foam F is a map $c\colon F^2 \to \mathcal{P}(\mathbb{P})$ such that
  1. $|c(f)| = \ell(f)$ for all facets f.
  2. Around each binding $c(f_{thick}) = c(f_{thin_1}) \cup c(f_{thin_2})$
- Let $F_{ij}(c)$ be the surface which is formed by facets colored by i or j but not both.
- Let $P_f(X_{c(f)})$ be the evaluation of P_f in the variable $X_{c(f)}$.
- Let $Q(F,c) = \prod_{1 \leq i < j \leq N} (X_i - X_j)^{\frac{\chi(F_{ij}(c))}{2}}$

Review of foams and webs

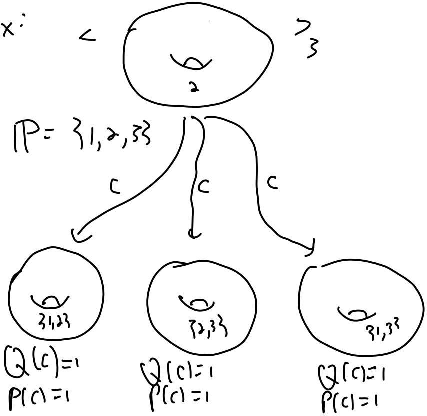
- $P(F, c) = \prod_f P_f(X_{c(f)})$.
- $\langle F, c \rangle = (-1)^{s(F,c)} \frac{P(F,c)}{Q(F,c)}$.
- $\langle F \rangle_N = \sum_c \langle F, c \rangle_N$.

Theorem (Robert-Wagner)

$\langle F \rangle_N \in \mathbb{Z}[E_1, \ldots, E_N] := R_N$.

Review of foams and webs

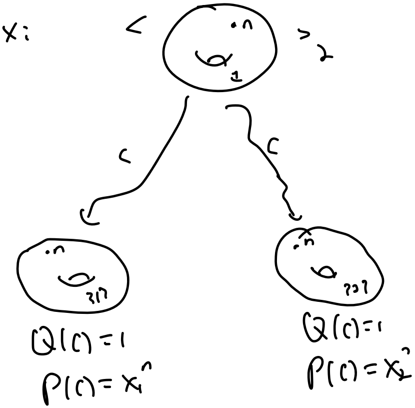
Ex:



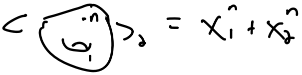
$\langle F \rangle_3 = 3$

Review of foams and webs

Ex:



$P = \{1,2\}$



Review of foams and webs

- Let $\mathcal{V}_N(\Gamma)$ be the free R_N-module generated by all foams going from the empty web to Γ.
- Let $\langle \cdot ; \cdot \rangle_N \colon \mathcal{V}_N(\Gamma) \times \mathcal{V}_N(\Gamma) \to R_N$ by $\langle F ; G \rangle_N = \langle \bar{G} \circ F \rangle_N$.
- Let $\mathcal{F}_N(\Gamma) = \mathcal{V}_N(\Gamma) / ker \langle \cdot ; \cdot \rangle_N$.
- This is an example of the universal construction due to Blanchet, Habegger, Masbaum, and Vogel.

Review of foams and webs

This gives a functor

$$\mathcal{F}_N \colon \mathit{Foam} \to R_N - \mathit{gmod}.$$

Theorem (Robert-Wagner)

$\mathcal{F}_N(\Gamma)$ categorifies $MOY_N(\Gamma)$.

Example

For $N = 2$, the state space of a circle is:

$\mathbb{Z}[E_1, E_2]\langle$  $\rangle \quad \bigoplus \quad \mathbb{Z}[E_1, E_2]\langle$  $\rangle$

$\mathfrak{sl}_2$-action on foams

- For the next few sections, we will work over $\mathbb{C}$ instead of $\mathbb{Z}$.
- Let $t_1, t_2 \in \mathbb{C}$ and $\bar{t}_i = 1 - t_i$.
- We will define operators e, f, h on basic foams as follows.

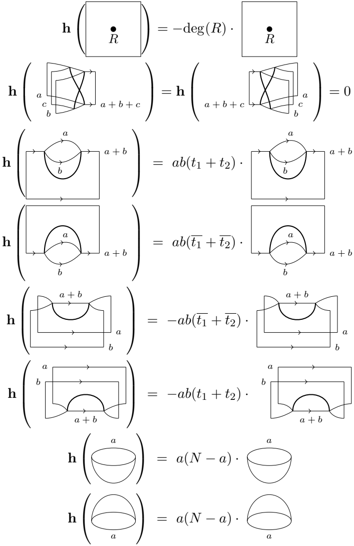
$$e(P_f) = -\sum_i \frac{\partial P_f}{\partial x_i}$$

,

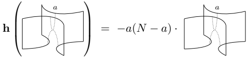
$$e(foam) = 0$$

.

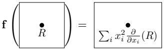
- h, f act as follows, where $\spadesuit_i$ is the ith power sum symmetric function and $\hat{\spadesuit}_i = P_i - \spadesuit_i$:

h
$= -\deg(R) \cdot$
R
R
h
a
c
b
$a+b+c$
$= \mathbf{h}$
$a+b+c$
a
c
b
$= 0$
h
a
b
$a+b$
$= ab(t_1 + t_2) \cdot$
a
b
$a+b$
h
a
b
$a+b$
$= ab(\overline{t_1} + \overline{t_2}) \cdot$
a
b
$a+b$
h
$a+b$
a
b
$= -ab(\overline{t_1} + \overline{t_2}) \cdot$
$a+b$
a
b
h
a
b
$a+b$
$= -ab(t_1 + t_2) \cdot$
a
b
$a+b$
h
a
$= a(N-a) \cdot$
a
h
a
$= a(N-a) \cdot$
a

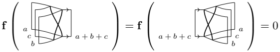
(31)



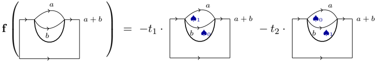
(32)



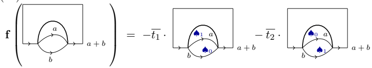
(33)



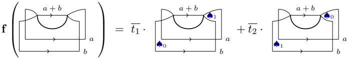
(34)



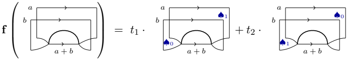
(35)



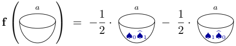
(36)



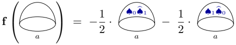
(37)



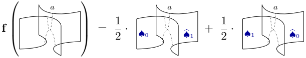
(38)



(39)



(40)



$\mathfrak{sl}_2$-action on foams

Lemma

$\mathfrak{sl}_2$ acts on the set of all foams in good position.

Proof.

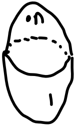
This is a straightforward but lengthy check. ☐

Theorem

If $g \in \mathfrak{sl}_2$, then $\langle g \cdot F \rangle_N = g \cdot \langle F \rangle_N$.

$\mathfrak{sl}_2$-action on foams

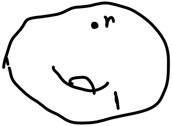
Ex:

$\langle$  $\rangle_{N=2} = h_{n-1}(x_1, x_2) = \dfrac{x^n - y^n}{x-y}$

$f \downarrow \qquad\qquad\qquad\qquad \downarrow f$

$\langle n$  $- E_1$  $\rangle = n\, h_n - E_1 h_{n-1} = n\, h_n - h_1 h_{n-1}$

$\mathfrak{sl}_2$-action on foams

Ex: $\langle$ 

$\rangle_{N=2} = p_n(x_1,x_2) = x_1^n + x_2^n$

$\downarrow f$ $\qquad\qquad$ $\downarrow f$

$\langle\, n$ 

$\rangle = n\,p_{n+1}(x_1,x_2)$

$\mathfrak{sl}_2$-action on foams

Corollary

If Γ is a closed web, then $\mathcal{F}_N(\Gamma)$ is an $\mathfrak{sl}_2$-module.

Proof.

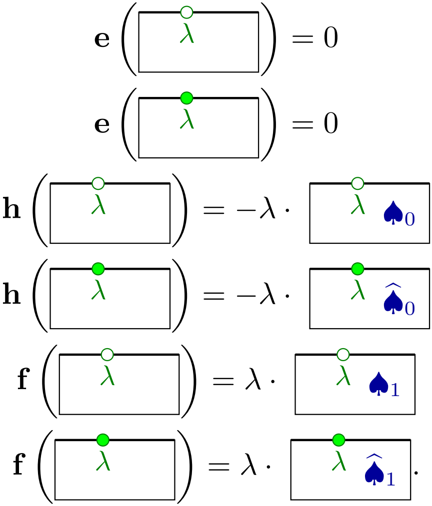
Suppose $F = 0$ in the state space. Then $\langle G; F \rangle = 0$ for all foams G. Then for $g \in \mathfrak{sl}_2$,

$$0 = g \cdot \langle G; F \rangle = \langle g \cdot G; F \rangle + \langle G; g \cdot F \rangle = \langle G; g \cdot F \rangle$$

for all G. Thus $g \cdot F$ is zero in the state space. □

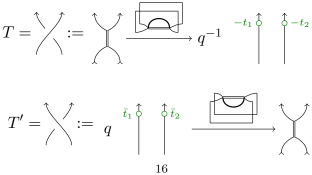
$\mathfrak{sl}_2$-action on foams

We often need to twist the action encoded by green dots.

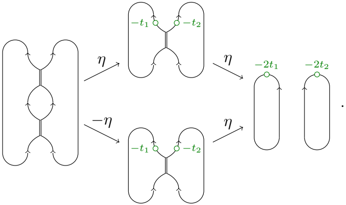


$\mathfrak{sl}_2$-action on link homology

For a link L, resolve each crossing in the following ways:



This gives rise to a complex where the objects are webs and the differentials are foams.



$\mathfrak{sl}_2$-action on link homology

Applying the functor $\mathcal{F}_N$ yields a complex of R_N-modules.

Theorem (Robert-Wagner)

The homology of this complex is the Khovanov-Rozansky homology $KR_N(L)$.

Theorem

$KR_N(L)$ is an $\mathfrak{sl}_2$-representation.

This action has recently been extended to a representation of a partial Witt algebra by Guerin–Roz.

$\mathfrak{sl}_2$-action on link homology

- For an integer $\lambda \in \mathbb{Z}$, there is an $\mathfrak{sl}_2$-module $M(\lambda)$ which is isomorphic to $\Bbbk[f]$ as a $\Bbbk[f]$-module with $e \cdot 1 = 0$, and $h \cdot 1 = \lambda$. This object is known as a Verma module with highest weight λ.
- For $\lambda \in \mathbb{Z}$, we let $L(\lambda)$ denote the irreducible $\mathfrak{sl}_2$-module whose highest weight is λ. If $\lambda \geq 0$, $L(\lambda)$ is finite-dimensional.
- Let $P(\lambda)$ and $P(-\lambda-2)$ be the indecomposable projective covers of $L(\lambda)$ and $L(-\lambda-2)$ respectively.

$\mathfrak{sl}_2$-action on link homology

The Verma module $M(\lambda)$ is isomorphic to $P(\lambda)$ and $P(-\lambda-2)$ can be described as an extension of Verma modules

$$0 \to M(\lambda) \to P(-\lambda-2) \to M(-\lambda-2) \to 0 \ .$$

We will also use the short exact sequence

$$0 \to L(-\lambda-2) \to M(\lambda) \to L(\lambda) \to 0 \ .$$

Taking the dual of this yields the short exact sequence

$$0 \to L(\lambda) \to M^*(\lambda) \to L(-\lambda-2) \to 0$$

since the simple objects are preserved by the contravariant duality functor $*$. These sequences describe the unique extensions between simples.

$\mathfrak{sl}_2$-action on link homology

- The Zuckerman functor $\Gamma\colon \mathfrak{sl}_2\text{-mod} \to \mathfrak{sl}_2\text{-mod}$ is the functor which takes an $\mathfrak{sl}_2$-module M and returns its maximally locally finite submodule ΓM under the action of $\mathfrak{sl}_2$.
- The Bernstein functor (sometimes known as the dual Zuckerman functor)

  $$\mathcal{Z}\colon \mathfrak{sl}_2\text{-mod} \to \mathfrak{sl}_2\text{-mod}$$

  is the functor which takes an $\mathfrak{sl}_2$-module M and returns its maximally locally finite quotient $\mathcal{Z}M$ under the action of $\mathfrak{sl}_2$.

$\mathfrak{sl}_2$-action on link homology

- $\Gamma L(\lambda) = L(\lambda)$ if $\lambda \geq 0$.
- $\Gamma L(\lambda) = 0$ if $\lambda < 0$.
- $\Gamma M(\lambda) = 0$.
- $\Gamma M^*(\lambda) = L(\lambda)$ if $\lambda \geq 0$.
- $\Gamma M^*(\lambda) = 0$ if $\lambda < 0$.
- $\Gamma P(\lambda) = 0$.

$\mathfrak{sl}_2$-action on link homology

Example

Let $N = 2$ and let L be the unknot. Then as an $\mathfrak{sl}_2$-representation,

$$KR_2(L) \cong P(-3) \oplus M(-1) \oplus \bigoplus_{r=3}^{\infty} M(1-2r)$$

$$\Gamma(KR_2(L)) = Z(KR_2(L)) = 0.$$

$\mathfrak{sl}_2$-action on link homology

Example

Let $N = 2$ and let L be the Hopf link Then as an $\mathfrak{sl}_2$-representation,

$$\begin{aligned} KR_2(L) \cong & t^0(M^*(0) \oplus \bigoplus_{r=1}^{\infty} M(-2r)) \oplus \\ & t^2(M^*(4) \oplus M^*(2) \oplus P(-2) \oplus \bigoplus_{r=0}^{\infty} M(-8-2r)). \end{aligned}$$

$$\Gamma(KR_2(L)) = t^0(L(0)) \oplus t^2(L(2) \oplus L(4)), \qquad Z(KR_2(L)) = 0.$$

$\mathfrak{sl}_2$-action on link homology

General $(2, n)$-torus links were computed by Felix Roz.

Conjecture

The homology of any link has a dual Verma filtration.

$\mathfrak{sl}_2$-action on link homology

The Rasmussen invariant $s(K)$ has a representation-theoretic interpretation.

- There is an action of $\mathbb{C}[x]$ on $KR_2(K)$ by choosing a basepoint on a knot K and considering a small unknot near the basepoint.
- Let $KR_2^{free}(K)$ denote the torsion free part with respect to this action.

Theorem

$KR_2^{free}(K)$ is an $\mathfrak{sl}_2$-representation. Assume $t_1 + t_2 = 1$. Then $s(k) = \mu(K) - 1$ where $\mu(K)$ is the highest weight of $KR_2^{free}(K)$.

Characteristic p

- The $\mathfrak{sl}_2$-action in characteristic 0 requires us to work over the equivariant ring $\mathbb{Z}[\frac{1}{2}][E_1, \ldots, E_N]$.
- Over $\mathbb{Z}[\frac{1}{2}]$, the unknot has homology $\mathbb{Z}[\frac{1}{2}][x]/(x^N)$.
- However $e(x^N) = -Nx^{N-1}$ so the ideal isn't preserved with this coefficient ring.
- But in characteristic $N = p$, one then has $e(x^N) = 0$.

We'll assume throughout that we were working over a field $\mathbb{k}$ of characteristic $p > 2$.

Characteristic p

Let $u(\mathfrak{sl}_2)$ denote the restricted enveloping algebra. That is, we quotient by $e^p = f^p = 0$ and $h^p = h$.

There are baby Verma $\Delta(\lambda)$ and dual baby Verma $\nabla(\Delta)$ modules

$$\begin{tikzcd}
\Delta(\lambda): & u_0 \arrow[loop above, "h=\lambda"] \arrow[r, bend left, "f=1"] & u_1 \arrow[loop above, "h=\lambda-2"] \arrow[l, bend left, "e=\lambda"] \arrow[r, bend left, "f=2"] & \cdots \arrow[l, bend left, "e=\lambda-1"] \arrow[r, bend left, "f=p-2"] & u_{p-2} \arrow[loop above, "h=\lambda-2p+4"] \arrow[l, bend left, "e=\lambda-(p-3)"] \arrow[r, bend left, "f=p-1"] & u_{p-1} \arrow[loop above, "h=\lambda-2p+2"] \arrow[l, bend left, "e=\lambda-(p-2)"]
\end{tikzcd}$$

$$\begin{tikzcd}
\nabla(\lambda): & v_0 \arrow[loop above, "h=\lambda"] \arrow[r, bend left, "f=-\lambda"] & v_1 \arrow[loop above, "h=\lambda-2"] \arrow[l, bend left, "e=p-1"] \arrow[r, bend left, "f=1-\lambda"] & \cdots \arrow[l, bend left, "e=p-2"] \arrow[r, bend left, "f=-3-\lambda"] & v_{p-2} \arrow[loop above, "h=\lambda-2p+4"] \arrow[l, bend left, "e=2"] \arrow[r, bend left, "f=-2-\lambda"] & v_{p-1} \arrow[loop above, "h=\lambda-2p+2"] \arrow[l, bend left, "e=1"]
\end{tikzcd}$$

$\Delta(p-1) \cong \nabla(p-1)$ are known as Steinberg modules.

Characteristic p

The reduced homology of a link is defined by:

- Choose a basepoint b.
- This gives rise to an action of the algebra $\Bbbk[x_b]/(x_b^p)$.
- Take homology of the subcomplex $\overline{C}_p(L) := x_b^{p-1} C_p(L)$.

Using the operator e, one has:

Theorem

The reduced homology $\overline{KR}_p(L)$ is independent of the basepoint.

- This was first proved by Shumakovitch for $p = 2$.
- It was generalized to any prime p by Wang.

Characteristic p

Theorem

$KR_p(L)$ has a dual baby Verma filtration.

Proof.

- Choose a basepoint b. Then there is an action of $\Bbbk[x_b]/(x_b^p)$.
- Consider the subcomplex for the link $x_b^{p-1}C(D)$.
- This subcomplex is then a $\Bbbk[f]/(f^p)$-module.
- Filter this module such that subquotients are 1-dimensional.
- Induce up to get an action of $u(\mathfrak{sl}_2)$ such that the subquotients are dual baby Verma modules.

$\square$

Characteristic p

Theorem

If K is a slice knot then $KR_p(K)$ contains a Steinberg summand.

Proof.

Since K is slice, there is a genus 0 cobordism from the unknot to K.

This induces a non-zero map from the homology of the unknot to K.

Since the homology of the unknot is simple, this map is injective.

Since the homology of the unknot is injective, it must be a direct summand.

☐

Characteristic p

- Suppose a link L has two components L_1, L_2.
- Choose basepoints on the two components.
- As a result there are actions of $\Bbbk[x_i]/(x_i^p)$ for $i = 1, 2$.

The homology of each component carries an $\mathfrak{sl}_2$-action satisfying:

$$h_i(x_j) = -\delta_{i,j} 2x_j, \qquad e_i(x_j) = -\delta_{i,j}, \qquad f_i(x_j) = \delta_{i,j} x_j^2$$

Theorem

There is an $\mathfrak{sl}_2 \times \mathfrak{sl}_2$*-action satisfying the above properties if and only if* L_1 *and* L_2 *are separated by an embedded sphere.*

Characteristic p

Proof.

This follows from a theorem of Wang who proved that the link is split if and only if as a $\Bbbk[x_1, x_2]/(x_1^p, x_2^p)$-module, the homology is free.

One could show that with the hypotheses of the $u(\mathfrak{sl}_2)$-action that the homology is free over $\Bbbk[x_1, x_2]/(x_1^p, x_2^p)$. ☐

- Lipshitz–Sarkar earlier proved that Khovanov homology detects split links.

Characteristic p

- Let $KR_p^{i,j}(L)$ denote the homology in homological degree j and q-degree i.
- Let
$$\operatorname{gdim} KR_p(L) = \sum_i \sum_j \dim KR_p^{i,j}(L) t^j q^i$$
- Let
$$\begin{aligned}\operatorname{gdim}_p KR_p(L) &= \sum_{i=0}^{2p-1} \sum_j \dim \sum_k KR_p^{i+2kp,j}(L) t^j q^i \\ &= \sum_{i=0}^{2p-1} \sum_j \lambda_{i,j} q^i t^j\end{aligned}$$

Characteristic p

A sequence $a_0, a_1, \ldots, a_n$ is symmetric unimodal if

$$a_0 \leq a_1 \leq a_{\frac{n-1}{2}} = a_{\frac{n+1}{2}} \geq a_{n-1} \geq a_n, \qquad a_i = a_{n-i}$$

Theorem

The homology of a web satisfies a unimodality property:

$$\lambda_{0,j} = \lambda_{2,j} = \cdots = \lambda_{2p-2,j}$$
$$\lambda_{1,j} = \lambda_{3,j} = \cdots = \lambda_{2p-1,j}$$

Proof.

This follows from the fact that the homology has a dual baby Verma filtration. ☐

Characteristic p

In order to categorify when q is a root of unity, one should first find a monoidal category whose Grothendieck group is isomorphic to $\mathbb{O}_p = \mathbb{Z}[q]/(\Phi_p(q^2))$.

- Let $\Bbbk$ be a field of characteristic p.
- Let $H_p = \Bbbk[\partial]/(\partial^p)$ be a graded algebra where the degree of ∂ is 2.
- H_p has a unique simple module (up to isomorphism and grade shift).
- Let L be the 1-dimensional module concentrated in degree 0.

Characteristic p

This implies that $K_0(H_p\text{-gmod}) \cong \mathbb{Z}[q, q^{-1}]$ where $[L\langle r\rangle] \mapsto q^r$.

As a module over itself, H_p has a filtration with subquotients $L, L\langle 2\rangle, \ldots, L\langle 2(p-1)\rangle$.

Thus in the Grothendieck group $[H_p] = q^{2(p-1)} + \cdots + 1$.

In order to categorify $\mathbb{O}_p$ we need a category where $H_p \cong 0$.

Characteristic p

Let $H_p\text{-}\underline{\mathrm{gmod}}$ be the stable category of H_p-modules.

Objects: same as $H_p\text{-}\mathrm{gmod}$.

Morphisms:

$$\mathrm{Hom}_{H_p\text{-}\underline{\mathrm{gmod}}}(M, N) = \mathrm{Hom}_{H_p\text{-}\mathrm{gmod}}(M, N)/I(M, N)$$

where $I(M, N)$ is the subspace of maps which factor through H_p.

Since the identity map of H_p is in the subspace we get the following result due to Khovanov.

Theorem

$K_0(H_p\text{-}\underline{\mathrm{gmod}}) \cong \mathbb{O}_p$.

Characteristic p

Theorem

Taking $\partial = e$ or $\partial = f$,

- *In the Grothendieck group of the stable category, $[KR_p(L)]$ is the $\mathfrak{gl}_p$ Reshetikhin-Turaev invariant at a 2pth root of unity.*
- *For $\partial = e$, in the stable category $KR_p(L) = 0$.*

Proof.

The second part follows from the fact that the homology has a dual baby Verma filtration. ☐